Twistor Newsletter

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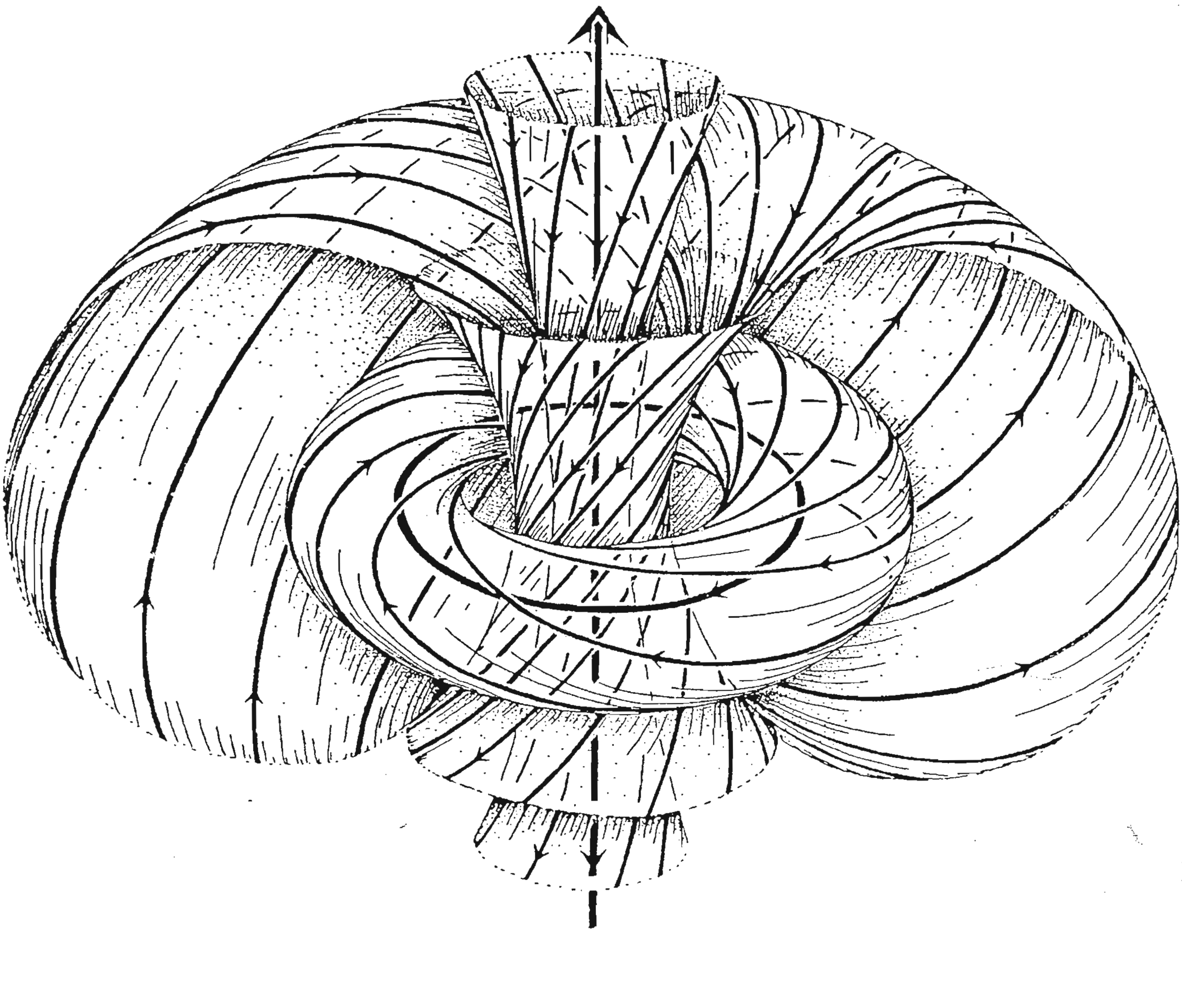

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New Ideas for the Googly Graviton

In my articles in $\mathbb{T}$N 31(pp.6–8), 32(1–5), 38(1–9), 39(1–5), in *Gravitation and Modern Cosmology* (eds. A. Zichichi, N. de Sabbata, & N. Sánchez; Plenum 1991), in *Twistor Theory* (ed. S. Huggett; Marcel Dekker 1995); and in L.J.M. & R.P. $\mathbb{T}$N 37(1–6), J.F. $\mathbb{T}$N 37(7–9), J-P.N. $\mathbb{T}$N 39(6–10), and elsewhere, it has been suggested that the appropriate twistor-space procedure for encoding the structure of an arbitrary vacuum space-time $\mathcal{M}$ would be to find the space $\mathcal{T}$ of "charges" for helicity $\frac{3}{2}$ massless fields — or, rather, for potentials $\sigma_{A'B'}^{\ C}$ for such fields. This suggestion was based on the following two observations:

(1) the field equations (Dirac or Rarita-Schwinger) for $\sigma_{A'B'}^{\ C}$ are <u>consistent</u> iff $\mathcal{M}$ is Ricci-flat,

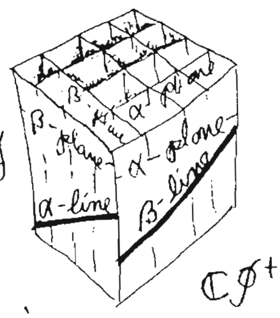
(2) in Minkowski space $\mathbb{M}$, the space of <u>charges</u> for $\sigma_{A'B'}^{\ C}$ is standard twistor space $\mathbb{T}$.

The programme, then, is to try to find the appropriate notion of "charge" for such fields for a general Ricci-flat $\mathcal{M}$, and then to try to see how to reconstruct $\mathcal{M}$ (with vacuum equations automatically satisfied) from the "twistor space" $\mathcal{T}$ of such charges.

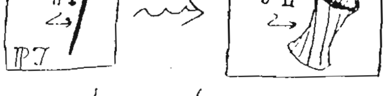
In general, there appear to be severe difficulties standing in the way of constructing an appropriate twistor space in accordance with these ideas (cf. particularly R.P. in $\mathbb{T}$N 33(1–6) and the above ref. in *Grav. & Mod. Cosmol.* 1991). The most plausible procedure for an asymptotically flat (analytic) $\mathcal{M}$ seems to be to go to infinity, say to $\mathscr{I}^+$, and see how best to define the space of helicity 3/2 charges in terms of "glitches" in the potentials at $\mathscr{I}^+$ or $\mathbb{C}\mathscr{I}^+$. These glitches would have to be not only in the first potential $\sigma_{A'B'}^{\ C}$ but also in a second potential $\rho_{A'}^{\ BC}$, all restricted to $\mathscr{I}^+$ (or $\mathbb{C}\mathscr{I}^+$). Recent work by Jorge Frauendiener, Ted Newman, and Joy Ghose has explored the nature of $\mathcal{T}$ constructed in this way, and seems to confirm my general expectation that $\mathcal{T}$ ought to turn out to be the <u>asymptotic twistor space</u> of $\mathcal{M}$.

However, according to the understanding that has existed to date, the asymptotic twistor space of $\mathcal{M}$ encodes only the anti-self-dual (ASD) part of the radiation field of $\mathcal{M}$ — according to the $\mathcal{H}$-space/non-linear graviton construction — and not the self-dual (SD) part. (It should be borne in mind that all this information is to be encoded holomorphically, the operation of complex conjugation, or reality structure, not being something that one may call upon. Thus, we are, for the time being at least, operating within the framework of complex general relativity.) Accordingly, it appears to be essential to re-examine the googly programme for ascertaining how the SD part of the radiation field may also be holomorphically encoded in the structure of $\mathcal{T}$ (and not simply in that of the dual asymptotic twistor space $\mathcal{T}^*$). Ultimately, some information about sources for the gravitational field would also have to be encoded somehow. This article describes various ideas that, I believe, point to some significant new progress towards a solution of the googly problem.

Let us first recall the way in which asymptotic twistor space arises in terms of $\mathbb{C}\mathscr{I}^+$. We may regard $\mathbb{C}\mathscr{I}^+$ as a union of α-planes, or alternatively as a union of β-planes. There are null geodesics on $\mathbb{C}\mathscr{I}^+$ of three types: α-lines, β-lines, and generators, the generators being limiting cases of α-lines and also of β-lines. Each α-line lies on a unique β-plane and each β-line on a unique α-plane (all on $\mathbb{C}\mathscr{I}^+$), each generator lying on one of each. The "finite" part of projective asymptotic twistor space $\mathbb{P}\mathcal{T}^b$ is simply the space of α-lines (and the corresponding dual twistor space $\mathbb{P}\mathcal{T}^{*b}$ is the space of β-lines).



In the flat case of $\mathbb{M}$, $\mathbb{P}\mathcal{T}^b = \mathbb{P}\mathbb{T}^b$, which is the ordinary twistor space $\mathbb{P}\mathbb{T}$, but with the line $\mathbb{P}I$ (representing infinity) removed. The entire space $\mathbb{P}\mathbb{T}$ arises if we include the α-planes on $\mathbb{C}\mathscr{I}^+$ in addition to the α-lines, these various α-planes providing the points of $\mathbb{P}I$. This procedure could be adopted in the case of a general $\mathbb{C}\mathscr{I}^+$ also, but then it turns out that the resulting space $\mathbb{P}\mathcal{T}$ is not a smooth complex manifold, being singular along $\mathbb{P}I$. In some appropriate sense, it is the nature of this singularity along $\mathbb{P}I$ that contains all the googly gravitational information.

One standard way to proceed is to blow up the twistor space, so the line $\mathbb{P}I$ becomes a quadric surface $\mathbb{P}\mathbb{I}$:
In the flat case, this is achieved by $\frac{1}{z} \rightsquigarrow \frac{1}{z\,\bar{z}}$], 

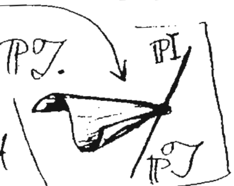
i.e. by

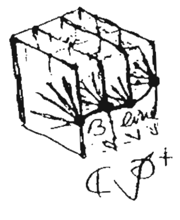
$$(\omega^A, \pi_{A'}) \rightsquigarrow (\omega^A \pi^{B'}, \pi_{A'}\pi^{B'}) = (\omega^{AB'}, \pi_{A'}{}^{B'})$$

where $\omega^{AA'}\pi_{A'}{}^{B'} = 0$, $\pi_{A'}{}^{A'} = 0$.

The space $\mathbb{T}^{\#}$ of blown up twistors is the space of such $(\omega^{AB'}, \pi_{A'}{}^{B'})$. For the space $\mathbb{PT}^\#$ of projective blown up twistors, we factor out by an overall factor. $\mathbb{PT}^\#$ is a (non-singular) complex 3-manifold in $\mathbb{CP}^6$. The projective blown-up twistors (elements of $\mathbb{PT}^\#$) represent α-lines on $\mathbb{C}\mathscr{I}^+$ together with their limits, the generators of $\mathbb{C}\mathscr{I}^+$. This limiting procedure is achieved by applying $\gamma \to 0$ to $(\gamma^{-1}\omega^A, \gamma\pi_{A'})$, which leaves us with blown-up twistors of the form $(\omega^A\pi^{A'}, 0)$, where $\pi^{A'}$ specifies a β-plane and ω^A an α-plane on $\mathbb{C}\mathscr{I}^+$ these intersecting in the generator in question. The α-lines and generators are scaled in terms of parallelly propagated covectors "pointing along" them (a conformally invariant notion), which gives the scalings needed to define non-projective blown-up twistors (elements of $\mathbb{T}^\#$).

In the curved case, this definition in terms of scaled α-lines and scaled generators works just as well as in the flat case, and provides us with the asymptotic (blown-up) twistor space $\mathscr{T}^\#$ (provisional definition). Dropping the scaling, we get $\mathbb{P}\mathscr{T}^\#$. It is not hard to see that $\mathbb{P}\mathscr{T}^\#$ is a non-singular complex manifold. The trouble, however, is that now the (googly) information of the SD radiation field seems to have got lost. This information lies in the location of the β-lines on $\mathbb{C}\mathscr{I}^+$. A β-line can be represented by the family of α-lines which meet it, i.e. by the corresponding "crinkly cone" in $\mathbb{P}\mathscr{T}$.





The fact that these crinkly cones cannot be consistently represented as having flat tangent planes at their vertices on $\mathbb{PI}$ is a manifestation of the fact that $\mathbb{PI}$ is singular for $\mathbb{P}\mathscr{T}$. When we blow-up, we get smooth surfaces looking like

What is being proposed here is that if we specify the appropriate structure on $\mathbb{P}\mathscr{T}$ or $\mathbb{P}\mathscr{T}^\#$, then these surfaces will be singled out. It is not expected that this structure will manifest itself on $\mathscr{T}^\flat$, but rather in the way that $\mathbb{I}$ (or whatever is appropriate) is attached to $\mathscr{T}^\flat$ to yield $\mathscr{T}^\#$.

In TN 23(1-4), the forms

$$\delta = I Z dZ \quad \text{and} \quad \mathcal{D} = \frac{1}{6} I Z dZ \wedge dZ \wedge dZ,$$

both well defined on $\mathcal{T}^b$, with

$$\delta_\wedge \mathcal{D} = 0\,, \quad \delta_\wedge d\delta = 0\,,$$

were considered. It was noted that in the flat case the quantity

$$\Gamma = \delta \otimes \mathcal{D}$$

(a $\boxplus$-tableau quantity) is regular (and non-zero) on $\mathbb{T}^\#$, whereas $\mathcal{D}$ blows up at $\mathbb{I}$ and δ goes to zero there. We can see this by choosing local coordinates ($w = 0$ giving $\mathbb{I}$):

$$w = (Z^2)^2 = \pi_{0'}\pi_{0'}\,, \quad x = \frac{Z^1}{Z^0} = \frac{\omega^1}{\omega^0}\,, \quad y = Z^0 Z^2 = \omega^0 \pi_{0'}\,, \quad z = \frac{Z^3}{Z^2} = \frac{\pi_{1'}}{\pi_{0'}}$$

whence

$$\delta = w\, dz\,, \quad \mathcal{D} = \frac{y}{w}(w\,dy - y\,dw)_\wedge dx_\wedge dz\,,$$

$$\Gamma = dz \otimes dz_\wedge (w\,dy - y\,dw)_\wedge y\,dx_\wedge dz\,.$$

In TN 23, the hope was expressed that some differential operator might be found, which is coordinate independent for objects of the nature of Γ, and whose kernel would, in some appropriate sense, be objects defined by twistor functions homogeneous of degree -6. In fact, there *is* such an operator, defined as follows. I shall use the notation $\textcircled{\scriptsize\wedge}$ for a product of a 2-form with an n-form defined by

$$(dp_\wedge dq) \textcircled{\scriptsize\wedge} \alpha = dp \otimes dq_\wedge \alpha - dq \otimes dp_\wedge \alpha$$

and extended by linearity to arbitrary 2-forms. Then the required operator is given by

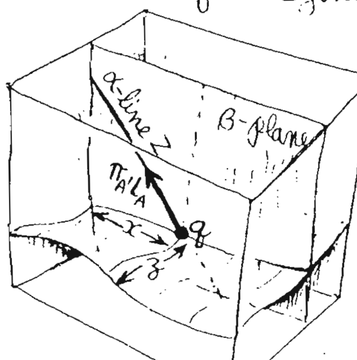
$$D(\delta \otimes \mathcal{D}) = d\delta \textcircled{\scriptsize\wedge} \mathcal{D} - \delta \otimes d\mathcal{D}$$

whence (by $\delta_\wedge \mathcal{D} = 0$)

$$D((\lambda\delta) \otimes \mathcal{D}) = D(\delta \otimes (\lambda \mathcal{D}))\,.$$

It follows that D does indeed operate on Γ itself, not depending on the particular way in which it is represented. $D\Gamma$ is a $\boxplus$-tableau tensor. We also have $D(f\Gamma) = 0$ iff f has homogeneity degree -6. We might hope to see a role for this in the encoding of googly information into the structure of $\mathcal{T}^\#$, but this has so far remained somewhat elusive. (For example, we might have found that there is some ambiguity in Γ in its natural extension to $\mathbb{I}$, but that $D\Gamma$ is defined unambiguously there.) However, Γ appears also to be unambiguously regular on $\mathscr{I}^{\#}$ (as here defined) in the curved case.

