43

A Nonlinear Graviton from the Sine-Gordon equation

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In **TN** 30 [1] Lionel Mason pointed out that certain integrable systems may be encoded within a geometry of the nonlinear graviton. This is true at least for integrable equations arising from $SL(2,C)$ anti-self-dual Yang-Mills equations (ASDYM). The purpose of this note is to describe how this construction works, illustrating the method with the example of the Sine-Gordon equation.
Let $\mathcal{M}$ be a real four-manifold with a volume form ν. Let $V_i = (W, \widetilde{W}, Z, \widetilde{Z})$ be real, independent, volume preserving vector fields on $\mathcal{M}$. Define $f^2 = \nu(W,\ \widetilde{W},\ Z,\ \widetilde{Z})$. Mason and Newman's [2] form of the nonlinear graviton theorem states that if

$$L = W - \lambda\widetilde{Z}, \qquad M = Z - \lambda\widetilde{W} \tag{1}$$

commute for each value of $\lambda \in CP^1$, then $f^{-1}V_i$ is a normalized null tetrad for an anti self dual vacuum metric. The metric has a signature $(++--)$. ASDYM possess a Lax formulation isomorphic to that of the ASD Einstein. We use $x^\mu = (w, \tilde{w}, z, \tilde{z})$ as coordinates on R^4 that are independent and real for signature $(++--)$. ASDYM are equivalent to the commutativity of the Lax pair

$$L = D_w - \lambda D_{\tilde{z}}, \quad M = D_z - \lambda D_{\tilde{w}}. \tag{2}$$

Here $D_\mu = \partial_\mu - A_\mu$ is a covariant derivative. We fix the gauge group to be $SU(2)$ and put $A_\mu = A^a{}_\mu \sigma_a$, where

$$\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

We impose symmetries in the ∂_w and $\partial_{\tilde{w}}$ directions so that, in an invariant gauge, $A^a{}_\mu$ are functions of $(z, \tilde{z})$. The passage to ASD gravity is based on re-expressing σ_a as real hamiltonian vector fields X_{h_a} on $\Sigma = CP^1$ with respect to a symplectic structure $\Omega_\Sigma = i(1+\xi\bar{\xi})^{-2}(d\xi \wedge d\bar{\xi})$. Hamiltonians h_a are found from the Möbius action of $SU(2)$ on a Riemann sphere,

$$h_1 = -\frac{\xi+\bar{\xi}}{1+\xi\bar{\xi}}, \quad h_2 = -i\frac{\xi-\bar{\xi}}{1+\xi\bar{\xi}}, \quad h_3 = \frac{2}{1+\xi\bar{\xi}}, \qquad X_{h_b}(h_a) = 2\epsilon_{ab}{}^c h_c. \tag{3}$$

Now D_μ become volume preserving vector fields on $\mathcal{M} = R^2 \times CP^1$ with $\nu = dz \wedge d\tilde{z} \wedge \Omega_\Sigma$. We identify D_μ with V_i;

$$W = -A^a{}_w X_{h_a}, \quad \widetilde{W} = -A^a{}_{\tilde{w}} X_{h_a}, \quad Z = \partial_z - A^a{}_z X_{h_a}, \quad \widetilde{Z} = \partial_{\tilde{z}} - A^a{}_{\tilde{z}} X_{h_a}.$$

A covariant metric on $\mathcal{M}$ is conveniently expressed by a dual frame e_{V_i} such that $e_{V_i}(V_j) = \delta_{ij}$. A simple calculation yields

$$\begin{aligned}
f^2 &= 2A^a{}_w A^b{}_{\tilde{w}} \epsilon_{ab}{}^c h_c, \\
e_W &= -f^{-2} A^b{}_{\tilde{w}} (\partial_\xi h_b d\xi + \partial_{\bar{\xi}} h_b d\bar{\xi} - 2\epsilon_{ab}{}^c h_c (A^a{}_z dz + A^a{}_{\tilde{z}} d\tilde{z})) \\
&= -f^{-2} A^a{}_{\tilde{w}} D_\Sigma h_a, \\
e_{\widetilde{W}} &= f^{-2} A^b{}_w (\partial_\xi h_b d\xi + \partial_{\bar{\xi}} h_b d\bar{\xi} - 2\epsilon_{ab}{}^c h_c (A^a{}_z dz + A^a{}_{\tilde{z}} d\tilde{z})) \\
&= f^{-2} A^a{}_w D_\Sigma h_a, \\
e_Z &= dz, \quad e_{\widetilde{Z}} = d\tilde{z},
\end{aligned} \tag{4}$$

