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Coherent States and Fubini-Study Geometry

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In this article I derive the geometry of the submanifold of coherent states for a bosonic quantum field theory. The approach taken is to view the coherent state submanifold, which we shall denote by $\mathcal{C}$, as embedded in an infinite dimensional complex projective space $CP^{\aleph_0}$ – the quantum-mechanical state space, or projective Fock space. The metric on the ambient state space is the familiar Fubini-Study metric, and we use this to calculate the induced metric on $\mathcal{C}$. There are some technicalities associated with the fact that in physics one is dealing typically with an infinite dimensional Fock space of states, which itself is built up from an infinite dimensional single particle Hilbert space $\mathcal{H}^1$. However it is in practice reasonable to assume that the underlying single particle Hilbert space is separable, that is to say any vector can be decomposed along countably and possibly infinitely many basis states, as for example occurs in the Fourier series analysis of an oscillator with boundary conditions. In the case of a state not built up in this way one can always argue via continuity in the relevant function space.

We begin with the basic notation and definitions.

1. Fock Space and Abstract Index Notation

Let V denote the Hilbert space of real solutions to some classical linear field equation, and define the single particle Hilbert space as

$$\mathcal{H} = V \otimes C.$$

Now introduce the notation for the n-fold tensor product of $\mathcal{H}$ with itself,

$$\mathcal{H}^n \equiv \mathcal{H}^{\otimes^n}$$

where $\otimes$ denotes the symmetric tensor product for bosons and antisymmetric tensor product for fermions. Then $\mathcal{H}^n$ is said to be the n-particle Hilbert space, and we define Fock space as the Hilbert space

$$\mathcal{F} = C \oplus \mathcal{H} \oplus \mathcal{H}^2 \oplus ... \oplus \mathcal{H}^n \oplus ...$$

in which C represents the vacuum state or $\mathcal{H}^0$. Now we define $\mathcal{H}^n_\pm$ to be the positive and negative frequency n-particle Hilbert spaces respectively,

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$$

giving rise to the associated positive and negative frequency Fock spaces $\mathcal{F}_\pm$. We shall use the abstract index notation for elements of $\mathcal{H}$ so that we write

$$\xi^a \in \mathcal{H}.$$

We take $\mathcal{H}$ to have a *countably* infinite basis, so that the index on ξ can be thought of as running over the natural numbers. For a positive frequency field we shall use an unprimed Greek index so that for example $\xi^\alpha \in \mathcal{H}^1_+$.

Now a state vector in Fock space $\mathcal{F}_+$ can be written

$$|\xi\rangle = (\xi, \xi^\alpha, \xi^{\alpha\beta}, ...)$$

where $\xi^{\alpha\beta} \in \mathcal{H}^2_+$ and so on.

The evaluation of the squared Hilbert space norm of a vector $|\xi\rangle$ in $\mathcal{F}$ is according to

$$\|\xi\|^2 = \xi\bar{\xi} + \xi^\alpha\bar{\xi}_\alpha + \xi^{\alpha\beta}\bar{\xi}_{\alpha\beta} + ...$$

(for further details of this index notation and the relation to the complex structure involved in the frequency splitting see Geroch 1971). For any $\sigma^\alpha \in \mathcal{H}^1_+$ we define creation and annihilation operators $\hat{C}_\alpha$, $\hat{A}^\alpha$ respectively, according to

$$\hat{C}_\alpha \sigma^\alpha |\xi\rangle \equiv \hat{C}(\sigma)\psi = (0, \sigma^\alpha \xi, \sqrt{2}\sigma^{(\alpha}\xi^{\beta)}, \sqrt{3}\sigma^{(\alpha}\xi^{\beta\gamma)}, \ldots)$$