To see this, let us go to $\mathbb{C}\mathscr{I}^+$, a suitable conformal factor Ω having been chosen to make the metric and ε-spinors finite. We can (locally) make the induced metric on $\mathbb{C}\mathscr{I}^+$ flat and (locally) choose spinors ι^A, $\tilde{\iota}^{A'}$ constant on $\mathbb{C}\mathscr{I}^+$, where the vectors $\iota^A\tilde{\iota}^{A'}$ are tangent to the generators of $\mathbb{C}\mathscr{I}^+$. However, the spinors o^A, $\tilde{o}^{A'}$, needed to complete a spinor basis at each point of $\mathbb{C}\mathscr{I}^+$, cannot be chosen constant unless the Bondi-Sachs news function vanishes. There is curvature in the connection along α-planes on $\mathbb{C}\mathscr{I}^+$ if there is SD gravitational radiation and along β-planes if there is ASD gravitational radiation. Choose (locally) an arbitrary cut $\mathscr{C}$ of $\mathbb{C}\mathscr{I}^+$ (smooth cross-section of the generators) and mark the point q where an α-line or generator intersects $\mathscr{C}$. We adopt a local twistor description of the α-lines, so that along each α-line a twistor corresponding to that α-line has the description



$$Z^\alpha = (0, \pi_{A'}),$$

each tangent vector the α-line being proportional to $\iota^A\pi^{A'}$. (It is important to bear in mind that, because of the conformal rescaling, this "$\pi_{A'}$" is quite different from the "$\pi_{A'}$" used as a flat-twistor-space coordinate above, where we had ($\underline{\text{old}}\,\pi_{A'}$) $Z = (0, \pi^{A'})$.) We recall (R.P. & W.R. Spinors & Space-Time Vol. 2, pp. 376, 65; TN 23, p2) that on $\mathbb{C}\mathscr{I}^+$ the infinity twistor takes the form

$$I = I_{\alpha\beta} = \begin{bmatrix} 0 & -i\,\iota_A\tilde{\iota}^{A'} \\ i\,\tilde{\iota}^{A'}\iota_B & 0 \end{bmatrix}$$

($P' = 0$, $A = 1$, being chosen here, in the notation of S.&S.-T.). Thus $\frac{1}{z}\Xi$ has the local twistor description

$$Z^\alpha I_{\alpha\beta} = (\iota_B(i\tilde{\iota}^{A'}\pi_{A'}),\ 0) = i\pi_{1'}(\iota_B,\ 0),$$

so the covector associated with $\frac{1}{z}\Xi$ is

$$p_a = (i\,\pi_{1'})\,\iota_A\pi_{A'}\ .$$

Local coordinates for $\mathcal{T}^\#$ (for the α-line non-tangent to $\mathcal{C}$) can be taken to be the two independent components of p_a and the two coordinates (x, y) specifying the point q on $\mathcal{C}$. (See figure on previous page). Indeed, we obtain local coordinates (w, x, y, z), closely analogous to (and direct generalizations of) those introduced for $\mathbb{T}^\#$ above, by putting

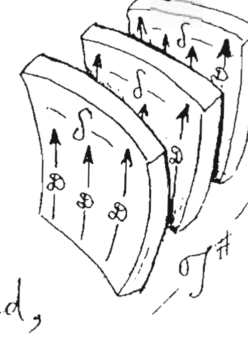
$$w = i\,\pi_{1'}\pi_{1'}\ ,\quad y = i\,\pi_{1'}\pi_{0'}\ ,$$

the ratio $w/y = \pi_{1'}/\pi_{0'}$ of these two quantities being a coordinate defining the direction of the α-line through q (in the β-plane on $\mathbb{C}\mathscr{I}^+$ through q). Provided that the scalings for z and x are chosen appropriately in relation to ι^A and $\tilde{\iota}^{A'}$, we find that, just as before,

$$\delta = y\,dz \quad \text{and} \quad \mathcal{D} = \frac{y}{w}(w\,dy - y\,dw)\wedge dx\wedge dz\,.$$

It follows that there can be no "googly information" solely in the behaviour of the forms δ and $\mathcal{D}$ on $\mathcal{T}^\#$, with the definition of $\mathcal{T}^\#$ so far given. (It is of interest to note that the ratio y/w of the $\pi_{A'}$ components used here define the <u>slope</u> of the α-line, with $\pi_{1'} = 0$ for a generator. This may be contrasted with the "standard" $\pi_{A'}$ used earlier, for flat twistor space $\mathbb{T}$, the ratio of whose components fixed the choice of β-plane, this ratio being essentially the quantity z.

The geometrical meaning of the tensor Γ up to proportionality — equivalently, the pair of forms δ and $\mathcal{D}$ up to proportionality — is that it defines a structure on $\mathcal{T}^\#$ that is the same at each point. This structure is a foliation by curves (the integral curves of the Euler vector field Υ, defined as the dual of $\mathcal{D}$, dualized with respect to $\frac{1}{4}d\mathcal{D}$) which lie in 3-surfaces (whose tangent spaces are duals to δ). The Frobenius relation $\delta \wedge d\delta = 0$ ensures that these 3-surfaces are locally integrable. Moreover $\delta \wedge \mathcal{D} = 0$, as we have here, is the condition that the curves lie on the 3-surfaces. As a point of twistor philosophy, one might take the view that all information should be stored in global structure, the neighbourhood of any point being indistinguishable from that of the neighbourhood of any other. It is not clear how closely such a viewpoint can be followed, but the use of the space $\mathcal{T}^\#$ is in accordance with its spirit. We note that even in the flat case the non-blown-up space $\mathbb{T}$ does not qualify, since the points of I do not have neighbourhoods with the above structure, whereas the blown-up space $\mathbb{T}^\#$ does.



In the standard nonlinear graviton construction, one uses slightly more than the above local structure since the space-time metric itself, as opposed to merely the conformal metric, uses the forms δ and $\mathcal{D}$ themselves and not just their proportionality classes. More precisely, it is the ratio $\mathcal{D}/\delta\otimes\delta$ that comes in to define the metric (for δ defines $I_{\alpha\beta}$ which is, in effect, $\varepsilon^{A'B'}$ and $\delta/\mathcal{D}$ defines $I^{\alpha\beta}$ which is, in effect, ε^{AB}; thus $\delta\otimes\delta/\mathcal{D}$ in effect defines g^{ab}). We may imagine assigning a new $\hat{\delta}$ and $\hat{\mathcal{D}}$ in the neighbourhood of some point of $\mathcal{T}^b$, compatible with the foliation structure, so $\hat{\delta}$ and $\hat{\mathcal{D}}$ are scalar multiples of δ and $\mathcal{D}$. For the metric of $\mathcal{M}$ to be preserved, we must have

$$\hat{\delta} = \chi\delta, \quad \hat{\mathcal{D}} = \chi^2\mathcal{D},$$

whence

$$\hat{\Gamma} = \chi^3\Gamma.$$

If we allow for a conformal factor (needed for $\mathbb{C}\mathcal{I}^+$, in any case), so that

$$\hat{g}_{ab} = \Omega^2 g_{ab},$$

we have

$$\hat{\mathcal{S}} = \Omega^{-1}\chi\,\mathcal{S}, \quad \hat{\mathcal{D}} = \chi^2\mathcal{D}, \quad \hat{\Gamma} = \Omega^{-1}\chi^3\,\Gamma$$

corresponding to $\hat{\varepsilon}_{AB} = \Omega\chi\,\varepsilon_{AB}$, $\hat{\varepsilon}_{A'B'} = \Omega\chi^{-1}\varepsilon_{A'B'}$. However, we should bear in mind that these rescalings apply at points of $\mathcal{T}^{\#}$ rather than at individual points of $\mathcal{M}$. (A conformal rescaling of $\mathcal{M}$ does not preserve $\mathcal{S}$, even up to proportionality; the rescaled $\mathcal{M}$ would not be Ricci-flat.)

Now suppose that a rescaling is introduced which makes both of $\hat{\mathcal{S}}$ and $\hat{\mathcal{D}}$ regular and non-zero at $\mathbb{I}$. As things stand, there is too much choice for such scalings. We want somehow to restrict this choice so that the googly information is encoded. It may be noted that some choices are distinguished by the fact that they yield a 2-form $\mathcal{S}$ that is closed, so locally

$$\hat{\mathcal{S}} = d\zeta.$$

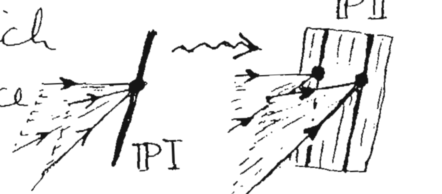
In the flat case, we may take

$$\Omega\chi^{-1} = \omega, \quad \zeta = z$$

and we recall that $z = \pi_{1'}/\pi_{0'}$ where $\pi_{0'}, \pi_{1'}$ are the ordinary flat-space $\pi_{A'}$-components. $SL(2,\mathbb{C})$ transformations of these coordinates are generated by the two types of replacement

$$\zeta \mapsto \zeta + \text{const.}, \quad \zeta \mapsto -\zeta^{-1}.$$

For the first of these, the rescaling factor $\Omega^{-1}\chi$ is unchanged; for the second, it is replaced by $\Omega^{-1}\chi\,\zeta^{-2}$. This procedure applies to any rescaling factor for which $\hat{\mathcal{S}}$ is closed, so each such choice provides us with a local (primed) spin-space. As described in TN 23, this family of primed spin-spaces encodes, in a sense, the entire googly information. Thus, we require to know the appropriate "preferred" rescaling factors. Each such factor would have to depend upon the actual direction in which a point of $\mathbb{P}\mathcal{I}$ is approached, from within $\mathbb{P}\mathcal{T}^b$, which still leaves some directional dependence at individual points of $\mathbb{P}\mathbb{I}$: ⟶



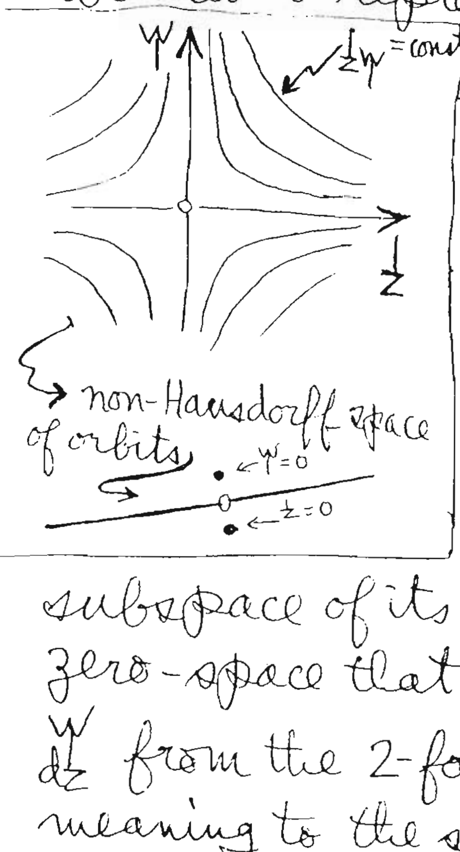
The rescaling factor $\Omega^{-1}\chi$ has to have homogeneity -2 in order for $d\hat{S}=0$ to hold. We are not at liberty to incorporate twistor scale-dependence into the conformal factor Ω, so taking that dependence to lie in χ, we find that the homogeneity degree for the scale factor for $\hat{\Gamma} = \Omega^{-1}\chi^3\Gamma$ is -6. This has some suggestiveness for a role for the googly twistor function, but the matter is not entirely clear, as yet.

In any case, we still have the problem of fixing the appropriate scale factors in terms of the complex geometry of $\mathcal{T}^\#$. As specified so far, the geometry of $\mathcal{T}^\#$ does not seem to contain this information, and a further idea is needed. From the twistor point of view, there is something a little unnatural about forming the space of products $\mathrm{Z}\bar{\mathrm{Z}}$ to replace the space of twistors Z. I wish to adopt a subtly different viewpoint here. We are to think of the product $\mathrm{Z}\bar{\mathrm{Z}}$ as a particular case of a product $\mathrm{Z}\mathrm{W}$, where the dual twistor W specifies a covector $\frac{\mathrm{W}}{d\mathrm{Z}}$. The covector bundle $T^*\mathcal{T}^b$ of $\mathcal{T}^b$ is, of course, well defined; and one can easily specify a covector at a point Z of $\mathcal{T}^b$ in terms of $\mathbb{C}\mathscr{I}^+$, where W is represented as a local dual twistor which is local-twistor constant along the α-line representing $\mathbb{P}\mathrm{Z}$. The exterior derivative $d(\mathrm{W}\wedge d\mathrm{Z})$ of $\frac{\mathrm{W}}{d\mathrm{Z}}$ is the

2-form defining the symplectic structure of $T^*\mathcal{T}^b$. We are interested in the subspace of $T^*\mathcal{T}^b$ defined by $\underset{Z}{\overset{W}{I}} = 0$ (or possibly $\underset{Z}{\overset{W}{I}} = k$ – of relevance to APH's work?). Taking $\underset{Z}{\overset{W}{I}}$ as a Hamiltonian, we obtain the flow whose orbits are to be factored out by in order to obtain the relevant "reduced phase space", where

$$(\lambda W, Z) \equiv (W, \lambda Z)$$

(integral curves of $W\partial_W - Z\partial_Z$). The product ZW can be used to represent these equivalence classes, for the most part, but there is a subtle difference. There are those particular orbits for which either $Z=0$ or $W=0$, in addition, giving a reduced phase space which is non-Hausdorff. In fact, for an ordinary cotangent bundle, one may not take $Z=0$, whereas the space given by $W=0$ is the canonical copy of the original (twistor) space as a subspace of its cotangent bundle. Indeed, it is this zero-space that enables us to reconstruct the 1-form $W dZ$ from the 2-form $dW \wedge dZ$ in a unique way. To give meaning to the subspace $Z=0$, it would be necessary to have some identification between covectors at different Z-points as being, in an appropriate sense, the same W.



One "solution" to the googly problem, therefore, would be to attach a "$Z=0$ zero-set" to the cotangent bundle $T^*\mathcal{T}^b$. It seems to me, however, that this procedure would be more in the nature of "begging the question" than in providing a solution to the googly problem. It would really be stretching a point to consider this procedure to be providing us with an extended "twistor space". The resulting space would be completely symmetrical with respect to twistors and dual twistors and is really much more related to ambitwistor space than to either of $\mathcal{T}$ or $\mathcal{T}^*$ individually.