where $D_\Sigma = d_\Sigma - A^a{}_\mu dx^\mu \otimes X_{h_a}$ is the form valued operator acting on functions. Note that D_Σ decomposes to $\mathcal{D}_\Sigma + \overline{\mathcal{D}}_\Sigma$, where

$$\begin{aligned}
\mathcal{D}_\Sigma &= (d\xi - iA^a{}_\mu (1+\xi\bar{\xi})^2 \partial_{\bar{\xi}} h_a dx^\mu) \otimes \partial_\xi = d_A\xi \otimes \partial_\xi, \\
\overline{\mathcal{D}}_\Sigma &= (d\bar{\xi} + iA^a{}_\mu (1+\xi\bar{\xi})^2 \partial_\xi h_a dx^\mu) \otimes \partial_{\bar{\xi}} = \overline{d_A\xi} \otimes \partial_{\bar{\xi}}.
\end{aligned} \tag{5}$$

Finally we consider the metric

$$ds^2 = f^2 dz \otimes d\tilde{z} + f^{-2} A^a{}_{\tilde{w}} A^b{}_w D_\Sigma h_a \otimes D_\Sigma h_b. \tag{6}$$

This may be viewed as a metric on the total space of the Σ-bundle associated to a YM bundle. An infinitesimal gauge transformation $A^a{}_\mu \longrightarrow A^a{}_\mu + \tau(\epsilon_{ab}{}^c A^b{}_\mu F^c + \partial_\mu F^a)$, where F^a are functions of z and $\tilde{z}$, is equivalent to the diffeomorphism of $R^2 \times CP^1$ given by $y^\nu \longrightarrow y^\nu + \tau F^a X_{h_a}(y^\nu)$. Tensor objects transform by Lie derivative along $F^a X_{h_a}$. As a consequence D_Σ undergoes the same type of transformation as a covariant deriative .
Singularities in (6) which come from the base space may be removed by imposing the boundary conditions on the YM connection. However we are

left with singularities of the vector fields X_{h_a} for ξ being real or purely imaginary.

To establish the connection with Sine-Gordon, we express $A^a{}_\mu$ in terms of its solutions. This reduction was noted by a number of authors. We use a gauge choice due to Ablowitz and Chakravarty [3].

Set $A_{\tilde{z}} = 0$. ASDYM with $G = SU(2)$ are solved by anzatzs

$$A_w = \cos\phi\,\sigma_1 + \sin\phi\,\sigma_2, \quad A_{\tilde{w}} = \sigma_1, \quad A_z = 1/2\partial_{\tilde{z}}\phi\,\sigma_3 \tag{7}$$

provided that ϕ satisfies $\partial_{z\tilde{z}}\phi = 4\sin\phi$.

Using (7) we obtain: $f^2 = 2(1+\xi\bar{\xi})^{-1}\sin\phi$, $d_A\xi = d\xi + i\xi\partial_{\tilde{z}}\phi dz$, and

$$ds^2 = \frac{1}{1+\xi\bar{\xi}}\Big([(1-\bar{\xi}^2)^2\cot\phi + i(1-\bar{\xi}^4)]d_A\xi\otimes d_A\xi$$

$$+\Big(\cot\phi(1-\bar{\xi}^2)(1-\xi^2) + i[(1+\bar{\xi}^2)(1-\xi^2) - (1-\bar{\xi}^2)(1+\xi^2)]\Big)\,d_A\xi\otimes\overline{d_A\xi}$$

$$+[(1-\xi^2)^2\cot\phi + i(1-\xi^4)]\overline{d_A\xi}\otimes\overline{d_A\xi} + 2\sin\phi\,dz\otimes d\tilde{z}\Big) \tag{8}$$

Now we can use the whole machinery (Bäcklund transformations, topological conservation laws, etc.) known from the theory of Sine-Gordon to deal with (8). If one takes a solution describing the interaction of half kink and half anti-kink (two topological solitons traveling in $z-\tilde{z}$ direction and increasing from 0 to π as $z+\tilde{z}$ goes from $-\infty$ to ∞) then the singularity in $\sin\phi = 0$ may be absorbed by a conformal transformation of $z+\tilde{z}$.

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References

[1] Mason, L.J. (1990) $\mathcal{H}$-space, a universal integrable system?, Twistor Newsletter 30.

[2] Mason, L.J. & Newman, E.T. (1989) A connection between the Einstein and Yang-Mills equations, Comm. Math. Phys., **121**, 659-668.

[3] Chakravarty, S. & Ablowitz (1992) On reductions of self-dual Yang Mills equations, in *Painlevé transcendents*, D. Levi & P. Winternitz, Plenum Press, New York.