Cohomology of a Quaternionic Complex

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Abstract

We investigate the cohomology of a certain elliptic complex defined on a compact quaternionic-Kähler manifold with negative scalar curvature. We show that this particular complex is exact, with the possible exception of one term.

Let (M, g) be an oriented $4k$-dimensional compact quaternionic-Kähler manifold having negative scalar curvature. In [2] we proved a rigidity theorem for such manifolds which was, in essence, a consequence of the vanishing of a certain cohomolgy group on the twistor space Z, of M. The paper was largely devoted to the proof of a vanishing theorem for the cohomolgy of the first term of a particular complex on M, the required vanishing theorem on Z being deduced by the Penrose transform. This note is an extension of that work in that we use the same techniques to show that all the cohomology of the complex vanishes, with the possible exception of one term, so that the complex is exact except possibly at the right. We have not been able to establish whether the final term of the complex has non-trivial cohomology.

Since the preparation of this article it has been brought to our attention that a similiar result has been proved, though in a different way, by Nagato and Nitta.

As in [2], we shall make extensive use of the methods and techniques developed by Bailey and Eastwoood in [1], these being based on the abstract index notation of Roger Penrose [3]. In this notation the indices are used as 'place-markers' and do not indicate a choice of basis. Thus the E, H bundles of Salamon [4], are written as $\mathcal{O}^A, \mathcal{O}^{A'}$ respectively. The (complexified) tangent bundle of M is then $\mathcal{O}^a = \mathcal{O}^A \otimes \mathcal{O}^{A'} \stackrel{def}{=} \mathcal{O}^{AA'}$. The Levi-Civita connection on M is written $\nabla_a = \nabla_{AA'}$. We shall take the Riemann curvature tensor to be given by

$$2\nabla_{[a}\nabla_{b]}w_c = R_{abc}{}^d w_d \tag{1}$$

so that, in our convention, the standard metric on the sphere has positive curvature. (This differs from the convention adopted in [1] which was based on that of [3]. This latter is essentially a relativity book.) The square brackets in the definition of the Riemann tensor, denotes anti-symmetrisation. Round brackets will be used for symmetrisation. For more details on the notation and techniques used we refer the reader to [3].

When written in abstract index notation, the complex we consider is defined on M by the following

$$0 \longrightarrow \Gamma(M, \mathcal{O}^{(A'_1\cdots A'_n)}) \xrightarrow{\nabla^{A'_{n+1}}_{B_1}} \Gamma(M, \mathcal{O}^{(A'_1\cdots A'_{n+1})}_{B_1}) \xrightarrow{\nabla^{A'_{n+2}}_{B_2}} \Gamma(M, \mathcal{O}^{(A'_1\cdots A'_{n+2})}_{[B_1 B_2]}) \longrightarrow \cdots$$

$$\cdots \xrightarrow{\nabla^{A'_{n+2k}}_{B_{2k}}} \Gamma(M, \mathcal{O}^{(A'_1\cdots A'_{n+2k})}_{[B_1\cdots B_{2k}]}) \longrightarrow 0 \qquad (2)$$

Here, for example, the bundle $\mathcal{O}^{(A'_1\cdots A'_{n+2})}_{[B_1 B_2]}$ is $\bigwedge^2 E^* \otimes S^{n+2} H$ (or its sheaf of smooth sections), and $\Gamma(M, V)$ is the space of global smooth sections of the bundle V over M. The map $\nabla^{A'_1}_{A}$ is then $\varepsilon^{A'_1 A'} \nabla_a$ with $\varepsilon^{A'_1 A'}$ being the (covariantly constant) symplectic form on H^*. This complex was studied by Salamon in [5] and shown by him to be an elliptic complex. Since M is compact we may use Hodge theory [6] to examine the cohomology of this complex. It is well known that the cohomology of the first term vanishes for negative scalar curvature. The vanishing of the cohomology of the second term was proved in [2], but can easily be obtained from the proof of theorem (0.3) with the appropriate changes. For terms other than the last Hodge theory implies that each cohomology class has a unique representative f which satisfies

$$\begin{array}{ll} \text{(a)} & f \in \Gamma(M, \mathcal{O}^{(A'_1\cdots A'_p)}_{[B_1\cdots B_m]}) \\ \text{(b)} & \nabla^{(A'_0}_{[B_0} f^{A'_1\cdots A'_p)}_{B_1\cdots B_m]} = 0 \\ \text{(c)} & \nabla_{AC'} f^{AC'A'_2\cdots A'_p}_{B_2\cdots B_m} = 0 \end{array} \qquad (3)$$

where $1 < m < 2k$ and $p - m = n$. If we let $D_0 = \nabla^{A'_{n+1}}_{B_1}$ and $D_1 = \nabla^{A'_{n+2}}_{B_2}$ in (2), then with $m = 1$, (b) is the abstract index version of $D_1 f = 0$ and (c) is that of $D_0^* f = 0$, D_0^* being the adjoint of the mapping D_0.