Nevertheless, I believe that the above ideas provide an appropriate setting for what one should really be doing to construct $\mathcal{T}^\#$. What is required, initially, is the sub-bundle of $T^*\mathcal{T}^b$ for which the covector Ψ has the special form $\underset{x}{\urcorner}$. The "Hamiltonian" $\underset{Z}{W}$ now becomes $\underset{xZ}{\urcorner}$; accordingly we appear to have to factor out by

$$(\lambda \underset{x}{\urcorner}, \tfrac{1}{Z}) \equiv (\underset{x}{\urcorner}, \lambda \tfrac{1}{Z})$$

where $\underset{xZ}{\urcorner} = 0$, i.e. where $\underset{x}{\urcorner}$ and $\underset{Z}{\urcorner}$ are proportional. To a first approximation, this indeed gives the blown-up twistor space $\mathcal{T}^\#$, as defined above. Ignoring the issue of zero-sets, for the moment, we can think of a point of this space as having elements of the form

$$\tfrac{1}{Z}\underset{x}{\urcorner},$$

with $\underset{x}{\urcorner}$ proportional to $\underset{Z}{\urcorner}$, where $\underset{x\,dZ}{\urcorner}$ is a covector at the point $\tfrac{1}{Z}$. We have the freedom to choose the proportionality factor by putting it equal to unity, so our space (the $\mathcal{T}^\#$ defined above) consists of objects

$$\tfrac{1}{Z}\underset{Z}{\urcorner},$$

where $\underset{Z}{\urcorner}$ defines a (restricted) covector:

$$\delta = \underset{Z\,dZ}{\urcorner}.$$

However, we need to consider the zero sets, and it seems to me that the googly information ought to be contained therein. Rather than merely considering the two alternatives $\times\!\rceil = 0$ and $Z = 0$ in $(\lambda \times\!\rceil, Z) \equiv (\times\!\rceil, \lambda Z)$ with $\times Z = 0$, it may be that we ought to view all this slightly differently. The space of $(\times\!\rceil, Z)$ is not, after all, a non-degenerate symplectic manifold, the 2-form $d\times\!\rceil \wedge dZ$ having rank 4 on a 6-dimensional space. For a start, we may examine the zero-sets in relation to the "π-spaces" $(\times\!\rceil, Z\!\rceil)$ alone. Putting (flat case) $\psi = \times\!\rceil = (0, \xi^{A'})$, $Z = (\omega^A, \pi_{A'})$, we have

$$d\psi \wedge dZ = d\times\!\rceil \wedge dZ = d\xi^{A'} \wedge d\pi_{A'}.$$

Taking the Hamiltonian to be $\xi^{A'}\pi_{A'}$ $(= \times Z)$, we are led to contemplate the equivalence

$$(\lambda \xi^{A'}, \pi_{A'}) \equiv (\xi^{A'}, \lambda \pi_{A'})$$

with zero-sets given by $\xi^{A'} = 0$ and $\pi_{A'} = 0$. However, all of this is parameterized by ω^A in some way. It should be noted that the limits like

$$(\omega^A, \lambda \pi_{A'}) \rightarrow (\omega^A, 0)$$

given when $\lambda \rightarrow 0$ are just those of the kind in which an α-line rotates about a point of $\mathbb{C}\mathcal{I}^+$ to become a generator, as considered earlier. More work needs to be done to discover the detailed nature of the appropriate zero-sets and how quantities like Γ are to be defined and extended to these zero-sets. The condition that such extensions to the zero-set regions exist could well provide the necessary restrictions on, say, Γ that encode the full googly information — and perhaps, if this is successful, the required source information also. Work very much in progress....

Thanks to L.M.J., E.T.N., J.F., ...

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Two Point Norms for Massless Fields

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In **TN 39**, I described the action of the *quantum complex structure* J on any field or potential satisfying the wave equation in flat spacetime, as an integral over an arbitrary curved spacelike hypersurface. In the present article I use this result to demonstrate the relationship between the twistor scalar product for massless fields, with 2-point norms in the bosonic cases of a massless scalar field, electromagnetism and linear gravity. The 2-point norms are automatically positive definite for fields of *mixed* frequency, and invariant under the 15-parameter conformal group. They have the added advantage that their computation involves no extraction of potentials or positive/negative frequency parts from the principal dynamical fields.

In the case of gravity I give an interpretation in terms of coherent states and superpositions of spacetime geometries.

I also describe the corresponding two point integral for the fermionic case of a massless neutrino field.

The Hermitian, Symplectic, Complex and Metric Structures

To begin with we introduce $\mathcal{F}$ as the standard Hermitian Fierz scalar product for a spin $\frac{1}{2}n$ field ϕ, given by

$$\mathcal{F}(\chi,\phi) := \kappa \int_\Sigma \overset{(n-1)}{\phi}{}_L^{A'B'\ldots K'} \bar{\chi}_{A'B'\ldots L'} d^3\Sigma^{LL'}$$

or for two scalar positive frequency fields ϕ and χ

$$\kappa \int_\Sigma \phi\nabla_a\bar{\chi} - \bar{\chi}\nabla_a\phi, \qquad \kappa = i.$$

Here $\overset{(n)}{\phi}$ is the n^{th} Dirac potential for ϕ, and κ is equal to 1 or i according as n is odd or even. Now $\mathcal{F}$ is the Hermitian structure. Denoting the symplectic, quantum complex and metric structures by Ω, J and $\langle\ ,\ \rangle$ respectively, the scheme for these three basic structures is

$$\mathcal{F}(\underset{1}{\varphi},\underset{2}{\varphi}) \to \Omega(\underset{1}{\varphi},\underset{2}{\varphi}) := -i[\mathcal{F}(\underset{1}{\varphi},\underset{2}{\varphi}) - \mathcal{F}(\underset{1}{\varphi},\underset{2}{\varphi})]$$

and the positive definite *metric* or norm, for a *mixed* frequency bosonic state is now,

$$\langle\underset{1}{\varphi}|\underset{2}{\varphi}\rangle = \Omega(\underset{1}{\varphi}, J[\underset{2}{\varphi}]).$$

Remark on locality and complexification. It is worth noting the two distinct types of possible complexification for spacetime fields, with regard to locality. The complexification of a field itself (to allow for a distinct antiparticle) is *local*, as is the splitting of a field into its self-dual and anti-self-dual parts. However, as the results above show, the splitting of any field into its positive and negative frequency parts is *non-local* – that is to say, J is a *non-local operator*. It is non-local to the extent that it requires data over an entire Cauchy spacelike hypersurface.

The Scalar Case

We consider a real Klein-Gordon field of mixed frequency satisfying the massless wave equation. Then the symplectic form is given by

$$\Omega(\phi,\psi) = \int_\Sigma (\psi\nabla_a\phi - \phi\nabla_a\psi) d^3\Sigma^a$$

which is compatible with the complex structure in that $\Omega(\phi,\psi) \equiv \Omega(J\phi, J\psi)$. Now the Hermitian scalar product $\mathcal{F}$ is

$$\mathcal{F}(\phi,\psi) = \Omega(\phi, J[\psi]) + i\Omega(\phi,\psi)$$

and the symmetric positive definite metric is

$$\langle\phi,\psi\rangle \equiv \langle\psi,\phi\rangle = \Omega(\phi, J[\psi]) \propto \int\int_{\Sigma} [2\frac{(x-x')_a}{(x-x')^4}\phi(x')\nabla_b\psi(x) - \frac{\nabla'_a\phi(x')\nabla_b\psi(x)}{(x-x')^2}]\, d^3x'^a \cdot d^3x^b$$

where Σ is a once differentiable spacelike hypersurface. The 2-point integrand above represents the probability current in the *product* of two configuration spaces. For bosonic fields it is *not possible* in the relativistic theory to construct a single point current vector $j_a(x)$, which is future pointing, so that $j_a \cdot d^3\Sigma^a$ would represent the *positive* probability flux. (This is in contrast to the standard non-relativistic Schrödinger prescription of forming the symplectic probability 3-current $\mathbf{j} = \psi\nabla\psi^* - \psi^*\nabla\psi$ for some wave function ψ.) Thus we see that the particle number concept, which we obtain from the metric $\langle\ ,\ \rangle$, is non-local in the single configuration space but *local in* $\Sigma(x)\times\Sigma(y)$ – two *copies* of a single configuration space. This important idea as we shall see carries over to the electromagnetic and linear gravity cases also.

The Photon Case

We consider a generally complex electromagnetic field,

$$F_{ab} = \varphi_{AB}\varepsilon_{A'B'} + \psi_{A'B'}\varepsilon_{AB}$$

(so that in an alternative frequently adopted notation $\tilde{\varphi} \equiv \psi$). The splitting of F into its self-dual and anti-self-dual parts is given simply by,

$$^{(\mp)}F_{ab} = \frac{1}{2}(F_{ab} \pm i^*F_{ab})$$

where $*$ denotes the duality transformation, and the splitting into positive and negative frequency parts is given by,

$$F^{\pm} = \frac{1}{2i}(i \pm J)[F].$$

Remark on photon states. A physical photon state can be given in two ways: (1) as a real electromagnetic field with the relevant complex structure J at hand, or equivalently (2) a positive frequency field with independent left and right handed parts.
Remark on self-duality and helicity. The (anti-)self-dual part of a massless field is given always by its (un)primed spinor part – it is important to realise that the right (left) handed part of the field is given by the (un)primed spinor part *only* if the field is of *positive* frequency.
Remark on complex fields. These allow for the case that the particles associated to the field are *not* identical with their antiparticles. In the electromagnetic case the photon is its own antiparticle, so that F_{ab} is real for a physical field. However for its mathematical interest we study the general case of a complex field tensor.

Now the conserved Fierz scalar product is given by,

$$\mathcal{F}(\varphi,\varphi) = \int_{\Sigma} \bar{\varphi}_{A'B'}\eta_A^{A'}\, d^3x^{AB'}$$

where η is the vector potential, subject to the Lorenz gauge condition $\nabla^A_{B'}\eta^{A'}_A = 0$ (GL), and Σ is a spacelike hypersurface.

Note on the gauge condition: the spinor gauge GL implies that η satisfies the wave equation $\nabla^2\eta = 0$ and that the divergence of η vanishes $\nabla_a\eta^a = 0$. Later on we will need to assume the *Coulomb* gauge: $\eta^a = (0,\mathbf{A}) = \int(0,\tilde{\mathbf{A}}(k))e^{ikx}d^4k$ – such η clearly satisfies the wave equation since the integration is over $\{k^2 = 0\}$ – to obtain consistency with GL we need that $\mathbf{k}\cdot\mathbf{A} = 0$ – the 3-vector potential $\mathbf{A}$ and the electric field are parallel, and the electric, magnetic and momentum 3-vectors form a right-handed set.

Now from **TN 39** the repolarised potential is given by,

$$J[\eta_A^{A'}](x) = \frac{1}{2\pi^2}\int_{\Sigma'\ni x}\frac{1}{K^2}(\overleftarrow{\nabla}^b - \overrightarrow{\nabla}^b)\,\eta_A^{A'}(x')\,d^3x'_b.$$

In the general case,

$$\mathcal{F}(\varphi, J[\varphi]) \propto \iint_{\Sigma\times\Sigma'}\bar{\varphi}_{A'B'}(x)[\frac{\nabla'_{CC'}\eta_A^{A'}(x')\,d^3x'^{CC'}}{K^2(x,x')}]\,d^3x^{AB'}$$

$$+\iint_{\Sigma\times\Sigma'}\bar{\varphi}_{A'B'}(x)[(\nabla'_{CC'}\frac{1}{K^2})\eta_A^{A'}(x')\,d^3x'^{CC'}]\,d^3x^{AB'}$$

where Σ, Σ' are identical, but in general curved spacelike hypersurfaces. In the case that Σ is a flat 3-plane the second term above vanishes since this contains the factor $(x-x')^a$ orthogonal to the integration measure. The first term above upon interchanging the upper indices A', C' is equal to

$$-\iint_{\Sigma\times\Sigma'}\frac{\bar{\varphi}_{A'B'}(x)}{K^2}\nabla'_{CC'}\eta_A^{C'}(x')\,d^3x'^{CA'}\,d^3x^{AB'}\qquad\text{(term 1)}$$

$$-\iint_{\Sigma\times\Sigma'}\frac{\bar{\varphi}_{A'B'}(x)}{K^2}\nabla'_{CC'}\varepsilon^{A'C'}\eta_{D'A}(x')\,d^3x'^{CD'}\,d^3x^{AB'}\qquad\text{(term 2)}.$$

We claim that (term 2) above is zero : this term contains $\nabla'^{A'}_C\eta_{D'A}(x')d^3x'^{CD'}$ which by GL equals $\nabla'^{A'}_A\eta_{D'C}(x')d^3x'^{CD'}$ which vanishes in the Coulomb gauge. Since the original Fierz expression was gauge independent subject to GL holding, and since GL is consistent with the Coulomb gauge, we may assume the latter gauge to hold when we gauge fix Σ to be a flat 3-plane. We are left then with (term 1) which from the Dirac chain is equal to

$$-\iint_{\Sigma\times\Sigma'}\frac{\bar{\varphi}_{A'B'}(x)\varphi_{AC}(x')}{K^2}d^3x'^{CA'}\,d^3x^{AB'}.$$

Now define

$$P(x,y) = \mathbf{E}(x)\cdot\mathbf{E}^*(y) + \mathbf{B}(x)\cdot\mathbf{B}^*(y).$$

Then $P^*(x,y) = P(y,x)$ and $P(x,x)$ is real. Define

$$Q(x,y) = \mathbf{E}(x)\cdot\mathbf{B}^*(y) - \mathbf{B}(x)\cdot\mathbf{E}^*(y).$$

Then $Q(y,x) = -Q(x,y)^*$ and $Q(x,x)$ is purely imaginary. Also define,

$$\mathbf{C} = \mathbf{E} - i\mathbf{B}.$$

Now introduce the following generalised *two-point* stress-energy tensor for electromagnetism:

$$T_{ab}(x,y) = \varphi_{AB}(x)\bar{\varphi}_{A'B'}(y).$$

Note that, even in the 2-point case this tensor retains its symmetry and its trace-free property, since the field spinors φ and ψ are symmetric. Now,

$$\varphi_{AB}(x)\bar{\varphi}_{A'B'}(y)n^a n^b = \varphi_{00}(x)\bar{\varphi}_{0'0'}(y) + 2\varphi_{01}(x)\bar{\varphi}_{0'1'}(y) + \varphi_{11}(x)\bar{\varphi}_{1'1'}(y)$$

$$= \frac{1}{2}\mathbf{C}(x)\cdot\mathbf{C}^*(y)$$

$$= \frac{1}{2}[P(x,y) + iQ(x,y)]$$

Now to calculate the corresponding ψ part: we have

$$\psi_{0'0'} = \frac{1}{2}(D_1 + iD_2)$$

$$\psi_{0'1'} = -\frac{1}{2}D_3$$

$$\psi_{1'1'} = -\frac{1}{2}(D_1 - iD_2)$$

where now $\mathbf{D} = \mathbf{E} + i\mathbf{B}$, noting the change in the sign in front of i (of course the electric and magnetic vectors here are the same as in the φ case – there is only a single complex electromagnetic field under consideration).

Now, introducing new variables $\{X, Y\} = \{x, y\}$ but with a free ordering,

$$\psi_{A'B'}(Y)\bar{\psi}_{AB}(X)n^a n^b$$

$$= \bar{\psi}_{00}(X)\psi_{0'0'}(Y) + 2\bar{\psi}_{01}(X)\psi_{0'1'}(Y) + \bar{\psi}_{11}(X)\psi_{1'1'}(Y)$$

$$= \frac{1}{2}\mathbf{D}^*(X) \cdot \mathbf{D}(Y)$$

$$= \frac{1}{2}[P^*(X, Y) + iQ^*(X, Y)].$$

Then,

$$[\varphi_{AB}(x)\bar{\varphi}_{A'B'}(y) + \psi_{A'B'}(Y)\bar{\psi}_{AB}(X)]n^a n^b \quad (\dagger)$$

$$= \frac{1}{2}(P + iQ)(x, y) + \frac{1}{2}(P^* + iQ^*)(X, Y)$$

which in the case of ordering $(X, Y) = (x, y)$

$$= \frac{1}{2}[P(x, y) + P(y, x)] + \frac{1}{2}i[Q(x, y) - Q(y, x)]$$

which integrates against symmetric K to

$$\int \frac{P(x, y)}{K(x, y)}.$$

Alternatively in the case $(X, Y) = (y, x)$, $(\dagger) = P(x, y)$. Thus both orderings give the same result for the photon norm $\langle\varphi|\varphi\rangle$ below. This is a reflection of the fact that the photons obey *Bose statistics*. More precisely the 2-photon state represented by $F_{ab,cd}(x, y)$ satisfies Maxwell's equations on the first two indices with respect to x and on the second two indices with respect to y, and Bose statistics implies $F_{ab.cd}(x, y) = F_{cd,ab}(y, x)$. Our norm is in fact the Hilbert space norm for Quantum Electrodynamics, which is closely related to the expectation of the number operator $\hat{N}$ in the Fock space state coherent to the classical state (φ, ψ). In summary we have shown

$$\langle\varphi|\varphi\rangle \propto \iint \frac{\mathbf{E}(x) \cdot \mathbf{E}^*(y) + \mathbf{B}(x) \cdot \mathbf{B}^*(y)}{|\mathbf{x} - \mathbf{y}|^2} \geq 0 \ \forall\, \mathbf{E}, \mathbf{B}$$

and that this is identical to the Fierz/Twistor scalar product with the relevant complex structure inserted. (I have shown that the generic 2-point 3-space integral $\int \phi(x)\phi(y)/|x - y|^2$ is convergent if and only if $\phi \sim 1/r^{1+\gamma}$ for $\gamma > 1$ at spatial infinity, though I omit the proof of this here.)

