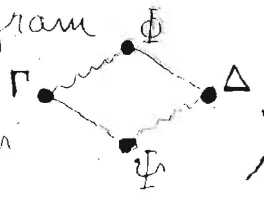
Incidence between Complex Null Rays

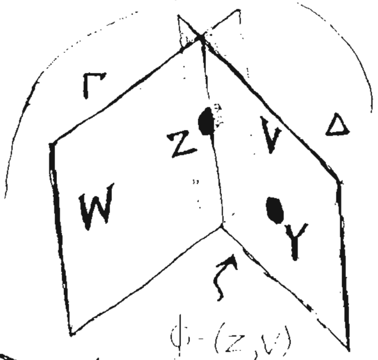
The incidence relations between complex null rays exhibit certain curious properties. I shall try to illuminate these here. The structures that arise have relevance to the geometry of ambitwistor space. The discussion will be phrased in terms of compactified complex Minkowski space $\mathbb{C}\mathbb{M}^{\#}$ and its associated (projective) twistor space $\mathbb{PT}$, in the first instance, with some generalizations to curved space-time indicated at the end.

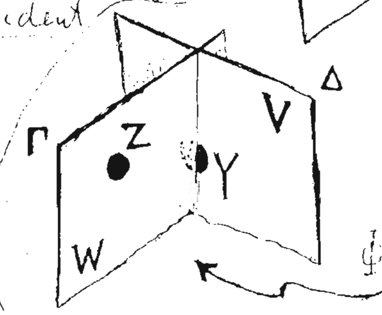
By a <u>ray</u> I mean a (generally complex) null geodesic. A ray γ in $\mathbb{C}\mathbb{M}^{\#}$ has a twistor interpretation in terms of a pair $\Gamma = (Z, W)$, where $Z \in \mathbb{PT}$, $W \in \mathbb{PT}^*$ with Z incident with W (i.e. $Z^\alpha W_\alpha = 0$). I shall refer to Γ as a <u>pencil</u>, in reference to the plane pencil of lines lying in the plane $W \subset \mathbb{PT}$ and passing through the point $Z \in \mathbb{PT}$, these lines representing the points of the ray $\gamma \subset \mathbb{C}\mathbb{M}^{\#}$. The point $Z \in \mathbb{PT}$ is called the <u>vertex</u> of the pencil and W is its plane.

There are various kinds of incidence relation holding between rays in $\mathbb{C}\mathbb{M}^{\#}$, which I shall call "weak incidence", "strong incidence", "α-incidence", and "β-incidence". Two pencils $\Gamma = (Z, W)$ and $\Delta = (Y, V)$ are <u>α-incident</u> if $Y = Z$ (diagram $\overset{\Gamma}{\bullet}\!\sim\!\sim\!\sim\!\overset{\Delta}{\bullet}$); they are <u>$\beta$-incident</u> if $V = W$ (diagram $\overset{\Gamma}{\bullet}\cdots\cdots\overset{\Delta}{\bullet}$); they are <u>weakly incident</u> if the join $Y \sqcup Z$ of Y and Z meets the intersection $V \cap W$ of V and W (diagram $\overset{\Gamma}{\bullet}\!\sim\!\sim\!\overset{\Phi}{\bullet}\!-\!-\!\overset{\Delta}{\bullet}$ or $\overset{\Gamma}{\bullet}\!-\!-\!\overset{\Phi}{\bullet}\!\sim\!\sim\!\overset{\Delta}{\bullet}$ for some Φ); they are <u>strongly incident</u> if $Y \sqcup Z = V \cap W$ (diagram



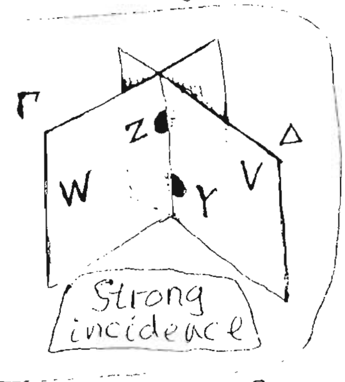
for some Φ, Ψ).