We shall take $u^A \overline{v}_A$ to be the positive definite inner- product, where $\overline{v}$ is the quaternionic conjugate of v, on the bundle $\mathcal{O}^A$, so that $u^A \overline{u}_A$ is positive if $u \neq 0$, and similarly for other tensors. The L^2-norm of a tensor $v = v^{AB'}{}_{CD'}$ for example, is then

$$\| v \|^2 = \int v^{AB'CD'} \overline{v}_{AB'CD'} \qquad (4)$$

with the integral taken over the manifold M. With this convention, one can use integration by parts over M and quickly obtain (c) above as the abstract index version of $D_0^* f = 0$, when $m = 1$. We shall show that these terms all vanish, so that the above complex (2) is exact, except possibly at the right. We note the following elementary facts.

Lemma 0.1 *If $S^{A'_0\cdots A'_p}$ is symmetric in its final p indices and $U_{B_0\cdots B_m}$ is antisymmetric in* *its final m indices, then*

$$(p+1)S^{(A'_0\dots A'_p)} = (p+1)S^{A'_0\dots A'_p} - \sum_{i=1}^{i=p}\varepsilon^{A'_0A'_i}S_{C'}{}^{C'A'_1\dots\hat{A'}_i\dots A'_p} \qquad (5)$$

$$(m+1)U_{[B_0\dots B_m]} = U_{B_0\dots B_m} + \sum_{i=1}^{i=m}(-1)^iU_{B_iB_0\dots\hat{B}_i\dots B_m} \qquad (6)$$

It is then quite easy to obtain the following.

Lemma 0.2 *If f satisfies the conditions of (3) then*

$$(p+1)\parallel \nabla^{A'_0}_{[B_0}f^{A'_1\dots A'_p}_{B_1\dots B_m]}\parallel^2 = p\parallel \nabla_{A'[B_0}f^{A'A'_2\dots A'_p}_{B_1\dots B_m]}\parallel^2$$

and $\nabla^{BA'_0}f^{A'_1\dots A'_p}_{BB_2\dots B_m}$ is symmetric in its primed indices.

Proof For the first part we put $S^{A'_0\dots A'_p} = \nabla^{A'_0}_{[B_0}f^{A'_1\dots A'_p}_{B_1\dots B_m]}$ in (5) and using (b) of (3) we get

$$(p+1)\nabla^{A'_0}_{[B_0}f^{A'_1\dots A'_p}_{B_1\dots B_m]} = \sum_{i=1}^{i=p}\varepsilon^{A'_0A'_i}\nabla_{A'[B_0}f^{A'A'_2\dots A'_p}_{B_1\dots B_m]}$$

Contracting both sides of the above with $\nabla^{[B_0}_{A'_0}\bar{f}^{B_1\dots B_m]}_{A'_1\dots A'_p}$ and integrating the result over M, one quickly obtains the first part. The second part can be obtained by putting $S^{A'_0\dots A'_p} = \nabla^{BA'_0}f^{A'_1\dots A'_p}_{BB_2\dots B_m}$ in (5) and using (b) of (3).

We can now state our main result.

Theorem 0.3 *Let M be a compact quaternionic-Kähler manifold of dimension $4k$, for $k > 1$, and let $\Lambda = R/8k(k+2)$, where R is its (non-zero) scalar curvature. If f is a harmonic element of $\Gamma(M, \mathcal{O}^{(A'_1\dots A'_p)}_{[B_1\dots B_m]})$, i.e. satisfies the conditions of(3), and if $1 < m < 2k$, then*

$$\frac{(p+2)}{2(p+1)}\parallel \nabla^{[B_0}_{A'}f^{B_1\dots B_m]A'A'_2\dots A'_p}\parallel^2 + \frac{m}{2(m+1)}\parallel \nabla^{C'}_{B}f^{BB_2\dots B_mA'_1\dots A'_p}\parallel^2 = \Lambda\frac{(p+2)}{(m+1)}(2k-m)\parallel f\parallel^2 \qquad (7)$$

Thus if $R < 0$ then $f = 0$.