Remarks. The invariance under the restricted conformal group $C_+^\uparrow$

$$SU(2, 2) \xrightarrow{2-1} SO(2, 4) \xrightarrow{2-1} C_+^\uparrow$$

follows automatically from the twistor translation of the Fierz scalar product, and that the action of J commutes with any conformal transformation. In particular the 2-point norm is Lorentz and translation invariant, as in fact follows immediately from the above argument – the current vector in the Fierz scalar product is divergence free so one is free to boost and translate Σ, before J is performed. The photon norm is thus a property of a given complex electromagnetic field – not of the particular time slice. In this sense it contains information about the time evolution of the field, *via* the requirement that it be constant. It is also worth noting that the 2-point integral enables one to compute the norm of the field *without* having to extract any potentials or positive/negative frequency parts.

The remarks above carry over naturally to the spin-2 case, that is linear gravity, as we demonstrate below.

The Gravitational Case

To begin with we integrate by parts once in the Fierz scalar product given earlier to obtain,

$$\mathcal{F}(\psi,\psi) = -i\int_\Sigma \bar{\Gamma}^A_{B'C'D'}(x)\chi^{B'C'}_{AD}(x)\, d^3x^{DD'}$$

where Γ and χ are the first and second potentials for the linearised Weyl spinor respectively. It follows that, with $h = \chi + \bar{\chi}$

$$\Omega(\psi, J[\psi]) \propto \iint_{\Sigma\times\Sigma'} \bar{\Gamma}^A_{B'C'D'}(x)\cdot[\frac{1}{(x-x')^2}\cdot\nabla'_{EE'}h^{B'C'}_{AD}(x') - 2\frac{(x-x')_{EE'}}{(x-x')^4}h^{B'C'}_{AD}(x')]d^3x'^{EE'}\cdot d^3x^{DD'}$$

where Σ, Σ' are in general curved but coincident. In the special case where Σ is flat this simplifies to

$$\Omega(\psi, J[\psi]) \propto \iint_{\Sigma\times\Sigma'} \bar{\Gamma}^A_{B'C'D'}(x)\cdot\frac{1}{(x-x')^2}\cdot\nabla'_{EE'}h^{B'C'}_{AD}(x')d^3x'^{EE'}\, d^3x^{DD'}.$$

Here ψ represents a real spin-2 field of *mixed* frequency, and we assume the transverse gauge $h_{ab}\cdot d^3x^a = 0$. Then we have

$$\nabla'_{EE'}h^{B'C'}_{AD}(x')d^3x'^{EE'}\, d^3x^{DD'}$$

$$= \nabla'_{EE'}h^{B'E'}_{AD}(x')d^3x'^{EC'}\, d^3x^{DD'} + \nabla'_{EE'}\epsilon^{C'E'}h^{B'}_{F'AD}(x')d^3x'^{EF'}\, d^3x^{DD'}$$

in which the second term vanishes since we can interchange the lower indices A and E by the spinor gauge condition, leaving a term involving $h^{B'}_{DEF'}(d')^{EF'}$ which vanishes in the transverse gauge. Then using the Dirac chain we are left in the transverse gauge simply with

$$\mathcal{F}(\psi, J[\psi]) \propto \iint_{\Sigma\times\Sigma'} \frac{\bar{\Gamma}^A_{B'C'D'}(x)\Gamma^{B'}_{EAD}(x')(d')^{EC'}(d)^{DD'}}{(x-x')^2}.$$

A tensorial version of this expression can be obtained as follows. In the transverse gauge the *linearised* extrinsic curvature is given by

$$\Pi_{ab} = \frac{1}{2}q^c_a q^d_b n^e\nabla_e h_{cd}$$

in which the derivative of h arises from the perturbed covariant derivative $\hat{\nabla}$.

We now proceed to demonstrate that,

$$\bar{\Gamma}^A_{B'C'D'}(x)\Gamma^{B'}_{EAD}(x')\,(d')^{EC'}(d)^{DD'} = \Pi_{ab}(x)\Pi^{ab}(x')\, d^3x\, d^3x'.$$

Clearly,

$$4\Pi_{ab}(x)\Pi^{ab}(x') = (g^{cs} - n^c n^s)(g^{dt} - n^d n^t)n^e n^r(\nabla_e h_{cd})(x)(\nabla_r h_{st})(x')$$

which by the gauge choice reduces to,

$$n^e n^r(\nabla_e h_{ct})(x)(\nabla_r h^{ct})(x').$$

The spinor part of our assertion is,

$$\nabla_{BB'}\bar{\chi}^{AB}_{C'D'}(x)\nabla_{EE'}\chi^{B'E'}_{AD}(x')n^{EC'}n^{DD'}.$$

By the spinor gauge condition on χ (from the Dirac chain) we have,

$$\nabla_{EE'}\chi^{B'E'}_{AD}n^{EC'} = \nabla_{AE'}\chi^{B'E'}_{ED}n^{EC'} = \nabla_{AE'}\chi^{B'C'}_{ED}n^{EE'} = \nabla_{EE'}\chi^{B'C'}_{AD}n^{EE'}$$

and also,

$$\nabla_{BB'}\bar{\chi}^{AB}_{C'D'}n^{DD'} = \nabla_{BD'}\bar{\chi}^{AB}_{C'B'}n^{DD'}$$

which together give a contribution

$$(\nabla_{EE'}\chi^{B'C'}_{AB})(x')n^{EE'}(\nabla_{DD'}\bar{\chi}^{AD}_{C'B'})(x)n^{BD'}.$$

Now we can interchange the upper indices B and D since this just gives an extra term

$$(\nabla_{EE'}\chi^{B'C'}_{AB})(x')n^{EE'}(\nabla_{DD'}\bar{\chi}^{A}_{C'B'F})(x)n^{FD'}\varepsilon^{DB}$$

in which we may interchange the lower indices D' and C' by the spinor gauge condition on $\bar{\chi}$, and then this vanishes since $h\cdot n = 0$. Up to an overall numerical factor therefore, our assertion is established.

The conformal invariance of this 2-point gravity norm follows from the same arguments given for the photon case above.

In terms of Laplacians we have,

$$\langle \underset{1}{\psi} \,|\, \underset{2}{\psi}\rangle \propto \int \underset{1}{\Pi}^{ab}(x)\Delta^{-1/2}\underset{2}{\Pi}_{ab}(x) = \frac{i}{2\pi^2}\int\!\!\int \frac{\underset{1}{\Pi}^{ab}(x)\underset{2}{\Pi}_{ab}(x')}{(x-x')^2} \quad (\dagger)$$

in which $\Delta^{-1/2}$ acts as a non-local operator. In the vacuum case invariance under the conformal group holds so that in particular the expression is conserved under time translations. It is noteworthy that one could calculate this integral *in the presence of matter* where the Einstein vacuum field equations fail to hold. For the integrand only requires a foliation of the spacetime. However in the non-vacuum case the norm is certainly not conserved in general. Only in the special case that there exists a timelike Killing vector field $\partial/\partial t$ along which the Lie derivative of Π vanishes is the above norm guaranteed to be conserved in the presence of matter.

For two distinct fields the above expression could be interpreted as the 'overlap' of two neighbouring spacetime geometries, where the constituent fields themselves are normalised. It is this overlap which one could interpret as a measure of the probability of a transition $|\underset{1}{\Pi}\rangle \mapsto |\underset{2}{\Pi}\rangle$ where the Dirac kets denote the *coherent* states in the projective Fock space (state space) PF. Now any state $|\psi\rangle$ in PF has an unique decomposition into *coherent states*, and thus the submanifold C of coherent states in PF is *non-linear*. That is to say the complex projective line L in state space joining the coherent states $|\underset{1}{\Pi}\rangle$ and $|\underset{2}{\Pi}\rangle$ lies entirely off the coherent state submanifold apart from at the points $|\underset{1}{\Pi}\rangle$ and $|\underset{2}{\Pi}\rangle$ where it intersects C transversally. Thus we think of the superposition of two neighbouring spacetime geometries as a point $|\psi\rangle$ on $L\backslash C$ and then the probability of transition to $|\underset{i}{\Pi}\rangle$ is the distance $d(\psi, \underset{i}{\Pi})$ with respect to the standard Fubini-Study metric restricted to the projective line joining $|\psi\rangle$ to $|\underset{i}{\Pi}\rangle$. It is important to realize that the nonlinearity of C is fully present even in the *linear* gravity. This is a desired feature of our description, since in the presence of weak gravitational fields nature in the classical domain does not support superpositions of neighbouring well defined classical geometries. It is precisely the space C which corresponds under exponentiation to these classical geometries – the solutions $\underset{i}{\Pi}$ of the classical field equation. Moreover, as the early work of Schrödinger shows, C is unitarily invariant under a wide class of Hamiltonians, in particular all harmonic oscillator Hamiltonians. Thus a free coherent state of the gravitational field remains coherent under its time evolution and is associated at each time to an unique classical geometry. Since the neighbouring coherent states overlap, according to (†), the coherent state is free to wander within C under unitary evolution, and so under time evolution the geometry may change, but *retain* its classical nature.

The Fermionic Case

Let us now consider the corresponding situation for the case of a massless neutrino field ν_A. The Fierz scalar product is now

$$\mathcal{F}(\nu, \bar{\nu}) = \int_\Sigma \bar{\nu}_A \nu_{A'} d^3\Sigma^{AA'} \quad (*).$$

Note here that the current 4-vector $j = \nu\bar{\nu}$ is automatically a *future* pointing null vector so that the expression above is positive definite for all ν. We can apply the same repolarisation procedure as before to give

$$\mathcal{F}(\bar{\nu}, J[\nu]) \propto \iint_{\Sigma^2} -2\frac{(x-y)^{BB'}}{(x-y)^4}\bar{\nu}_A(x)\nu_{A'}(y) - \frac{1}{(x-y)^2}\bar{\nu}_A(x)\nabla^{BB'}_{(y)}\nu_{A'}(y) d^{3AA'}_{(x)} \cdot d^3_{(y)BB'} \quad (**)$$

where Σ is a single once differentiable spacelike hypersurface, which is in general curved. In the case that Σ is flat we are left simply with

$$-\iint_{\Sigma^2} \frac{\bar{\nu}_A(x)\nabla^{BB'}_{(y)}\nu_{A'}(y)}{(x-y)^2} d^{3AA'}_{(x)} \cdot d^3_{(y)BB'}.$$

We remark on the physical significance of the above formula. Suppose we have a neutrino field consisting of a *mixture* of both left and right-handed particles. Firstly note that for such a general mixture we only use a *single* spinor field ν. More precisely both ν_A and $\bar{\nu}_{A'}$ have positive frequency parts corresponding to the left and right-handed fields respectively. The standard Fierz 1-point integral (*), because of the Fermi statistics which ensures the current vector is always future pointing, represents for all ν the positive definite norm which corresponds to the *sum* of the numbers L and R of left and right-handed neutrinos. The effect of the quantum complex structure in (**), as one can see from the repolarising action of J in (*), is simply to produce the *difference* $L - R$. The situation for Bose statistics is reversed in the sense described earlier.

Concluding remarks

I have been able to show that these two-point norms can be expressed in terms of Dirac's CP^5 calculus, and also as twistor diagrams. In the latter case I have shown using RP's article in **TN 27** that the action of the quantum complex structure J is expressible as an holomorphic link integral in twistor space. Thus also, from the integral representation of J, is the scalar product of a field with the Green's function for the wave operator, a suggestion of Atiyah which is alluded to in RP's article.

Details of these issues will appear later.

Thanks to Roger Penrose for suggesting this topic, and to Lane Hughston for many useful discussions.

Tim Field.

The Rarita-Schwinger Equation for Einstein-Maxwell Space-times

It will be a familiar fact to readers of Twistor Newsletter that the Rarita-Schwinger equation (equation (1) below) has "as many" solutions in a space-time **M** as it has in flat space-time if **M** satisfies the vacuum equations. Also, the Rarita-Schwinger equation has gauge-solutions, generated by an arbitrary choice of spinor field α^A, again provided **M** is vacuum. From these facts have arisen various attempts to use the Rarita-Schwinger equation as a means to generate vacuum solutions.

It is natural to ask (and the question arose after David Robinson's seminar in Oxford on 16/1/96) whether there is a generalisation of the Rarita-Schwinger equation which stands in the same relation to the Einstein-Maxwell equations as the original does to the vacuum equations. In fact there is, as is implicit in the theory of N=2 supergravity and as is therefore probably well-known to supergravity aficionados. My purpose here is simply to note, in the formalism of 2-component spinors, what these generalised Rarita-Schwinger equations are.

The Rarita-Schwinger equation for a spinor field $\psi^{BAA'}$ may be written as the pair of equations:

$$\nabla_{A(B'}\psi^{BA}{}_{A')} = 0 \qquad \text{1a}$$

$$\delta_A{}^C\,\nabla^{(B}{}_{B'}\psi_C{}^{A)B'} = 0 \qquad \text{1b}$$

and gauge solutions are obtained from an arbitrary spinor field α^A according to

$$\psi^B{}_{AA'} = \nabla_{AA'}\alpha^B \qquad \text{2}$$

It can be checked that (2), for arbitrary α^A, solves equation (1a) iff the trace-free Ricci tensor vanishes, and satisfies (1b) iff the Ricci scalar vanishes. To generalise (1) we seek equations for a pair of Rarita-Schwinger fields (or a Dirac-spinor-valued 1-form) whose gauge solutions are generated by a pair of spinor fields $\alpha^A, \beta^{A'}$ (or a Dirac spinor) according to

$$\psi^B{}_{AA'} = \nabla_{AA'}\alpha^B - \kappa\,\phi_A{}^B\,\beta_{A'} \qquad \text{3a}$$

$$\chi^{B'}{}_{AA'} = \nabla_{AA'}\beta^{B'} + \kappa\,\bar{\phi}_{A'}{}^{B'}\,\alpha_A \qquad \text{3b}$$

Here K is a constant to be determined and ϕ_{AB} is a Maxwell field. These equations are taken from my paper Phys.Lett.**121B** (1983) 241 where I obtained them by translating a 4-component-spinor expression given by Gibbons and Hull Phys.Lett.**109B** (1982) 190.

By eliminating the Maxwell field between (3a) (respectively (3b)) and the derivative of (3b) (respectively (3a)) we guess the candidate generalised Rarita-Schwinger equations:

$$\nabla_{A(B'}\psi^{BA}{}_{A')} + K\phi^{AB}\chi_{(A'B')A} = 0 \qquad 4a$$

$$\nabla_{A'(B}\chi^{B'A'}{}_{A)} - K\bar{\phi}^{A'B'}\psi_{(AB)A'} = 0 \qquad 4b$$

$$\delta_A{}^C\nabla^{(B}{}_{B'}\psi_C{}^{A)B'} + \frac{1}{2}K\phi_A{}^B\chi^{B'A}{}_{B'} = 0 \qquad 4c$$

$$\delta_{A'}{}^{C'}\nabla^{(B'}{}_B\chi_{C'}{}^{A')B} - \frac{1}{2}K\bar{\phi}_{A'}{}^{B'}\psi^{BA'}{}_B = 0 \qquad 4d$$

Now the point is that (3a,b) are in fact the gauge solutions of the generalised Rarita-Schwinger equations (4a-d) *provided that the source-free Einstein-Maxwell equations are satisfied,* i.e. provided that

$$\Phi_{ABA'B'} = K^2\phi_{AB}\bar{\phi}_{A'B'};\ \Lambda = 0;\ \nabla_{A'}{}^A\phi_{AB} = 0 \qquad 5$$

(so that, in the usual conventions, $K^2 = 2G$).

The equations (4a-d) seem to be the translation into the 2-component-spinor language of equations given long ago in the supergravity literature by S.Ferrara and P.van Nieuwenhuizen (Phys.Rev.Lett.**37** (1977) 1669; see also T.Dereli and P.C.Aichelburg (Phy.Lett.**80B** (1979) 357) and P.C.Aichelburg and R.Güven (Phys.Rev.**D24** (1981) 2066).