$$\hat{A}^\alpha \bar{\sigma}_\alpha |\xi\rangle \equiv \hat{A}(\bar{\sigma})\psi = (\xi^\mu \bar{\sigma}_\mu, \sqrt{2}\xi^{\mu\alpha}\bar{\sigma}_\mu, \sqrt{3}\xi^{\mu\alpha\beta}\bar{\sigma}_\mu, \ldots)$$

and these obey the CCR

$$[\hat{C}(\sigma), \hat{C}(\sigma')] = 0$$

$$[\hat{A}(\bar{\sigma}), \hat{A}(\bar{\sigma}')] = 0$$

$$[\hat{A}^\alpha, \hat{C}_\beta] = \delta^\alpha_\beta \Longleftrightarrow [\hat{A}(\bar{\sigma}), \hat{C}(\sigma)] = (\sigma \cdot \bar{\sigma})I.$$

The creation and annihilation operators are adjoints of each other, so that for any vector $|\phi\rangle \in \mathcal{F}$ we have

$$\langle \hat{C}(\sigma)\psi, \phi\rangle = \langle \psi, \hat{A}(\bar{\sigma})\phi\rangle.$$

2. Coherent States

Now we define the space of coherent states. There are many different characterizations of coherent states within the context of quantum field theory. To begin with we give the definition in terms of *exponentiation* of the single particle Hilbert space. We begin with a vector

$$\xi^a \in \mathcal{H}^1$$

and we wish to infer from this, uniquely, an element of state space $P\mathcal{F}$, said to be the state *coherent* to ξ^a and which we shall denote $P|\xi_c\rangle$. Here P is the quotient map by complex scalar multiples. The exponential map e is defined by

$$\xi^a \mapsto e(\xi^a) = (1, \xi^a, \xi^a\xi^b/\sqrt{2!}, \ldots, \xi^a\xi^b\ldots\xi^d/\sqrt{n!}, \ldots) =: |\xi_c\rangle \in \mathcal{F}$$

in which the term containing $\sqrt{n!}$ has n indices. Now $|\xi_c\rangle$ is said to be a coherent state *vector* and P takes this to the associated complex ray in Fock space. It is crucial to understand that

$$P \circ e(\lambda\xi^a) = P \circ e(\mu\xi^a)$$

for $\xi^a \neq 0$ holds if and *only* if $\lambda = \mu$. For the vacuum parts of both vectors $e(\lambda\xi^a)$ and $e(\mu\xi^a)$ are unity and so if these vectors are proportional then we must have $(\mu - \lambda)\xi^a$ vanishing, which implies the property claimed. Note however, that although $\lambda\xi^a$, $\mu\xi^b$ define different vectors in $\mathcal{H}^1$ for $\lambda \neq \mu$, they define the same single particle states for all non-zero values of λ, μ. This is because the single particle states are elements of $P\mathcal{H}^1$ and not $\mathcal{H}^1$. In summary we must beware of the following property,

Changing the phase or scale of a single particle state vector changes its associated coherent state.

In terms of fibre bundles, for an $(n+1)$-dimensional single particle Hilbert space with corresponding state space CP^n we consider the *universal bundle* $\mathcal{U}$

$$\Pi : \mathcal{U} \to P\mathcal{H}^1$$

over the state space, defined so that the fibre above any point or state is precisely the ray in the Hilbert space it represents. Now the map e defined above is a map from the *bundle* to Fock space

$$e : \mathcal{U} \to \mathcal{F}$$

and this is non constant along each fibre $\Pi^{-1}(s)$ for all $s \in P\mathcal{H}^1$. At first sight this may seem a little pedantic but this idea is essential to the discussion which follows.