Proof We have

$$
\begin{aligned}
\| \nabla^{[B_0}_{A'} f^{B_1\dots B_m]A'A'_2\dots A'_p} \|^2 &= \int (\nabla^{[B_0}_{K'} f^{B_1\dots B_m]K'A'_2\dots A'_p})(\nabla_{C'[B_0} \bar{f}^{C'}_{B_1\dots B_m]A'_2\dots A'_p}) \\
&= -\int f^{B_1\dots B_m K'A'_2\dots A'_p} \nabla^{B_0}_{K'} \nabla_{C'[B_0} \bar{f}^{C'}_{B_1\dots B_m]A'_2\dots A'_p} \\
&= -\int f^{B_1\dots B_m K'A'_2\dots A'_p} (\nabla^{B_0}_{(K'} \nabla_{C')[B_0} \\
&\quad + \nabla^{B_0}_{[K'} \nabla_{C'][B_0}) \bar{f}^{C'}_{B_1\dots B_m]A'_2\dots A'_p} \qquad (8)
\end{aligned}
$$

where the second line above is obtained from the first by using integration by parts over the manifold. The latter integral on the right is

$$
\begin{aligned}
\frac{1}{2}\int f^{B_1\dots B_m K'A'_2\dots A'_p} \varepsilon_{K'C'} \nabla^{B_0}_{D'} \nabla^{D'}_{[B_0} \bar{f}^{C'}_{B_1\dots B_m]A'_2\dots A'_p} &= \frac{1}{2}\int f^{B_1\dots B_m A'_2\dots A'_p}_{C'} \nabla^{B_0}_{D'} \nabla^{D'}_{[B_0} \bar{f}^{C'}_{B_1\dots B_m]A'_2\dots A'_p} \\
&= -\frac{1}{2} \| \nabla^{A'_0}_{[B_0} f^{A'_1\dots A'_p}_{B_1\dots B_m]} \|^2
\end{aligned}
$$

the final expression being obtained from the previous one by using integration by parts again. Using the first part of lemma (0.2) and rearranging, (8) becomes

$$
\frac{(p+2)}{2(p+1)} \| \nabla^{[B_0}_{A'} f^{B_1\dots B_m]A'A'_2\dots A'_p} \|^2 = -\int f^{B_1\dots B_m K'A'_2\dots A'_p} \nabla^{B_0}_{(K'} \nabla_{C')[B_0} \bar{f}^{C'}_{B_1\dots B_m]A'_2\dots A'_p} \qquad (9)
$$

Examining the latter integrand we see that

$$
\begin{aligned}
\nabla^{B_0}_{(K'} \nabla_{C')[B_0} \bar{f}^{C'}_{B_1\dots B_m]A'_2\dots A'_p} &= \frac{1}{m+1}(\nabla^{B_0}_{(K'} \nabla_{C')B_0} \bar{f}^{C'}_{B_1\dots B_m A'_2\dots A'_p} \\
&\quad + \sum_{i=1}^{i=m} (-1)^i \nabla^{B_0}_{(K'} \nabla_{C')B_i} \bar{f}^{C'}_{B_0 B_1\dots \hat{B}_i\dots B_m A'_2\dots A'_p}) \qquad (10)
\end{aligned}
$$

Now by definition

$$
\nabla^{B_0}_{(K'} \nabla_{C')B_i} - \nabla_{B_i(K'} \nabla^{B_0}_{C')} = \Box_{K'C'}{}^{B_0}{}_{B_i}.
$$

(For the definition and properties of the curvature operators see [1]) The second operator on the left above is $\frac{1}{2}(\nabla_{B_iK'}\nabla^{B_0}_{C'} + \nabla_{B_iC'}\nabla^{B_0}_{K'})$ and the first part of this sum annihilates $\bar{f}$ by (c) of (3). Equation (9) can now be rewritten

$$
\begin{aligned}
\frac{(p+2)}{2(p+1)} \| \nabla^{[B_0}_{A'} f^{B_1\dots B_m]A'A'_2\dots A'_p} \|^2 &= -\frac{1}{m+1}\int f^{B_1\dots B_m K'A'_2\dots A'_p} (\Box_{K'C'}{}^{B_0}{}_{B_0} \bar{f}^{C'}_{B_1\dots B_m A'_2\dots A'_p}) \\
&\quad -\frac{1}{(m+1)} \sum_{i=1}^{i=m} (-1)^i \int f^{B_1\dots B_m K'A'_2\dots A'_p} (\Box_{K'C'}{}^{B_0}{}_{B_i} \bar{f}^{C'}_{B_0 B_1\dots \hat{B}_i\dots B_p A'_2\dots A'_p}) \\
&\quad -\frac{1}{2(m+1)} \sum_{i=1}^{i=m} (-1)^i \int f^{B_1\dots B_m K'A'_2\dots A'_p} (\nabla_{B_iC'} \nabla^{B_0}_{K'} \bar{f}^{C'}_{B_0 B_1\dots \hat{B}_i\dots B'_m A'_2\dots A'_p})
\end{aligned}
$$