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Geometric issues in the foundation of science

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(First announcement)

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We are planning a symposium entitled "Geometric issues in the foundations of science" to celebrate, in the year of his 65th birthday, Professor Sir Roger Penrose's outstanding contributions to many different areas of mathematics and physics and the 23 years so far during which he has been enlivening and shaping Oxford mathematics. The conference will be broadly based and draw together the fields to which Roger has contributed. We attach a provisional programme with a list of those who have agreed to speak.

Applications to participate in the symposium may be made on the enclosed forms. Please address all correspondence, by post, fax or e-mail, to the addresses above. Completed application forms should be sent to the above address as soon as possible; we would like to have the list of applicants in a reasonably final form by 29th February. We welcome forms sent by e-mail also. We have to restrict the number of participants to about 150.

Up to date information will be posted on the website:

`http://www.maths.ox.ac.uk/geom96/geom96.html`

where copies of these documents can be obtained also.

Location and accommodation: The symposium will be held at St John's College, Oxford, in the conference facilities in their new Garden Quad buildings.

Board and lodging can be obtained in St Johns. The prices (including VAT) are as follows:

One night's accommodation:	£26.50
Breakfast:	£4
Lunch:	£6
Dinner:	£16
Banquet:	£30.

We append some addresses of local bed and breakfasts and hotels in case you wish to make other arrangements.

Attendance and Registration: The number of participants that we can accommodate is limited. Although we shall do what we can to accommodate everyone who applies, we may not be able to.

There will be a registration fee of 50 UK pounds if submitted by May 1st which will increase to 75 UK pounds after that date. The registration will be half price for research students and the financially challenged.

Financial support: There may be some limited support for participants who will otherwise not be able to attend—we do not know how much, if anything, will be available at this stage, but let us know what you need in case we are able to do something.

Workshop submissions: There will be a limited number of slots for speakers in the parallel sessions. If you wish to speak, please submit an abstract by February 29th.

The local organizing committee:

Steve Huggett, Lionel Mason, Paul Tod, Tsou Sheung Tsun, Nick Woodhouse.

The scientific organizing committee:

Abhay Ashtekar, Gary Gibbons, Lionel Mason, Ted Newman, Paul Tod, and Nick Woodhouse

The honourary committee:

Sir Michael Atiyah, Sir Hermann Bondi, Martin Gardner, Dennis Sciama, John Wheeler.

Accommodation elsewhere:

Judith & John Goodwin, Tara, 10 Holywell St Oxford OX1 3SA
Tel: 01865 202953 or 244786

E.Tong, 54 St John St, Oxford OX1 2LQ
Tel: 01865 54894

Norham Guest House, 16 Norham Rd, Oxford OX2 6SF
Tel: 01865 515352

St Michael's Guest House, 26 St Michael's St, Oxford OX1 2EB
Tel: 01865 242101

More expensive accommodation

Bath Place Hotel, 4 & 5 Bath Place, Oxford OX1 3SU
Tel: 01865 791812

Even more expensive accommodation

The Old Parsonage Hotel, 1 Banbury Rd, OX2 6NN
Tel: 310210

Oxford Information Centre 01865 726871

The provisional program

Tuesday, 25th June:

Reception
Evening Lecture by Sir Michael Atiyah (Trinity Cambridge)

Wednesday, 26th:

9-10: N.J.Hitchin (University of Cambridge)
10-11: S.K.Donaldson (University of Oxford)
11.30-12.30: A.Connes (College de France)
Lunch
2-3: A.Ashtekar (Pennsylvania State University)
3.30-4.30: G.Veneziano (CERN)

Thursday, 27th:

9-10: H.Friedrich (Max Planck Insitute, Potsdam)
10-11: R.S.Ward (University of Durham)
11.30-12.30: C.R.LeBrun (SUNY at Stonybrook)
Afternoon: Parallel sessions: 2-5
1. Mathematical aspects of Twistor Theory.
2. Fundamental questions in quantum mechanics.

Friday, 28th:

9-10: A.Ekert (The Clarendon Laboratory, Oxford)
10-11: S.Hameroff (Tucson, Arizona)
11.30-12.30: A.Shimony (Boston University)
Afternoon: Parallel sessions: 2-5
3. Approaches to Quantum Gravity.
4. Geometry and gravity.
Evening: Conference dinner

Saturday, 29th:

9-10: P.Steinhardt (University of Pennsylvania)
10-11: D.W.Sciama (S.I.S.S.A, Trieste)
11.30-12.30: S.W.Hawking (University of Cambridge)
Afternoon:
2-3: G.B.Segal (University of Cambridge)

Geometric issues in the foundation of science

St. John's College, Oxford, June 25-29 1996

Application and accommodation reservation form

NAME:____

DEPARTMENT:____________________________________

INSTITUTION:____________________________________

ADDRESS:______________________________________

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TELEPHONE: (office) ____________(home) ____________(FAX) ____________

E-MAIL:____

Please specify on which nights you would like accommodation in St John's College:

Arrival date:______________________________________

Departure date:____________________________________

Total number of nights:________________________________

Total cost of accommodation at £26.50 per night: £____________________

Please specify the meals you would like with an 'x' below:

Breakfast (£4 each): Wednesday ( ), Thursday ( ), Friday ( ), Saturday ( ), Sunday ( ).

Lunch (£6 each): Wednesday ( ), Thursday ( ), Friday ( ), Saturday ( ).

Dinner (£16 each): Wednesday ( ), Thursday ( ), Saturday ( ).

Conference dinner at £30: Friday( )

Please state any special dietary requirements:________________________

Total cost of meals: £

Registration fee: £

Total sum: £

We hope to be able to accommodate all who apply. If we can accommodate you, we will write and invoice you for the above sum and expect you to confirm by paying the registration fee and an accommodation deposit of 50 pounds by May 1st 1996. We will let you know as soon as possible if there are any difficulties.

In search of a twistor correspondence for the KP equations

S.R.Barge (barge@maths.ox.ac.uk)

1 Introduction

This article summarises work done in an attempt to develop a twistor correspondence for the Kadomtsev-Petviashvili equations,

$$(4u_3 + 6uu_1 - u_{111})_1 - 3\sigma^2 u_{22} = 0,$$

where $\sigma^2 = -1$ for the KPI equation and $\sigma^2 = +1$ for the KPII equation, and other related equations as yet unattainable as reductions of the anti-self-dual Yang-Mills equations. The motivation comes from [2] and the work done on developing the Dirac operator and the nonlocal Riemann-Hilbert problem is adapted from work in [1] and [5].

We shall take as our twistor space $\mathcal{O}(n)$, the twisted line bundle of Chern class n, fibred over $\mathbb{CP}^1$, which is viewed stereographically as $\mathbb{C} \cup \infty$. The Riemann sphere will be given coordinates λ on $\mathbb{C}$ and $\lambda' = \lambda^{-1}$ on $\mathbb{C}' = \mathbb{C} \cup \infty - 0$. The bundle $\mathcal{O}(n)$ can then be given coordinates (μ, λ) on the fibres over $\mathbb{C}$ and $(\mu', \lambda') = (\mu\lambda^{-n}, \lambda^{-1})$ on the fibres over $\mathbb{C}'$. It can be shown that an element of the space of holomorphic sections of the bundle is of the form

$$\mu_n = \sum_{i=0}^{n} t_i \lambda^i, \quad (t_i) \in \mathbb{C}^{n+1}.$$

For the KPII case, we consider the coordinates (t_i) to be real.

2 The Dirac Operator

The standard Ward construction involves solving the equation $\bar{\partial}_E f = 0$ on a vector bundle E. The idea of Mason, in [2], was to replace the $\bar{\partial}_E$-operator with a Dirac operator $\not{D}_\alpha$, where

$$\not{D}_\alpha \underline{\phi} := \begin{pmatrix} \partial_{\bar\lambda} & \alpha \\ \bar\alpha & \partial_\lambda \end{pmatrix} \begin{pmatrix} \phi \\ \tilde\phi \end{pmatrix} = 0. \tag{1}$$

where α is a smooth function on $\mathcal{O}(n)$. Mason and Woodhouse, in [3], show that if we impose certain symmetry conditions on α and $\underline{\phi}$, the we can derive the equations of the KP hierarchy. These conditions are;

$$\begin{array}{ll} i) & \tilde\phi = \bar\phi, \\ ii) & \phi(\lambda,\bar\lambda) = \overline{\phi(\bar\lambda,\lambda)}, \\ iii) & \alpha = \exp(\bar\mu - \mu)\alpha_0(\lambda,\bar\lambda). \end{array}$$

In the next section, we proceed in the opposite direction, and summarise a method of deriving the Dirac operator with the given symmetries from the operators of the KP hierarchy.

3 Derivation of the Dirac operator

Consider the equation $L\psi = 0$ where L is the first operator in the KP hierarchy, $L = \partial_2 - \partial_1^2 + u$, where $u(t) \in L^1 \cap L^2(\mathbb{R}^2)$ and $t = (t_1, t_2)$. If we assume that $u(t)$ is zero in a neighbourhood of $|t| = \infty$, then $\psi \sim \exp \mu_2$ as $|t| \to \infty$. If we write $\psi = e^{\mu_2}\phi$, then ϕ satisfies

$$\left[(\partial_1 + \lambda)^2 - (\partial_2 + \lambda^2)\right] \phi(t;\lambda) = u(t)\phi(t;\lambda). \tag{2}$$

As detailed in Wickerhauser, [5], there is a unique solution to this equation, satisfying $\phi \to 1$ as $|t| \to \infty$.

We shall denote by $\hat\phi$ the Fourier transform of ϕ with respect to the variable $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, defined by

$$\hat\phi(\xi;\lambda)) = \int_{\mathbb{R}^2} e^{-i\xi\cdot t}\phi(t;\lambda)dt, \quad dt = dt_1 dt_2.$$

Define the polynomials $P(\xi) = \xi_1^2 - \xi_2$ and $P_\lambda(\xi) = P(i\xi_1 + \lambda, i\xi_2 + \lambda^2)$. By taking Fourier transforms of equation (2), one obtains

$$\phi(t;\lambda) = 1 + G\phi = 1 + \frac{1}{4\pi^2}\int_{\mathbb{R}^2} e^{i\xi\cdot t}\frac{[u\phi]^\wedge(\xi;\lambda)}{P_\lambda(\xi)}d\xi.$$

We can make $||G||$ less than unity by decreasing $||u||_{L^1} + ||u||_{L^2}$. We can then write $\phi = (I - G)^{-1}1$, and hence

$$\partial_{\bar{\lambda}}\phi = (I - G)^{-1}(\partial_{\bar{\lambda}}G)\phi.$$

After a little work, we obtain the expression

$$(\partial_{\bar{\lambda}}G)\phi = \alpha_0(\lambda)\exp(\bar{\mu_2} - \mu_2),$$

where

$$\alpha_0(\lambda) = \frac{-1}{2\pi i}\int_{\mathbb{R}^2}\exp(\mu_2 - \bar{\mu_2})u(t)\phi(t;\lambda)dt.$$

We now consider the behaviour of $\nu(t;\lambda) = (I - G)^{-1}\exp(\bar{\mu_2} - \mu_2)$. It is bounded and satisfies equation (2) with the boundary condition $\nu(t;\lambda) \sim \exp(\bar{\mu_2} - \mu_2)$ as $|t| \to \infty$. Define $\tilde{\nu}(t;\lambda) = \exp(\mu_2 - \bar{\mu_2})\nu(t;\lambda)$. This has the boundary condition $\tilde{\nu}(t;\lambda) \to 1$ as $|t| \to \infty$, and satisfies the equation

$$\left[(\partial_1 + \bar{\lambda})^2 - (\partial_2 + \bar{\lambda}^2)\right]\tilde{\nu}(t;\lambda) = u(t)\tilde{\nu}(t;\lambda). \quad (3)$$

But $\phi(t;\bar{\lambda})$ is the unique solution to this problem, therefore

$$\partial_{\bar{\lambda}}\phi(t;\lambda) - \exp(\bar{\mu_2} - \mu_2)\alpha_0(\lambda)\phi(t;\bar{\lambda}) = 0. \quad (4)$$

As u is real-valued, it follows that $\overline{\phi(\lambda)} = \phi(\bar{\lambda})$ and $\overline{\alpha_0(\lambda)} = \alpha_0(\bar{\lambda})$ and hence we obtain the conjugate equation

$$\overline{\exp(\bar{\mu_2} - \mu_2)\alpha_0(\lambda)}\phi(\lambda) + \partial_{\lambda}\phi(\bar{\lambda}) = 0.$$

Thus far we have only considered the variables t_1 and t_2. We now wish to include the t_3 dependence of ϕ, arising from the second operator in the KP hierarchy,

$$M = \partial_3 - \partial_1^3 + \frac{3}{2}u\partial_1 + \frac{3}{2}u_1 - v,$$

where v is defined in terms of u by the compatibility conditions. Writing equation (4) with $\mu_2 = \mu_3\exp(-\lambda^3 t_3)$ and multiplying both sides by $\exp\mu_3$, we then apply L_3 to both sides, giving that $\alpha_0(\lambda;t_3) = \alpha_0(\lambda)\exp-(\lambda^3 - \bar{\lambda}^3)t_3$ and hence

$$\partial_{\bar{\lambda}}\phi(t;\lambda) + \exp(\bar{\mu_3} - \mu_3)\alpha_0(\lambda)\phi(t;\bar{\lambda}) = 0, \quad (5)$$

where t is now an element of $\mathbb{R}^3$. It follows that if $\phi\exp\mu_n$ satisfies the first $(n-1)$ operators in the KP hierarchy, equation (5) holds, with n replacing the subscript 3.

4 The KPI equation

Whereas the inverse scattering for the KPII equation produces a d-bar problem relating $\phi(\lambda)$ and $\phi(\bar{\lambda})$, where ϕ is nowhere analytic in λ, the KPI equation gives a substantially different result. The first operator in the KPI hierarchy is given by $L = \partial_2 - \partial_1^2 + u$, so proceeding as in the KPII case, taking $\mu_3(\lambda) = i(\lambda t_1 + \lambda^2 t_2 - \lambda^3 t_3)$. The problem arising in the fact that for KPII, $P_\lambda(\xi)^{-1}$ has discrete singularities for all λ, but in the KPI case, $P_\lambda(\xi)^{-1}$ has continuous singularities for real λ. This causes the Green's function to be undefined on the real λ-axis. The resulting calculation leads to ϕ being a sectionally meromorphic function, satisfying a nonlocal Riemann-Hilbert problem on the real λ-axis,

$$(\phi_+ - \phi_-)(t;\lambda) = \int_{\mathbb{R}} F(\lambda,\kappa)\exp\{\mu(\kappa) - \mu(\lambda)\}\phi_-(\kappa)d\kappa,$$

$\phi_+(\phi_-)$ being holomorphic in the upper (lower) half-plane.

At this time it is unclear as to the connection between the two results. One feels that they may be parts of a larger construction (for complex t_i?) which reduces to the d-bar relation when $t_i \in \mathbb{R}$ and to the nonlocal Riemann-Hilbert problem when some t_i are imaginary.

5 The twistor correspondence

The main problem with this theory is its global nature. The Ward correspondence for the anti-self-dual Yang-Mills equations is a local construction, therefore it would be sensible to attempt to develop a localised version of the preceding theory. One possible direction is to use ideas from Segal and Wilson, [4], concerning the theory of Grassmannians and the KP hierarchy. A link exists between the theory, if we insist in equation (1) that α vanishes in a neighbourhood of infinity. Then in that neighbourhood, $\partial_{\bar{\lambda}}\phi = 0$ and ψ has the general form of the Segal-Wilson Baker function,

$$\psi(t;\lambda) = \exp\mu\left(1 + \sum_{i=1}^{\infty} a_i\lambda^{-i}\right).$$

It may well be possible to extend the Baker function to obtain the global ψ of the Dirac operator.

An alternative avenue of approach is to consider what happens if the coordinates t_i are complex. The Davey-Stewartson equations generate similar constructions to the KP equations, but with the t_1-coordinate being complex. In this sense, the DS-equations are more fundamental than the KP equations. By complexifying the other coordinates, it may be possible to generate higher-dimensional integrable systems.