We now examine the action of the creation and annihilation operators acting on coherent states. For $\tau^\alpha \in \mathcal{H}^1_+$ and $|\psi_c\rangle$ defined as before we have

$$\hat{A}(\bar{\tau})|\psi_c\rangle = (\xi \cdot \bar{\tau})|\psi_c\rangle$$

or equivalently

$$\hat{A}^\alpha|\psi_c\rangle = \psi^\alpha|\psi_c\rangle$$

so that *coherent states are eigenstates of the annihilation operator* . The action of the creation operator is via differentiation with respect to $\mathcal{H}^1$. For any $\sigma^\alpha \in \mathcal{H}^1_+$ we have

$$\hat{C}(\sigma)|\psi_c\rangle = (0, \sigma^\alpha, \sqrt{2}\,\sigma^{(\alpha}\psi^{\beta)}, ..., \sqrt{n}/\sqrt{(n-1)!}\,\sigma^{(\alpha}\psi^\beta ... \psi^{\sigma)}, ...)$$

where n factors appear in the general term, or equivalently

$$\hat{C}_\alpha|\psi_c\rangle = \frac{d|\psi_c\rangle}{d\psi^\alpha}.$$

For convenience we shall set

$$\Lambda := \xi^\alpha \bar{\xi}_\alpha \equiv \|\xi\|^2.$$

Then we have

$$\langle\psi_c|\psi_c\rangle = e^\Lambda$$

and we must divide by this factor to calculate the expected value of any operator from its matrix element with a coherent state vector.

Another important fact about coherent states is that they provide a *resolution of unity*

$$\int_{t\in\mathcal{A}} p_t|c_t\rangle\langle c_t| = \mathbf{1} \quad (*)$$

for some index set $\mathcal{A}$, and that the above equation determines p_t *uniquely* (see for example Klauder and Skagerstam 1985). This uniqueness property is in spite of the fact that the coherent states are not mutually orthogonal, and are indeed in a sense over complete. In fact, in our Hilbert space notation we have

$$\langle\xi_c|\psi_c\rangle = e^{\psi\cdot\bar{\xi}}$$

and clearly this can never equal zero. The unique resolution of unity implies that we can expand any state vector as a superposition of coherent states in a unique way. This implies the following.

Lemma 1. *The submanifold of coherent states is non-linear inside state space $P\mathcal{F}$. That is to say, given any two distinct coherent state vectors $|\xi_1\rangle$, $|\xi_2\rangle$, the superposition*

$$\lambda|\xi_1\rangle + \mu|\xi_2\rangle$$

is a coherent state vector if and only if exactly one of λ or μ vanishes.

Proof. This follows immediately from the uniqueness of decomposition of any state vector into coherent states.

3. The Fubini-Study Geometry

The projective form of the Fubini Study metric on CP^n which one most usually encounters in the context of quantum theory (see for example Hughston 1996) is given by

$$ds^2 = 8k^{-1}\frac{Z^{[\alpha}dZ^{\beta]}\bar{Z}_{[\alpha}d\bar{Z}_{\beta]}}{(Z^\gamma\bar{Z}_\gamma)^2}$$

where Z^α are homogeneous coordinates for CP^n with $\alpha = 0, 1, ..., n$, and k is the holomorphic sectional curvature, which in subsequent calculations we shall take to equal one. For our purposes in the application to coherent states it will be useful also to have this metric expressed in *non*-homogeneous coordinates. This is because if we set Z^0 to be the coordinate of the vacuum part of a coherent state then, as follows from above this coordinate is necessarily non-vanishing. Thus the coherent states form a submanifold of the *affine* part of the projective Fock space, the latter consisting of all elements of the form

$$\{(1, \xi^a, \xi^{ab}, ...)\} =: \mathcal{A} \supset \mathcal{C} \cong C^\infty$$

and whose *compactification* is

$$\{P(0, \psi^a, \psi^{ab}, ...)\} =: \mathcal{B} \cong CP^\infty,$$

that is to say states for which the probability of no quanta being present is zero. Note here that the image of any state vector $|\psi\rangle$ under the creation operator $\hat{C}(\sigma)$ always lies within the compactification,

$$\hat{C}(\sigma)|\psi\rangle \in \mathcal{B}_+ \quad \forall\, |\psi\rangle \in \mathcal{F}_+,\ \sigma^a \in \mathcal{H}_+.$$

From now on we deal with coherent states viewed as forming a submanifold of $\mathcal{A}$ and we shall make use of the following standard result.