Weak incidence

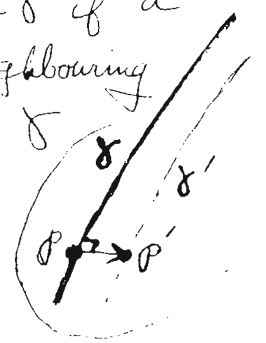
Strong incidence between Γ and Δ is the condition that the corresponding rays γ and δ in $\mathbb{C}\mathbb{M}^\#$ have a point in



common. (See also the discussion by T.R.F. in TN 38 pp. 30–34.)

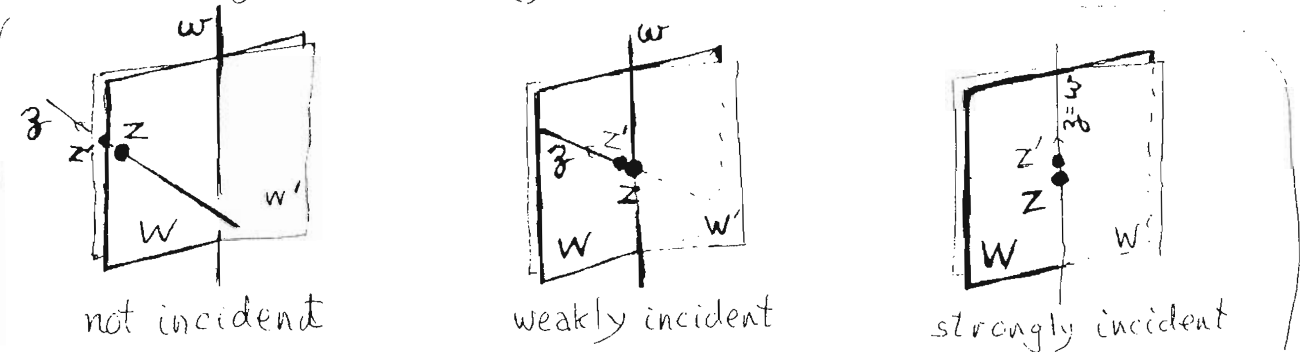
Note that the space $\mathcal{D}$ of those Δ that are weakly incident with a fixed Γ is the union of two smooth manifolds $\mathcal{D}_\alpha$, $\mathcal{D}_\beta$ (respectively), each being a $\mathbb{C}\mathbb{P}^2$ bundle over $\mathbb{C}\mathbb{P}^2$. This arises because each of the spaces of pencils Φ α-incident or β-incident with Γ is a $\mathbb{C}\mathbb{P}^2$, and we apply one step in each to arrive at weak incidence. The condition of strong incidence does not provide a smooth manifold, however, as we shall be seeing shortly.

Consider, next, weak and strong incidence with γ of a ray γ' neighbouring to γ (i.e. we are concerned with infinitesimal variations of rays away from γ). We find that the notion of weak incidence between γ' and γ is equivalent to abreastness. We say that γ' is abreast with γ if a connecting vector from a point p on γ to a neighbouring point p' of γ' is orthogonal to the direction of γ (cf. Penrose & Rindler, Spinors & Space-time Vol. 2 p. 176). This property is independent of the choice of p on γ and of the neighbouring point p' on γ'. An easy way to see that weak incidence between neighbouring rays implies abreastness is to observe that a way of stating the condition for two rays γ and δ to be weakly incident is that there be a ray meeting each of γ and δ orthogonally. This is the ray ϕ represented by the pencil Φ in $\overset{\Gamma}{\bullet}\!\!-\!\!\overset{\Phi}{\circ}\!\!-\!\!\overset{\Delta}{\bullet}$ or $\overset{\Gamma}{\bullet}\!\!-\!\!\overset{\Phi}{\circ}\!\!-\!\!\overset{\Delta}{\bullet}$.



(Note that any two rays in the same α-plane or in the same β-plane are orthogonal at their intersection point.) Hence, when δ becomes γ', neighbouring to γ, with p the intersection of ϕ with γ, we see that the condition for abreastness is satisfied. For the converse, we may simply note that weak incidence is a codimension 1 condition, so the neighbourhood of $\mathcal{D}$ at Γ cannot be smaller than the space of γ' abreast with γ, abreastness being an irreducible condition.