References

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[2] L.J. Mason: *Generalized twistor correspondences, d-bar problems and the KP equations*, Collection, 'Twistor Theory' ed. S.A. Huggett, Marcel Dekker (1995)

[3] L.J. Mason and N.M.J. Woodhouse: *Twistor theory, self-duality and integrability*, in proof, OUP (1995)

[4] G. Segal and G. Wilson: *Loop groups and equations of KdV type*, Inst. Hautes. Études. Sci. Publ. **61**, 5-65 (1985)

[5] M.V. Wickerhauser: *Inverse scattering for the heat operator and evolutions in 2+1 variables*, Comm. Math. Phys. **108**, 67-89 (1987)

On group theoretical aspects of the non-linear twistor transform

Sergey A. Merkulov (Glasgow)

0. Introduction. Let $X \hookrightarrow Y$ be a rational curve $X = \mathbb{CP}^1$ embedded into a complex 3-fold Y with normal bundle $N = \mathcal{O}(1) \oplus \mathcal{O}(1)$. According to Kodaira [7], the moduli space of all rational curves obtained by holomorphic deformations of X inside Y is a complex manifold M of dimension $h^0(X, N) = 4$. More strikingly, according to Penrose [9] this M comes also equipped with a canonically induced self-dual conformal structure, and, moreover, *any* local conformal self-dual structure arises in this way. Later several other manifestations of this strange phenomenon have been observed when a complex analytic data of the form $(X \hookrightarrow Y, N)$ gives rise to the full category of local geometric structures C_{geo} (more precisely, it is the successful choice of a pair (X, N) consisting of a complex homogeneous manifold X and a homogeneous vector bundle N on X which uniquely specifies C_{geo} — the choice of a particular ambient manifold Y corresponds to the choice of a particular object in C_{geo}).

The questions then arise: How is it possible that a complex analytic data of the form (X, N) can be used as the building block for such basic differential-geometric quantities as metrics, affine connections, curvatures? What is the guiding principle for making a *successful* choice of (X, N)? How general is this phenomenon?

In this note we attempt to illucidate these questions by unveiling very strong links between the (curved, or non-linear) twistor approach to differential geometry and Borel-Weil approach to representation theory (in this context, see also the works of Baston and Eastwood [5, 1] on applications of the Bott-Borel-Weil technique to the linear twistor transform between "flat" models).

1. Complex G-structures. Let M be an n-dimensional complex manifold and V a fixed reference n-dimensional vector space (typically, $V = \mathbb{C}^n$). Let $\pi : \mathcal{L}^*M \to M$ be the holomorphic bundle of V-valued coframes, whose fibres $\pi^{-1}(t)$ consist, by definition, of all $\mathbb{C}$-linear isomorphisms $e : T_tM \to V$, where T_tM is the cotangent space at $t \in M$. The space $\mathcal{L}^*M$ is a principle

right $\mathrm{GL}(V)$-bundle with the right action given by $R_g(e) = g^{-1} \circ e$. If G is a closed complex subgroup of $\mathrm{GL}(V)$, then a complex G-structure on M is a principal holomorphic subbundle $\mathcal{G}$ of $\mathcal{L}^*M$ with the group G. It is clear that there is a one-to-one correspondence between the set of G-structures on M and the set of holomorphic sections σ of the quotient bundle $\tilde{\pi} : \mathcal{L}^*M/G \to M$ whose typical fibre is isomorphic to $\mathrm{GL}(V)/G$. A G-structure on M is called *locally flat* if there exits a coordinate patch in the neighbourhood of each point $t \in M$ such that in the associated canonical trivialization of $\mathcal{L}^*M/G$ over this patch the section σ is represented by a constant $GL(m, \mathbb{C})/G$-valued function. A G-structure is called *k-flat* if, for each $t \in M$, the k-jet of the associated section σ of $\mathcal{L}^*M/G$ at t is isomorphic to the k-jet of some locally flat section of $\mathcal{L}^*M/G$. It is not difficult to show that a G-structure admits a torsion-free affine connection if and only if it is 1-flat (cf. [4]). A G-structure on M is called *irreducible* if the action of G on V leaves no non-zero invariant subspaces.

The notion of G-structure is a unifying idea for a variety of popular themes in differential geometry. For example, (i) if $G \subset \mathrm{GL}(n, \mathbb{C})$ is the standardly represented special orthogonal group $\mathrm{SO}(n, \mathbb{C})$, then the associated G-structure is nothing but a complex Riemannian structure; (ii) if $G = \mathrm{CO}(n, \mathbb{C}) \subset \mathrm{GL}(n, \mathbb{C})$, then G-structure coincides with the complex conformal structure; (iii) if $G = \mathrm{GL}(2,\mathbb{C})\mathrm{GL}(n,\mathbb{C}) \subset \mathrm{GL}(2n,\mathbb{C})$, $n \geq 3$, then G-structure is precisely almost quaternionic structure. In the first two examples G-structures are always 1-flat, in the third example this is not true — 1-flat $\mathrm{GL}(2,\mathbb{C})\mathrm{GL}(n,\mathbb{C})$-structures are called complex quaternionic structures.

The notion of G-structure is proved to very useful in the study of affine connections, especially in the context of classifying the irreducibly acting holonomies of torsion-free affine

connections [4]. Given an affine connection ∇ on a connected simply connected complex manifold M with the holonomy group G, the associated G-structure $\mathcal{G}_\nabla \subset \mathcal{L}^*M$ can be constructed as follows. Define two points u and v of $\mathcal{L}^*M$ to be equivalent, $u \sim v$, if there is a holomorphic path γ in M from $\pi(u)$ to $\pi(v)$ such that $u = P_\gamma(v)$, where $P_\gamma : \Omega^1_{\pi(v)}M \to \Omega^1_{\pi(u)}M$ is the parallel transport along γ. Then $\mathcal{G}_\nabla$ can be defined, up to an isomorphism, as $\{u \in \mathcal{L}^*M \mid u \sim v\}$ for some coframe v. The G-structure $\mathcal{G}_\nabla$ is the smallest subbundle of $\mathcal{L}^*M$ which is invariant under ∇-parallel translations.

The basic question about the local geometry of G-structures — what is the obstruction for a k-flat G-structure $\mathcal{G} \to M$ to be $(k+1)$-flat? — has been answered by Guillemin [6] and Singer and Sternberg [10]. The obstruction is given locally by a function on M with values in the Spencer cohomology $\mathrm{H}^{k,2}(\mathfrak{g})$ associated with the given representation of G in V (more accurately, with the associated representation of the Lie algebra $\mathfrak{g}$ in V). In the next paragraph we recall the definition of $\mathrm{H}^{k,l}(\mathfrak{g})$.

2. Spencer cohomology. Let V be a vector space and $\mathfrak{g}$ a Lie subalgebra of $gl(V) \simeq V \otimes V^*$. Define recursively the $\mathfrak{g}$-modules

$$\begin{aligned} \mathfrak{g}^{(-1)} &= V \\ \mathfrak{g}^{(0)} &= \mathfrak{g} \\ \mathfrak{g}^{(k)} &= [\mathfrak{g}^{(k-1)} \otimes V^*] \cap [V \otimes \odot^{k+1} V^*]\,,\ k = 1, 2, \ldots, \end{aligned}$$

and define the map

$$\mathfrak{g}^{(k)} \otimes \Lambda^{l-1} V^* \longrightarrow \mathfrak{g}^{(k-1)} \otimes \Lambda^l V^*$$

as the antisymmetrisation over the last l indices.

Since $\partial^2 = 0$, there is a complex

$$\mathfrak{g}^{(k)} \otimes \Lambda^{l-1} V^* \xrightarrow{\partial} \mathfrak{g}^{(k-1)} \otimes \Lambda^l V^* \xrightarrow{\partial} \mathfrak{g}^{(k-2)} \otimes \Lambda^{l+1} V^*$$

whose cohomology at the center term is denoted by $\mathrm{H}^{k,l}(\mathfrak{g})$ and is called the (k,l) Spencer cohomology group. In particular,

$$\begin{aligned} \mathrm{H}^{k,1}(\mathfrak{g}) &= 0 \\ \mathrm{H}^{k,2}(\mathfrak{g}) &= \frac{\mathrm{Ker} : \mathfrak{g}^{(k-1)} \otimes \Lambda^2 V^* \xrightarrow{\partial} \mathfrak{g}^{(k-2)} \otimes \Lambda^3 V^*}{\mathrm{Image} : \mathfrak{g}^{(k)} \otimes V^* \xrightarrow{\partial} \mathfrak{g}^{(k-1)} \otimes \Lambda^2 V^*}. \end{aligned} \qquad (1)$$

In addition to $\mathrm{H}^{k,2}(\mathfrak{g})$, the $\mathfrak{g}$-module $\mathfrak{g}^{(1)}$ also has a clear geometric meaning. If a G-structure $\mathcal{G} \to M$ is 1-flat then the set of all local torsion-free affine connections in $\mathcal{G}$ is the affine space modelled on the vector space of local functions on M with values in $\mathfrak{g}^{(1)}$. In particular, if $G \subseteq GL(V)$ is such that $\mathfrak{g}^{(1)} = 0$, then any G-structure admits at most one torsion-free affine connection. If $K(\mathfrak{g})$ is the $\mathfrak{g}$-module of formal curvature tensors of torsion-free affine connections with holonomy in $\mathfrak{g}$, i.e.

$$K(\mathfrak{g}) = [\mathfrak{g} \otimes \Lambda^2 V^*] \cap [\mathrm{Ker} : V \otimes V^* \otimes \Lambda^2 V^* \to V \otimes \Lambda^3 V^*],$$

then

$$\mathrm{H}^{2,2}(\mathfrak{g}) = \frac{K(\mathfrak{g})}{\partial(g^{(1)} \otimes V^*)}$$

i.e. the cohomology group $\mathrm{H}^{2,2}(\mathfrak{g})$ represents the part of $K(\mathfrak{g})$ which is invariant under $\mathfrak{g}^{(1)}$-valued shifts in a formal torsion-free affine connection with holonomy in $\mathfrak{g}$. For example, if $(G, V) = (\mathrm{CO}(n, \mathbb{C}), \mathbb{C}^n)$, then $\mathfrak{g}^{(1)} = V^*$ and $\mathrm{H}^{2,2}(\mathfrak{g})$ is the vector space of formal Weyl tensors.

If $\mathfrak{g}^{(1)} = 0$, then $\mathrm{H}^{2,2}(\mathfrak{g})$ is exactly $K(\mathfrak{g})$, the $\mathfrak{g}$-module which plays a key role in the theory of torsion-free affine connections with holonomy in $\mathfrak{g}$, especially in the Berger classification context [2, 3, 4]. The case $\mathfrak{g}^{(1)} = 0$ is generic — there are very few irreducibly acting Lie subgroups $\mathfrak{g} \subset gl(V)$ which have $\mathfrak{g}^{(1)} \neq 0$ and they are all known by now.

3. A simple group-theoretical explanation of the non-linear twistor transform. Let V be a finite dimensional complex vector space and $G \subseteq \mathrm{GL}(V)$ an irreducible representation of a reductive complex Lie group in V. Then G also acts irreducibly in V^* via the dual representation. Let $\tilde{X}$ be the G-orbit of a highest weight vector in $V^* \backslash 0$. Then the quotient $X := \tilde{X}/\mathbb{C}^*$ is a generalised flag variety (i.e. a compact complex homogeneous-rational manifold) canonically embedded into $\mathbb{P}(V^*)$ and there is a commutative diagram

$$\begin{array}{ccc} \tilde{X} & \hookrightarrow & V^* \backslash 0 \\ \downarrow & & \downarrow \\ X & \hookrightarrow & \mathbb{P}(V^*) \end{array}$$

In fact, $X = G_s/P$, where G_s is the semisimple quotient of G and P is the parabolic subgroup of G_s leaving a highest weight vector in V^* invariant up to a scale. Let L be the restriction of the hypèrplane section bundle $\mathcal{O}(1)$ on $\mathbb{P}(V^*)$ to the submanifold X. Clearly, L is an ample homogeneous line bundle on X.

In summary, there is a natural map

$$(G, V) \longrightarrow (X, L)$$

associating with any irreducibly acting reductive Lie group $G \subseteq \mathrm{GL}(V)$ a pair (X, L) consisting of a generelised flag variety X and an ample line bundle L on X. Can this map be reversed?

According to Borel-Weil (see, e.g., [1]), the representation space V can be reconstructed very easily:

$$V = \mathrm{H}^0(X, L).$$

What about G?

Theorem 1 (Onishchik 1962) *Let G be a simply connected simple complex Lie group, P a parabolic subgroup of G, and $X = G/P$ the associated generalised flag variety. Then*

$$\mathfrak{g} \subseteq \mathrm{H}^0(X, TX),$$

where $\mathfrak{g}$ *is the Lie algebra of* G. *Moreover,*

$$\mathfrak{g} = \mathrm{H}^0(X, TX)$$

unless one of the following holds

(i) G *is the representation of* $\mathrm{Sp}(n,\mathbb{C})$ *in* $\mathbb{C}^{2n}$ *(in this case* $\mathrm{H}^0(X,TX) \simeq sl(n,\mathbb{C})$*);*

(ii) G *is the representation of* G_2 *in* $\mathbb{C}^7$ *(*$\mathrm{H}^0(X,TX) \simeq so(7,\mathbb{C})$*);*

(iii) G *is the fundamental spinor representation of* $\mathrm{SO}(2n+1,\mathbb{C})$ *(*$\mathrm{H}^0(X,TX) \simeq so(2n+2, \mathbb{C})$*);*

Therefore, if $G \subset GL(V)$ is semisimple then, with a few exceptions, G can be reconstructed from (X, L). However, it is often undesirable to restrict oneself to semisimple groups only (especcially in the context of the Berger holonomy classification problem). There is a natural central extension of the Lie algebra $\mathrm{H}^0(X,TX)$ which is canonically associated with the pair (X, L).

Fact 2 *For any* (X, L) $\mathfrak{g} := \mathrm{H}^0(X, L \otimes (J^1L)^*)$ *is a reductive Lie algebra canonically represented in* $\mathrm{H}^0(X, L)$.

This fact is easy to explain — $\mathrm{H}^0(X, L \otimes (J^1L)^*)$ is exactly the Lie algebra of the Lie group G of all global biholomorphisms of the line bundle L which commute with the projection $L \to X$ [8].

In conclusion, with a given irreducible representation $G \subseteq \mathrm{GL}(V)$ there is canonically associated a pair (X, L) consisting of a generalised flag variety X and a very ample line bundle on X such that much of the original information about G can be restored from (X, L). In the twistor theory context, the crucial observation is that the $\mathfrak{g}$-modules $\mathfrak{g}^{(k)}$ and $\mathrm{H}^{k,2}(\mathfrak{g})$ also admit a nice description in terms of (X, L).

Theorem 3 *For any compact complex manifold* X *and any very ample line bundle* L *on* X*, there is an isomorphism*

$$\mathfrak{g}^{(k)} = \mathrm{H}^0(X, L \otimes \odot^{k+1} N^*), \quad k = 0, 1, 2, \ldots$$

and an exact sequence of $\mathfrak{g}$*-modules,*

$$0 \longrightarrow \mathrm{H}^{k,2}(\mathfrak{g}) \longrightarrow \mathrm{H}^1(X, L \otimes \odot^{k+2} N^*) \longrightarrow \mathrm{H}^1(X, L \otimes \odot^{k+1} N^*) \otimes V^*, \quad k = 1, 2, \ldots$$

where $\mathfrak{g} := \mathrm{H}^0(X, L \otimes (J^1L)^*)$, $N = J^1L$, *and* $\mathrm{H}^{k,2}(\mathfrak{g})$ *are the Spencer cohomology groups associated with the canonical representation of* $\mathfrak{g}$ *in the vector space* $V := H^0(X, L)$.