Lemma 2. *The equivalent non-projective form of the Fubini Study metric on CP^n is given by*

$$ds^2 = 4\frac{(1+\zeta^\alpha\bar{\zeta}_\alpha)d\zeta^\alpha d\bar{\zeta}_\alpha - (\zeta^\alpha d\bar{\zeta}_\alpha)(\bar{\zeta}_\alpha d\zeta^\alpha)}{(1+\zeta^\alpha\bar{\zeta}_\alpha)^2}$$

where $\zeta^\alpha = Z^\alpha/Z^0$ for $\alpha = 1, 2, ..., n$.

We shall also require the following standard

Definition 1. *A complex manifold M is said to be Kähler if it comes equipped with an Hermitian metric $h_{\alpha\beta'}$ with $ds^2 = h_{\alpha\beta'}dz^\alpha \otimes d\bar{z}^{\beta'}$ such that the associated real 2-form*

$$\Omega = ih_{\alpha\beta'}dz^\alpha \wedge d\bar{z}^{\beta'}$$

is closed. Then Ω is said to be a Kähler form for M.

This is equivalent to the existence on M of a Kähler scalar function K which is real valued, such that

$$\Omega = i\partial\bar{\partial}K \longleftrightarrow h_{\alpha\beta'} = \partial_\alpha\bar{\partial}_{\beta'}K \quad (*).$$

Thus we have the consequence that a complex submanifold N of a Kähler manifold M is itself Kähler, since one simply restricts the function K down to N to find the induced Kähler metric according to $(*)$. In the case of the Fubini-Study metric on CP^n the Kähler scalar function takes the form

$$K = 4k^{-1}\log[1 + k(|\zeta^1|^2 + |\zeta^2|^2 + ... + |\zeta^n|^2)]$$

where ζ^α are inhomogeneous coordinates on CP^n, that is $\zeta^\alpha = Z^\alpha/Z^0$ as above.

The following result also will be of interest with regard to the theorem to follow.

Lemma 3. *Let Ω be a positive $(1,1)$-form on a complex manifold M. Then Ω is a Kähler form for M if and only if for all $x_0 \in M$ there exist holomorphic 'Euclidean' coordinates $z^1, ...z^n$ around x_0 such that*

$$\Omega = ih_{\alpha\beta'}dz^\alpha \wedge d\bar{z}^{\beta'}$$

$$h_{\alpha\beta'} = \delta_{\alpha\beta'} + O(|z|^2) \quad \text{at } x_0,$$

and thus the Kähler metric osculates to the flat Euclidean metric to second order.

Proof. The implication towards Ω being a Kähler form is clear. In order to prove the reverse implication, begin with holomorphic coordinates $z^1, ..., z^n$ such that $dz^1, ..., dz^n$ give an orthonormal basis of T^*_{M,x_0}, the dual tangent space to M at x_0. Then this implies that

$$\Omega = i\tilde{h}_{\alpha\beta'} dz^\alpha \wedge d\bar{z}^{\beta'}$$

where

$$\begin{aligned}\tilde{h}_{\alpha\beta'} &= \delta_{\alpha\beta'} + O(|z|) \\ &= \delta_{\alpha\beta'} + \sum_{1\le\gamma\le n} (a_{\gamma\alpha\beta'} z^\gamma + a'_{\gamma'\alpha\beta'} \bar{z}^{\gamma'}) + O(|z|^2).\end{aligned}$$