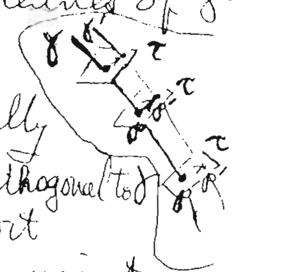
In terms of $\mathbb{PT}$, weak incidence between Γ and Γ' is understood as the condition that the join $z = Z \sqcup Z'$ meet the intersection $w = W \cap W'$. Unlike the case of finitely differing pencils, the condition now becomes symmetrical, w and z just being two lines in the plane $\mathbf{W}$ passing through the point $\mathbf{Z}$.

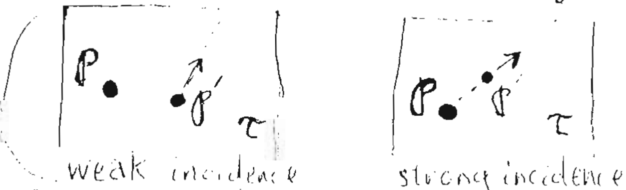


Incidence between neighbouring pencils.

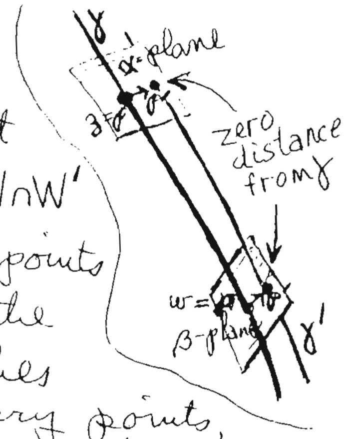
What about strong incidence? This is the condition that γ' actually "meets" γ', so that the pencils Γ and Γ' have a line in common. This line must be both z and w, so we have $z = w$ as the condition for strong incidence. Recall that with finitely differing pencils Γ, Δ, if both forms of weak incidence hold together, then strong incidence must hold. It is curious that for infinitesimally differing pencils Γ, Γ', the situation is quite different. The two forms of weak incidence for Γ, Γ' coincide, and the simultaneous holding of both forms is insufficient to imply strong incidence. I shall come to the explanation of this curious fact shortly.

In terms of space-time geometry, we can think of γ' as given by a Jacobi field of connecting vectors along γ. Referring this to a family of parallelly propagated 2-plane elements τ, at the points of γ, orthogonal to γ, all identified with one another by parallel transport along γ, we find that, for an abreast ray γ', the point p' in this plane executes a straight line in τ, uniformly with the affine parameter on γ.





If γ' is strongly incident with γ, this line passes through p; in the general case of weak incidence, it does not.



In the case of strong incidence, the point where p' coincides with p is the point $z \in \gamma$, represented by the line $z = Z \sqcup Z' = w = W \cap W'$ in $\mathbb{PT}$. In the general case of weak incidence, the points z and w are distinct, and they represent the locations of p at which the point p' reaches zero distance from p. These are imaginary points, when the rays are real, and they occur where γ' intersects the α-plane and the β-plane through γ, respectively.

In terms of twistor notation, we can write

$$Y^\alpha = Z^\alpha + \delta Z^\alpha, \quad V_\alpha = W_\alpha + \delta W_\alpha$$

where

$$Z^\alpha W_\alpha = 0, \quad Y^\alpha V_\alpha = 0$$

whence

$$W_\alpha \delta Z^\alpha + Z^\alpha \delta W_\alpha + \delta Z^\alpha \delta W_\alpha = 0.$$

To first order, we have

$$W_\alpha \delta Z^\alpha = - Z^\alpha \delta W_\alpha.$$

The condition for weak incidence is either $Z^\alpha V_\alpha = 0$, i.e.

$$Z^\alpha \delta W_\alpha = 0$$

or $Y^\alpha W_\alpha = 0$, i.e.