Proof. Since L is very ample, there is a natural "evaluation" epimorhism

$$V \otimes \mathcal{O}_X \longrightarrow J^1L \longrightarrow 0$$

whose dualization gives rise to the canonical monomorphism

$$0 \longrightarrow N^* \longrightarrow V^* \otimes \mathcal{O}_X.$$

Then one may construct the following sequences of locally free sheaves

$$0 \longrightarrow L \otimes \odot^{k+1} N^* \longrightarrow L \otimes \odot^k N^* \otimes V^* \longrightarrow L \otimes \odot^{k-1} N^* \otimes \Lambda^2(V^*) \tag{2}$$

and

$$0 \longrightarrow L \otimes \odot^{k+2} N^* \longrightarrow L \otimes \odot^{k+1} N^* \otimes V^* \longrightarrow L \otimes \odot^k N^* \otimes \Lambda^2(V^*) \longrightarrow L \otimes \odot^{k-1} N^* \otimes \Lambda^3(N^*). \tag{3}$$

One may notice that both these sequences are exact. [Hint: for any vector space W one has $W \otimes \Lambda^2 W \bmod \Lambda^3 W \simeq W \otimes \odot^2 W \bmod \odot^3 W$.]

Then computing $\mathrm{H}^0(X, \ldots)$ of (2) and using the inductive definition of $\mathfrak{g}^{(k)}$ as $[\mathfrak{g}^{(k-1)} \otimes V^*] \cap [V \otimes \odot^{k+1)} V^*]$ (with $\mathfrak{g}^{(0)} := \mathfrak{g}$ and $\mathfrak{g}^{(-1)} := V$) immediately implies the first statement of the Theorem.

The second statement of the theorem follows from (3) and the definition (1) of $\mathrm{H}^{k,2}(\mathfrak{g})$.

Indeed, define E_k by the exact sequence

$$0 \longrightarrow L \otimes \odot^{k+2} N^* \longrightarrow L \otimes \odot^{k+1} N^* \otimes V^* \longrightarrow E_k \longrightarrow 0$$

The associated long exact sequence implies the following *exact* sequence of vector spaces

$$0 \longrightarrow \mathrm{H}^0(X, E_k)/\partial[\mathfrak{g}^{(k)} \otimes V^*] \longrightarrow \mathrm{H}^1(X, L \otimes \odot^{k+2} N^*) \longrightarrow \mathrm{H}^1(X, L \otimes \odot^{k+1} N^*) \otimes V^*.$$

On the other hand, the exact sequence

$$0 \longrightarrow E_k \longrightarrow L \otimes \odot^k N^* \otimes \Lambda^2(V^*) \longrightarrow L \otimes \odot^{k-1} N^* \otimes \Lambda^3(N^*)$$

implies

$$\mathrm{H}^0(X, E_k) = \ker : \mathfrak{g}^{(k-1)} \otimes \Lambda^2(V^*) \xrightarrow{\partial} \mathfrak{g}^{(k-2)} \otimes \Lambda^3(V^*).$$

which in turn implies

$$\mathrm{H}^{k,2}(\mathfrak{g}) = \mathrm{H}^0(X, E_k)/\partial[\mathfrak{g}^{(k)} \otimes V^*].$$

This completes the proof of the second part of the Theorem. □

Therefore, the pair consisting from a homogeneous-rational manifold X and a holomorphic vector bundle bundle E on X is more than a natural building block for basic differential-geometric objects. If $\mathrm{rank} E \geq 2$, then, following a common practice in complex analysis, one should replace the pair (X, E) by an equivalent one $(\hat{X} = \mathbb{P}(E^*), L = \mathcal{O}(1))$ and apply Theorem 3 to find out which geometric category C_{geo} may correspond to the twistorial data (X, E). Applying this procedure, e.g., to the pair $(\mathbb{CP}^1, \mathbb{C}^k \otimes \mathcal{O}(1))$, $k \geq 3$, one immediately concludes that C_{geo} is the category of complexified quaternionic manifolds. Also, this purely group theoretical result suggests that there should exist a universal twistor construction for *all* torsion-free geometries. For details of this construction we refer to [8].

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Cohomology of a Quaternionic Complex

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Abstract

We investigate the cohomology of a certain elliptic complex defined on a compact quaternionic-Kähler manifold with negative scalar curvature. We show that this particular complex is exact, with the possible exception of one term.

Let (M, g) be an oriented $4k$-dimensional compact quaternionic-Kähler manifold having negative scalar curvature. In [2] we proved a rigidity theorem for such manifolds which was, in essence, a consequence of the vanishing of a certain cohomolgy group on the twistor space Z, of M. The paper was largely devoted to the proof of a vanishing theorem for the cohomolgy of the first term of a particular complex on M, the required vanishing theorem on Z being deduced by the Penrose transform. This note is an extension of that work in that we use the same techniques to show that all the cohomology of the complex vanishes, with the possible exception of one term, so that the complex is exact except possibly at the right. We have not been able to establish whether the final term of the complex has non-trivial cohomology.

Since the preparation of this article it has been brought to our attention that a similiar result has been proved, though in a different way, by Nagato and Nitta.

As in [2], we shall make extensive use of the methods and techniques developed by Bailey and Eastwoood in [1], these being based on the abstract index notation of Roger Penrose [3]. In this notation the indices are used as 'place-markers' and do not indicate a choice of basis. Thus the E, H bundles of Salamon [4], are written as $\mathcal{O}^{A}, \mathcal{O}^{A'}$ respectively. The (complexified) tangent bundle of M is then $\mathcal{O}^a = \mathcal{O}^{A} \otimes \mathcal{O}^{A'} \stackrel{def}{=} \mathcal{O}^{AA'}$. The Levi-Civita connection on M is written $\nabla_a = \nabla_{AA'}$. We shall take the Riemann curvature tensor to be given by

$$2\nabla_{[a}\nabla_{b]}w_c = R_{abc}{}^{d}w_d \tag{1}$$

so that, in our convention, the standard metric on the sphere has positive curvature. (This differs from the convention adopted in [1] which was based on that of [3]. This latter is essentially a relativity book.) The square brackets in the definition of the Riemann tensor, denotes anti-symmetrisation. Round brackets will be used for symmetrisation. For more details on the notation and techniques used we refer the reader to [3].

When written in abstract index notation, the complex we consider is defined on M by the following

$$0 \longrightarrow \Gamma(M, \mathcal{O}^{(A'_1\cdots A'_n)}) \overset{\nabla^{A'_{n+1}}_{B_1}}{\longrightarrow} \Gamma(M, \mathcal{O}^{(A'_1\cdots A'_{n+1})}_{B_1}) \overset{\nabla^{A'_{n+2}}_{B_2}}{\longrightarrow} \Gamma(M, \mathcal{O}^{(A'_1\cdots A'_{n+2})}_{[B_1 B_2]}) \longrightarrow \cdots$$
$$\cdots \overset{\nabla^{A'_{n+2k}}_{B_{2k}}}{\longrightarrow} \Gamma(M, \mathcal{O}^{(A'_1\cdots A'_{n+2k})}_{[B_1\cdots B_{2k}]}) \longrightarrow 0 \tag{2}$$

Here, for example, the bundle $\mathcal{O}^{(A'_1\cdots A'_{n+2})}_{[B_1 B_2]}$ is $\bigwedge^2 E^* \otimes S^{n+2} H$ (or its sheaf of smooth sections), and $\Gamma(M, V)$ is the space of global smooth sections of the bundle V over M. The map $\nabla^{A'_1}_{A}$ is then $\varepsilon^{A'_1 A'} \nabla_a$ with $\varepsilon^{A'_1 A'}$ being the (covariantly constant) symplectic form on H^*. This complex was studied by Salamon in [5] and shown by him to be an elliptic complex. Since M is compact we may use Hodge theory [6] to examine the cohomology of this complex. It is well known that the cohomology of the first term vanishes for negative scalar curvature. The vanishing of the cohomology of the second term was proved in [2], but can easily be obtained from the proof of theorem (0.3) with the appropriate changes. For terms other than the last Hodge theory implies that each cohomology class has a unique representative f which satisfies

$$\begin{aligned} &(a) \quad f \in \Gamma(M, \mathcal{O}^{(A'_1\cdots A'_p)}_{[B_1\cdots B_m]}) \\ &(b) \quad \nabla^{(A'_0}_{[B_0} f^{A'_1\cdots A'_p)}_{B_1\cdots B_m]} = 0 \\ &(c) \quad \nabla_{AC'} f^{AC' A'_2\cdots A'_p}_{B_2\cdots B_m} = 0 \end{aligned} \tag{3}$$

where $1 < m < 2k$ and $p - m = n$. If we let $D_0 = \nabla^{A'_{n+1}}_{B_1}$ and $D_1 = \nabla^{A'_{n+2}}_{B_2}$ in (2), then with $m = 1$, (b) is the abstract index version of $D_1 f = 0$ and (c) is that of $D_0^* f = 0$, D_0^* being the adjoint of the mapping D_0.

We shall take $u^A \overline{v}_A$ to be the positive definite inner- product, where $\overline{v}$ is the quaternionic conjugate of v, on the bundle $\mathcal{O}^A$, so that $u^A \overline{u}_A$ is positive if $u \neq 0$, and similarly for other tensors. The L^2-norm of a tensor $v = v^{AB'}{}_{CD'}$ for example, is then

$$\| v \|^2 = \int v^{AB'CD'} \overline{v}_{AB'CD'} \tag{4}$$

with the integral taken over the manifold M. With this convention, one can use integration by parts over M and quickly obtain (c) above as the abstract index version of $D_0^* f = 0$, when $m = 1$. We shall show that these terms all vanish, so that the above complex (2) is exact, except possibly at the right. We note the following elementary facts.

Lemma 0.1 *If $S^{A'_0\cdots A'_p}$ is symmetric in its final p indices and $U_{B_0\cdots B_m}$ is antisymmetric in* *its final m indices, then*

$$(p+1)S^{(A'_0\ldots A'_p)} = (p+1)S^{A'_0\ldots A'_p} - \sum_{i=1}^{i=p}\varepsilon^{A'_0A'_i}S_{C'}{}^{C'A'_1\ldots\hat{A'}_i\ldots A'_p} \qquad (5)$$

$$(m+1)U_{[B_0\ldots B_m]} = U_{B_0\ldots B_m} + \sum_{i=1}^{i=m}(-1)^i U_{B_iB_0\ldots\hat{B}_i\ldots B_m} \qquad (6)$$

It is then quite easy to obtain the following.

Lemma 0.2 *If f satisfies the conditions of (3) then*

$$(p+1)\parallel \nabla^{A'_0}_{[B_0}f^{A'_1\ldots A'_p}_{B_1\ldots B_m]}\parallel^2 = p \parallel \nabla_{A'[B_0}f^{A'A'_2\ldots A'_p}_{B_1\ldots B_m]}\parallel^2$$

and $\nabla^{BA'_0}f^{A'_1\ldots A'_p}_{BB_2\ldots B_m}$ *is symmetric in its primed indices.*

Proof For the first part we put $S^{A'_0\ldots A'_p} = \nabla^{A'_0}_{[B_0}f^{A'_1\ldots A'_p}_{B_1\ldots B_m]}$ in (5) and using (b) of (3) we get

$$(p+1)\nabla^{A'_0}_{[B_0}f^{A'_1\ldots A'_p}_{B_1\ldots B_m]} = \sum_{i=1}^{i=p}\varepsilon^{A'_0A'_i}\nabla_{A'[B_0}f^{A'A'_2\ldots A'_p}_{B_1\ldots B_m]}$$

Contracting both sides of the above with $\nabla^{[B_0}_{A'_0}\bar{f}^{B_1\ldots B_m]}_{A'_1\ldots A'_p}$ and integrating the result over M, one quickly obtains the first part. The second part can be obtained by putting $S^{A'_0\ldots A'_p} = \nabla^{BA'_0}f^{A'_1\ldots A'_p}_{BB_2\ldots B_m}$ in (5) and using (b) of (3).

We can now state our main result.

Theorem 0.3 *Let M be a compact quaternionic-Kähler manifold of dimension $4k$, for $k > 1$, and let $\Lambda = R/8k(k+2)$, where R is its (non-zero) scalar curvature. If f is a harmonic element of $\Gamma(M, \mathcal{O}^{(A'_1\ldots A'_p)}_{[B_1\ldots B_m]})$, i.e. satisfies the conditions of(3), and if $1 < m < 2k$, then*

$$\frac{(p+2)}{2(p+1)}\parallel \nabla^{[B_0}_{A'}f^{B_1\ldots B_m]A'A'_2\ldots A'_p}\parallel^2 + \frac{m}{2(m+1)}\parallel \nabla^{C'}_{B}f^{BB_2\ldots B_mA'_1\ldots A'_p}\parallel^2 = \Lambda\frac{(p+2)}{(m+1)}(2k-m)\parallel f\parallel^2 \qquad (7)$$

Thus if $R < 0$ then $f = 0$.

Proof We have

$$
\begin{aligned}
\| \nabla^{[B_0}_{A'} f^{B_1\dots B_m]A'A'_2\dots A'_p} \|^2 &= \int (\nabla^{[B_0}_{K'} f^{B_1\dots B_m]K'A'_2\dots A'_p})(\nabla_{C'[B_0} \bar{f}^{C'}_{B_1\dots B_m]A'_2\dots A'_p}) \\
&= -\int f^{B_1\dots B_m K'A'_2\dots A'_p} \nabla^{B_0}_{K'} \nabla_{C'[B_0} \bar{f}^{C'}_{B_1\dots B_m]A'_2\dots A'_p} \\
&= -\int f^{B_1\dots B_m K'A'_2\dots A'_p} (\nabla^{B_0}_{(K'} \nabla_{C')[B_0} \\
&\quad + \nabla^{B_0}_{[K'} \nabla_{C'][B_0}) \bar{f}^{C'}_{B_1\dots B_m]A'_2\dots A'_p} \qquad (8)
\end{aligned}
$$

where the second line above is obtained from the first by using integration by parts over the manifold. The latter integral on the right is

$$
\begin{aligned}
\frac{1}{2}\int f^{B_1\dots B_m K'A'_2\dots A'_p} \varepsilon_{K'C'} \nabla^{B_0}_{D'} \nabla^{D'}_{[B_0} \bar{f}^{C'}_{B_1\dots B_m]A'_2\dots A'_p} &= \frac{1}{2}\int f^{B_1\dots B_m A'_2\dots A'_p}_{C'} \nabla^{B_0}_{D'} \nabla^{D'}_{[B_0} \bar{f}^{C'}_{B_1\dots B_m]A'_2\dots A'_p} \\
&= -\frac{1}{2} \| \nabla^{A'_0}_{[B_0} f^{A'_1\dots A'_p}_{B_1\dots B_m]} \|^2
\end{aligned}
$$

the final expression being obtained from the previous one by using integration by parts again. Using the first part of lemma (0.2) and rearranging, (8) becomes

$$
\frac{(p+2)}{2(p+1)} \| \nabla^{[B_0}_{A'} f^{B_1\dots B_m]A'A'_2\dots A'_p} \|^2 = -\int f^{B_1\dots B_m K'A'_2\dots A'_p} \nabla^{B_0}_{(K'} \nabla_{C')[B_0} \bar{f}^{C'}_{B_1\dots B_m]A'_2\dots A'_p} \qquad (9)
$$

Examining the latter integrand we see that

$$
\begin{aligned}
\nabla^{B_0}_{(K'} \nabla_{C')[B_0} \bar{f}^{C'}_{B_1\dots B_m]A'_2\dots A'_p} &= \frac{1}{m+1}(\nabla^{B_0}_{(K'} \nabla_{C')B_0} \bar{f}^{C'}_{B_1\dots B_m A'_2\dots A'_p} \\
&\quad + \sum_{i=1}^{i=m} (-1)^i \nabla^{B_0}_{(K'} \nabla_{C')B_i} \bar{f}^{C'}_{B_0 B_1\dots \hat{B}_i \dots B_m A'_2\dots A'_p}) \qquad (10)
\end{aligned}
$$

Now by definition

$$
\nabla^{B_0}_{(K'} \nabla_{C')B_i} - \nabla_{B_i(K'} \nabla^{B_0}_{C')} = \square_{K'C'}{}^{B_0}{}_{B_i}.
$$