Using the symmetry in the primed indices given by the latter part of lemma (0.2) we may commute the final K' and C' in the final integral above and, after another application of integration by parts, this final integral becomes

$$-\int(\nabla_{B_iC'}f^{B_1\ldots B_mK'A_2'\ldots A_p'})(\nabla^{B_0C'}\bar{f}_{B_0B_1\ldots\hat{B}_i\ldots B_mK'A_2'\ldots A_p'})$$

$$=(-1)^i\int(\nabla^{C'}_{B_i}f^{B_iB_1\ldots\hat{B}_i\ldots B_mK'A_2'\ldots A_p'})(\nabla_{B_0C'}\bar{f}^{B_0}{}_{B_1\ldots\hat{B}\ldots B_mK'A_2'\ldots A_p'})$$

$$=(-1)^i\parallel\nabla^{C'}_{B}f^{BB_2\ldots B_mA_1'\ldots A_p'}\parallel^2$$

With this in mind we may now rearrange the above equation to obtain

$$\frac{(p+2)}{2(p+1)}\parallel\nabla^{[B_0}_{A'}f^{B_1\ldots B_m]A'A_2'\ldots A_p'}\parallel^2+\frac{m}{2(m+1)}\parallel\nabla^{C'}_{B}f^{BB_2\ldots B_mA_1'\ldots A_p'}\parallel^2=-\frac{1}{m+1}\int R(f) \tag{11}$$

where $R(f)$ is given by

$$R(f)=f^{B_1\ldots B_mK'A_2'\ldots A_p'}(\square_{K'C'}{}^{B_0}{}_{B_0}\bar{f}^{C'}{}_{B_1\ldots B_mA_2'\ldots A_p'}+\sum_{i=1}^{i=m}(-1)^i\square_{K'C'}{}^{B_0}{}_{B_i}\bar{f}^{C'}{}_{B_0B_1\ldots\hat{B}_i\ldots B_pA_2'\ldots A_p'})$$

The evaluation of the action of the curvature operators is an elementary exercise. We have

$$\square_{K'C'}{}^{B_0}{}_{B_0}\bar{f}^{C'}{}_{B_1\ldots B_mA_2'\ldots A_p'}=-2\Lambda k(p+2)\bar{f}_{B_1\ldots B_mK'A_2'\ldots A_p'}$$

and the other operator gives

$$\begin{aligned}
\square_{K'C'}{}^{B_0}{}_{B_i}\bar{f}^{A_1'\ldots A_p'}{}_{B_0\ldots\hat{B}_i\ldots B_p} &= -2\Lambda\epsilon^{B_0}{}_{B_i}\sum_{j=1}^{j=p}\varepsilon_{(K'}{}^{A_j'}\varepsilon_{C')Q'}\bar{f}^{Q'A_1'\ldots\hat{A}_j'\ldots A_p'}{}_{B_0\ldots\hat{B}_i\ldots B_m}\\
&= 2\Lambda\sum_{j=1}^{j=p}\varepsilon_{(K'}{}^{A_j'}\varepsilon_{C')Q'}\bar{f}^{Q'A_1'\ldots\hat{A}_j'\ldots A_p'}{}_{B_iB_1\ldots\hat{B}_i\ldots B_m}\\
&= (-1)^{i-1}2\Lambda\sum_{j=1}^{j=p}\varepsilon_{(K'}{}^{A_j'}\varepsilon_{C')Q'}\bar{f}^{Q'A_1'\ldots\hat{A}_j'\ldots A_p'}{}_{B_1\ldots B_m}\\
\square_{K'C'}{}^{B_0}{}_{B_i}\bar{f}^{C'}{}_{B_0\ldots\hat{B}_i\ldots B_mA_2'\ldots A_p'} &= (-1)^i\Lambda(p+2)\bar{f}_{B_1\ldots B_mK'A_2'\ldots A_p'}
\end{aligned}$$

so that $R(f)=-\Lambda(p+2)(2k-m)\bar{f}_{B_1\ldots B_mK'A_2'\ldots A_p'}$. Substituting this into (11) completes the proof. □

References

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