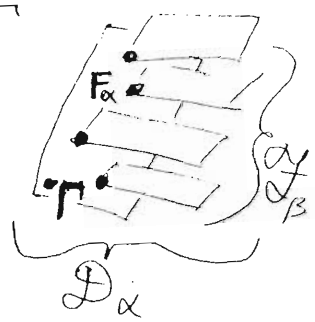
$$W_\alpha \delta Z^\alpha = 0$$

which, to first order, are indeed equivalent. To recognise strong incidence we must go to second order, the condition being that, in addition, the quadratic relation

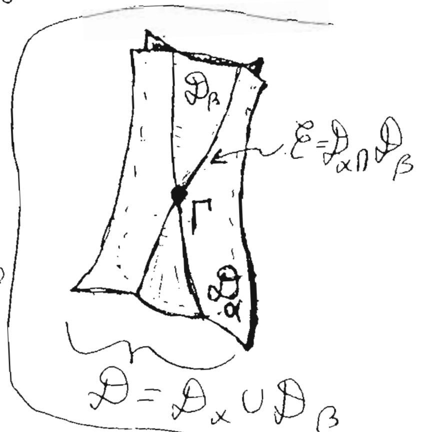
$$\delta Z^\alpha \delta W_\alpha = 0$$

must hold. (For real light rays the discussion is the same, but with $W_\alpha = \bar{Z}_\alpha$, $\delta W_\alpha = \delta \bar{Z}_\alpha$.)

Let us try to understand this curious situation in terms of the geometry of ambitwistor space $\mathbb{A}$, for $\mathbb{CM}^\#$. ($\mathbb{A}$ is the space of Γs, i.e. of complex rays in $\mathbb{CM}^\#$.) This space is **5**-dimensional. The space $\mathcal{D}$ of points of $\mathbb{A}$ that are weakly incident with Γ is, as we have seen, the union of two smooth manifolds $\mathcal{D}_\alpha$ and $\mathcal{D}_\beta$. We can think of $\mathcal{D}_\alpha$ as constructed as follows. There is a foliation $\mathcal{F}_\alpha$ of $\mathbb{A}$ by families of points of $\mathbb{A}$ that are α-incident with each other and another foliation $\mathcal{F}_\beta$ by families of points of $\mathbb{A}$ that are β-incident with each other. To construct $\mathcal{D}_\alpha$, we fix the member F_α of $\mathcal{F}_\alpha$ through Γ and sweep out a region of $\mathbb{A}$ by taking all the members of $\mathcal{F}_\beta$ through points of F_α. In an exactly similar way, we can construct $\mathcal{D}_\beta$, now by fixing F_β through Γ and allowing the $\mathcal{F}_\alpha$ members through points of F_β to vary.



Note that the difference between $\mathcal{D}_\alpha$ and $\mathcal{D}_\beta$, when constructed in this way is a <u>commutator</u>, and this difference is thus of second order. In other words, $\mathcal{D}_\alpha$ and $\mathcal{D}_\beta$ have the same tangent plane at Γ. This explains the symmetry between the two forms of weak incidence when we look only to first order. However, the <u>strong</u> incidence locus $\mathcal{E}$ is the intersection of $\mathcal{D}_\alpha$ and $\mathcal{D}_\beta$. Since $\mathcal{D}_\alpha$ and $\mathcal{D}_\beta$ touch at Γ, their intersection has a <u>conical singularity</u> at Γ. The equation, locally, of this cone is the quadratic relation $\delta Z^\alpha \delta W_\alpha = 0$.



These ideas can also be applied in a general curved space-time $\mathcal{M}$ in reference to a choice of hypersurface $\mathcal{H}$ (possibly $\mathcal{H} = \mathscr{I}^+$, for example). In this case $\mathbb{PT}$ and $\mathbb{PT}^*$ are, respectively, the α-curves and β-curves in $\mathbb{C}\mathcal{H}$. Incidence between α-curves and β-curves is the condition that they <u>intersect</u>. The other notions of incidence are built up from this. The geometry described above is virtually unchanged. (Of course all this is <u>relative</u> to the choice of $\mathcal{H}$. If $\mathcal{H}$ is moved within $\mathcal{M}$, the structure changes.)

Further details are to appear in an article in honour of Andrzej Trautman's 64th birthday.

~ R.P.