(For the definition and properties of the curvature operators see [1]) The second operator on the left above is $\frac{1}{2}(\nabla_{B_iK'}\nabla^{B_0}_{C'} + \nabla_{B_iC'}\nabla^{B_0}_{K'})$ and the first part of this sum annihilates $\bar{f}$ by (c) of (3). Equation (9) can now be rewritten

$$
\begin{aligned}
\frac{(p+2)}{2(p+1)} \| \nabla^{[B_0}_{A'} f^{B_1\dots B_m]A'A'_2\dots A'_p} \|^2 &= -\frac{1}{m+1}\int f^{B_1\dots B_m K'A'_2\dots A'_p} (\square_{K'C'}{}^{B_0}{}_{B_0} \bar{f}^{C'}_{B_1\dots B_m A'_2\dots A'_p}) \\
&\quad -\frac{1}{(m+1)} \sum_{i=1}^{i=m} (-1)^i \int f^{B_1\dots B_m K'A'_2\dots A'_p} (\square_{K'C'}{}^{B_0}{}_{B_i} \bar{f}^{C'}_{B_0 B_1\dots \hat{B}_i \dots B_p A'_2\dots A'_p}) \\
&\quad -\frac{1}{2(m+1)} \sum_{i=1}^{i=m} (-1)^i \int f^{B_1\dots B_m K'A'_2\dots A'_p} (\nabla_{B_iC'} \nabla^{B_0}_{K'} \bar{f}^{C'}_{B_0 B_1\dots \hat{B}_i \dots B'_m A'_2\dots A'_p})
\end{aligned}
$$

Using the symmetry in the primed indices given by the latter part of lemma (0.2) we may commute the final K' and C' in the final integral above and, after another application of integration by parts, this final integral becomes

$$
\begin{aligned}
&-\int(\nabla_{B_iC'}f^{B_1\ldots B_m K' A_2'\ldots A_p'})(\nabla^{B_0C'}\bar{f}_{B_0B_1\ldots\hat{B}_i\ldots B_m K' A_2'\ldots A_p'})\\
&=(-1)^i\int(\nabla_{B_i}^{C'}f^{B_iB_1\ldots\hat{B}_i\ldots B_m K' A_2'\ldots A_p'})(\nabla_{B_0C'}\bar{f}^{B_0}{}_{B_1\ldots\hat{B}\ldots B_m K' A_2'\ldots A_p'})\\
&=(-1)^i\parallel\nabla_B^{C'}f^{BB_2\ldots B_m A_1'\ldots A_p'}\parallel^2
\end{aligned}
$$

With this in mind we may now rearrange the above equation to obtain

$$
\frac{(p+2)}{2(p+1)}\parallel\nabla_{A'}^{[B_0}f^{B_1\ldots B_m]A'A_2'\ldots A_p'}\parallel^2+\frac{m}{2(m+1)}\parallel\nabla_B^{C'}f^{BB_2\ldots B_m A_1'\ldots A_p'}\parallel^2=-\frac{1}{m+1}\int R(f) \tag{11}
$$

where $R(f)$ is given by

$$
R(f)=f^{B_1\ldots B_m K' A_2'\ldots A_p'}(\square_{K'C'}{}^{B_0}{}_{B_0}\bar{f}^{C'}{}_{B_1\ldots B_m A_2'\ldots A_p'}+\sum_{i=1}^{i=m}(-1)^i\square_{K'C'}{}^{B_0}{}_{B_i}\bar{f}^{C'}{}_{B_0B_1\ldots\hat{B}_i\ldots B_p A_2'\ldots A_p'})
$$

The evaluation of the action of the curvature operators is an elementary exercise. We have

$$
\square_{K'C'}{}^{B_0}{}_{B_0}\bar{f}^{C'}{}_{B_1\ldots B_m A_2'\ldots A_p'}=-2\Lambda k(p+2)\bar{f}_{B_1\ldots B_m K' A_2'\ldots A_p'}
$$

and the other operator gives

$$
\begin{aligned}
\square_{K'C'}{}^{B_0}{}_{B_i}\bar{f}^{A_1'\ldots A_p'}{}_{B_0\ldots\hat{B}_i\ldots B_p} &= -2\Lambda\epsilon^{B_0}{}_{B_i}\sum_{j=1}^{j=p}\varepsilon_{(K'}{}^{A_j'}\varepsilon_{C')Q'}\bar{f}^{Q'A_1'\ldots\hat{A}_j'\ldots A_p'}{}_{B_0\ldots\hat{B}_i\ldots B_m}\\
&= 2\Lambda\sum_{j=1}^{j=p}\varepsilon_{(K'}{}^{A_j'}\varepsilon_{C')Q'}\bar{f}^{Q'A_1'\ldots\hat{A}_j'\ldots A_p'}{}_{B_iB_1\ldots\hat{B}_i\ldots B_m}\\
&= (-1)^{i-1}2\Lambda\sum_{j=1}^{j=p}\varepsilon_{(K'}{}^{A_j'}\varepsilon_{C')Q'}\bar{f}^{Q'A_1'\ldots\hat{A}_j'\ldots A_p'}{}_{B_1\ldots B_m}\\
\square_{K'C'}{}^{B_0}{}_{B_i}\bar{f}^{C'}{}_{B_0\ldots\hat{B}_i\ldots B_m A_2'\ldots A_p'} &= (-1)^i\Lambda(p+2)\bar{f}_{B_1\ldots B_m K' A_2'\ldots A_p'}
\end{aligned}
$$

so that $R(f)=-\Lambda(p+2)(2k-m)\bar{f}_{B_1\ldots B_m K' A_2'\ldots A_p'}$. Substituting this into (11) completes the proof. □

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A Nonlinear Graviton from the Sine-Gordon equation

Maciej Dunajski

In **TN** 30 [1] Lionel Mason pointed out that certain integrable systems may be encoded within a geometry of the nonlinear graviton. This is true at least for integrable equations arising from $SL(2,C)$ anti-self-dual Yang-Mills equations (ASDYM). The purpose of this note is to describe how this construction works, illustrating the method with the example of the Sine-Gordon equation.
Let $\mathcal{M}$ be a real four-manifold with a volume form ν. Let $V_i = (W, \widetilde{W}, Z, \widetilde{Z})$ be real, independent, volume preserving vector fields on $\mathcal{M}$. Define $f^2 = \nu(W, \widetilde{W}, Z, \widetilde{Z})$. Mason and Newman's [2] form of the nonlinear graviton theorem states that if

$$L = W - \lambda\widetilde{Z}, \qquad M = Z - \lambda\widetilde{W} \tag{1}$$

commute for each value of $\lambda \in CP^1$, then $f^{-1}V_i$ is a normalized null tetrad for an anti self dual vacuum metric. The metric has a signature $(++--)$.
ASDYM possess a Lax formulation isomorphic to that of the ASD Einstein. We use $x^\mu = (w, \tilde{w}, z, \tilde{z})$ as coordinates on R^4 that are independent and real for signature $(++--)$. ASDYM are equivalent to the commutativity of the Lax pair

$$L = D_w - \lambda D_{\tilde{z}}, \quad M = D_z - \lambda D_{\tilde{w}}. \tag{2}$$

Here $D_\mu = \partial_\mu - A_\mu$ is a covariant derivative. We fix the gauge group to be $SU(2)$ and put $A_\mu = A^a{}_\mu \sigma_a$, where

$$\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

We impose symmetries in the ∂_w and $\partial_{\tilde{w}}$ directions so that, in an invariant gauge, $A^a{}_\mu$ are functions of $(z, \tilde{z})$. The passage to ASD gravity is based on re-expressing σ_a as real hamiltonian vector fields X_{h_a} on $\Sigma = CP^1$ with respect to a symplectic structure $\Omega_\Sigma = i(1+\xi\bar{\xi})^{-2}(d\xi \wedge d\bar{\xi})$. Hamiltonians h_a are found from the Möbius action of $SU(2)$ on a Riemann sphere,

$$h_1 = -\frac{\xi+\bar{\xi}}{1+\xi\bar{\xi}}, \quad h_2 = -i\frac{\xi-\bar{\xi}}{1+\xi\bar{\xi}}, \quad h_3 = \frac{2}{1+\xi\bar{\xi}}, \qquad X_{h_b}(h_a) = 2\epsilon_{ab}{}^c h_c. \tag{3}$$

Now D_μ become volume preserving vector fields on $\mathcal{M} = R^2 \times CP^1$ with $\nu = dz \wedge d\tilde{z} \wedge \Omega_\Sigma$. We identify D_μ with V_i;

$$W = -A^a{}_w X_{h_a}, \quad \widetilde{W} = -A^a{}_{\tilde{w}} X_{h_a}, \quad Z = \partial_z - A^a{}_z X_{h_a}, \quad \widetilde{Z} = \partial_{\tilde{z}} - A^a{}_{\tilde{z}} X_{h_a}.$$

A covariant metric on $\mathcal{M}$ is conveniently expressed by a dual frame e_{V_i} such that $e_{V_i}(V_j) = \delta_{ij}$. A simple calculation yields

$$\begin{aligned}
f^2 &= 2A^a{}_w A^b{}_{\tilde{w}} \epsilon_{ab}{}^c h_c, \\
e_W &= -f^{-2} A^b{}_{\tilde{w}} (\partial_\xi h_b d\xi + \partial_{\bar{\xi}} h_b d\bar{\xi} - 2\epsilon_{ab}{}^c h_c (A^a{}_z dz + A^a{}_{\tilde{z}} d\tilde{z})) \\
&= -f^{-2} A^a{}_{\tilde{w}} D_\Sigma h_a, \\
e_{\widetilde{W}} &= f^{-2} A^b{}_w (\partial_\xi h_b d\xi + \partial_{\bar{\xi}} h_b d\bar{\xi} - 2\epsilon_{ab}{}^c h_c (A^a{}_z dz + A^a{}_{\tilde{z}} d\tilde{z})) \\
&= f^{-2} A^a{}_w D_\Sigma h_a, \\
e_Z &= dz, \quad e_{\widetilde{Z}} = d\tilde{z},
\end{aligned} \tag{4}$$

where $D_\Sigma = d_\Sigma - A^a{}_\mu dx^\mu \otimes X_{h_a}$ is the form valued operator acting on functions. Note that D_Σ decomposes to $\mathcal{D}_\Sigma + \overline{\mathcal{D}}_\Sigma$, where

$$\begin{aligned}
\mathcal{D}_\Sigma &= (d\xi - iA^a{}_\mu (1+\xi\bar{\xi})^2 \partial_{\bar{\xi}} h_a dx^\mu) \otimes \partial_\xi = d_A\xi \otimes \partial_\xi, \\
\overline{\mathcal{D}}_\Sigma &= (d\bar{\xi} + iA^a{}_\mu (1+\xi\bar{\xi})^2 \partial_\xi h_a dx^\mu) \otimes \partial_{\bar{\xi}} = \overline{d_A\xi} \otimes \partial_{\bar{\xi}}.
\end{aligned} \tag{5}$$

Finally we consider the metric

$$ds^2 = f^2 dz \otimes d\tilde{z} + f^{-2} A^a{}_{\tilde{w}} A^b{}_w D_\Sigma h_a \otimes D_\Sigma h_b. \tag{6}$$

This may be viewed as a metric on the total space of the Σ-bundle associated to a YM bundle. An infinitesimal gauge transformation $A^a{}_\mu \longrightarrow A^a{}_\mu + \tau(\epsilon_{ab}{}^c A^b{}_\mu F^c + \partial_\mu F^a)$, where F^a are functions of z and $\tilde{z}$, is equivalent to the diffeomorphism of $R^2 \times CP^1$ given by $y^\nu \longrightarrow y^\nu + \tau F^a X_{h_a}(y^\nu)$. Tensor objects transform by Lie derivative along $F^a X_{h_a}$. As a consequence D_Σ undergoes the same type of transformation as a covariant deriative .
Singularities in (6) which come from the base space may be removed by imposing the boundary conditions on the YM connection. However we are

left with singularities of the vector fields X_{h_a} for ξ being real or purely imaginary.
To establish the connection with Sine-Gordon, we express $A^a{}_\mu$ in terms of its solutions. This reduction was noted by a number of authors. We use a gauge choice due to Ablowitz and Chakravarty [3].
Set $A_{\tilde{z}} = 0$. ASDYM with $G = SU(2)$ are solved by anzatzs

$$A_w = \cos\phi\,\sigma_1 + \sin\phi\,\sigma_2, \quad A_{\tilde{w}} = \sigma_1, \quad A_z = 1/2\partial_{\tilde{z}}\phi\,\sigma_3 \qquad (7)$$

provided that ϕ satisfies $\partial_{z\tilde{z}}\phi = 4\sin\phi$.
Using (7) we obtain: $f^2 = 2(1+\xi\bar{\xi})^{-1}\sin\phi$, $d_A\xi = d\xi + i\xi\partial_{\tilde{z}}\phi dz$, and

$$ds^2 = \frac{1}{1+\xi\bar{\xi}}\Big([(1-\bar{\xi}^2)^2\cot\phi + i(1-\bar{\xi}^4)]d_A\xi\otimes d_A\xi$$

$$+\Big(\cot\phi(1-\bar{\xi}^2)(1-\xi^2) + i[(1+\bar{\xi}^2)(1-\xi^2) - (1-\bar{\xi}^2)(1+\xi^2)]\Big)\, d_A\xi\otimes\overline{d_A\xi}$$

$$+[(1-\xi^2)^2\cot\phi + i(1-\xi^4)]\overline{d_A\xi}\otimes\overline{d_A\xi} + 2\sin\phi\, dz\otimes d\tilde{z}\Big) \qquad (8)$$

Now we can use the whole machinery (Bäcklund transformations, topological conservation laws, etc.) known from the theory of Sine-Gordon to deal with (8). If one takes a solution describing the interaction of half kink and half anti-kink (two topological solitons traveling in $z-\tilde{z}$ direction and increasing from 0 to π as $z+\tilde{z}$ goes from $-\infty$ to ∞) then the singularity in $\sin\phi = 0$ may be absorbed by a conformal transformation of $z+\tilde{z}$.
Many thanks to Lionel Mason for much help and guidance.

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[1] Mason, L.J. (1990) $\mathcal{H}$-space, a universal integrable system?, Twistor Newsletter 30.

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Abstracts

Michael Murray and Michael Singer: "Spectral curves of non-integral hyperbolic monopoles." The twistor description of SU_n-monopoles on hyperbolic 3-space is developed and some consequences discussed. It is shown that in a precise sense, any appropriately decaying hyperbolic monopole is determined by its asymptotic values, and also that such monopoles determine and are determined by their spectral data (which reduces to a complex algebraic curve when $n = 2$). This paper provides the hyperbolic analogue of the work of Hitchin and others on euclidean monopoles and generalizes that of Atiyah and Braam-Austin on integral hyperbolic monopoles.

L.J.Mason: "The vacuum and Bach equations in terms of light cone cuts." Light cone cuts are the intersection of the light cones of space-time events with an initial data surface which is usually taken to be null infinity. These are determined by a "cut function," a function of the space-time coordinates and the sphere of generators of null infinity. The Bach equations and the conformal vacuum equations are shown to lead to a single scalar "main" equation on the cut function. In simplified cases, i.e., when the Weyl curvature is either small or self-dual, this equation can be integrated to give a simple scalar conformally invariant elliptic equation on the sphere of null directions. It has the remarkable property that the general solution of the Einstein vacuum equations in these cases can be derived from the solution to this auxiliary equation in two dimensions. Nevertheless, the main equation is still a necessary condition and it is conjectured that it is also sufficient to imply the Bach equations in general and supporting arguments are given. It is also shown how to extend the Kozameh-Newman framework to the case of light cone cuts of a spacelike Cauchy hypersurface. The analogous procedures for Yang-Mills are also discussed. A conformally invariant formalism for calculations on null infinity is described in an Appendix. This is used for all the calculations which are only described in brief form in the main text. *Copywrite 1995 American Institute of Physics.*

T.N.Bailey, M.G.Eastwood, A.R.Gover and L.J.Mason: "The Funk Transform as a Penrose Transform." The Funk transform is the integral transform from the space of smooth even functions on the unit sphere $S^2 \subset \mathbb{R}^3$ to itself defined by integration over great circles. One can regard this transform as a limit in a certain sense of the Penrose transform from $\mathbb{CP}_2$ to $\mathbb{CP}_2^*$. We exploit this viewpoint by developing a new proof of the bijectivity of the Funk transform which proceeds by considering the cohomology of a certain involutive (or formally integrable) structure on an intermediate space. This is the simplest example of what we hope will prove to be a general method of obtaining results in real integral geometry by means of complex holomorphic methods derived from the Penrose transform.

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