That Ω is real implies

$$a'_{\gamma'\alpha\beta'} = \bar{a}_{\gamma'\beta'\alpha}.$$

Furthermore the Kähler condition

$$\frac{\partial h_{\alpha\beta'}}{\partial z^\gamma}|_{x_0} = \frac{\partial h_{\gamma\beta'}}{\partial z^\alpha}|_{x_0}$$

implies that

$$a_{\alpha\gamma\beta'} = a_{(\alpha\gamma)\beta'}.$$

Now we define our holomorphic Euclidean coordinates $\hat{z}^\alpha$ to be

$$\hat{z}^\beta := z^\beta + \frac{1}{2} \sum_{\gamma,\alpha} a_{\gamma\alpha\beta'} \delta^{\beta\beta'} z^\gamma z^\alpha$$

which completes the proof of the lemma. qed

4. Geometry of Coherent States

We are now ready to state our main result. This is a global geometrical property which, from Lemma 3, is a special case of a flatness property which applies locally to any Kähler manifold.

Theorem 1. *The metric induced on the coherent state submanifold from the ambient Fubini-Study metric on the quantum mechanical state space is intrinsically flat. Moreover the coordinates ξ^a on the single particle Hilbert space $\mathcal{H}^1$ are complex Euclidean coordinates for the coherent state submanifold.*

Suppose instead we take the point of view that we begin with the coherent state submanifold and decide a priori to place on it the complex Euclidean metric, giving us $\mathcal{C}_E$ say. Then we have the following equivalent result.

Theorem 2. *The Euclidean coherent state submanifold has an isometric embedding into the Fubini-Study state manifold*

$$\mathcal{C}_E \overset{i}{\hookrightarrow} \mathcal{F}_{F.S.},$$

where i is the inclusion map, and is in this case an isometry.

We note that the theorems is independent of the details of the single particle Hilbert space $\mathcal{H}^1$. We give three proofs of this result, each of which uses a different technique.

Coordinate Proof. From the way that we defined coherent state vectors, we may regard $\xi^a \in \mathcal{H}^1$ as complex coordinate functions for the coherent state submanifold. It will be helpful to introduce some further notation. We define

$$\xi^{(n)} := \frac{\xi^{(\alpha}\xi^\beta \ldots \xi^{\delta)}}{\sqrt{n!}}$$

so that $\xi^{(n)}$ is the tensor contribution to the coherent state vector $|\xi_c\rangle$ which lies in $\mathcal{H}^n$, there being n factors in the symmetrized tensor product above. Similarly we define $\bar{\xi}_{(n)}$. Then setting $\Lambda = \xi^\alpha \bar{\xi}_\alpha$ we calculate

$$\xi^{(n)}\bar{\xi}_{(n)} = \frac{1}{n!}\Lambda^n.$$

Now *restricted* down to the coherent state submanifold $\mathcal{C}$ we can calculate the tangent vector to a coherent state induced by an element $d\xi$ of $T^*\mathcal{H}$. The component in $\mathcal{H}^n$ of the (dual) tangent vector is given by

$$d\xi^{(n)} = \frac{n}{\sqrt{n!}} d\xi^{(\alpha}\xi^{\beta}\xi^{\gamma}...\xi^{\delta)}$$

and similarly for the complex conjugate. Now to insert into the Fubini-Study line element we shall need to find the *coordinate* inner product of a tangent vector with itself. Note that the contribution to this of any pair of vectors lying in distinct $\mathcal{H}^n$ vanishes, as follows from our expression for the Hilbert space norm given earlier. The essential point here is that when evaluating the inner product of two Fock space vectors in the abstract index notation, one contracts over vectors and their conjugates with the same number of indices. Thus we have

$$d\xi^{(n)} d\bar{\xi}_{(m)} = \delta^{(n)}_{(m)} \cdot \frac{1}{(n-1)!}[\Lambda^{n-1} d\xi^\alpha d\bar{\xi}_\alpha + (n-1)\Lambda^{n-2}|\xi^\alpha d\bar{\xi}_\alpha|^2]$$

for all $n \geq 1$. We shall also need the coordinate inner product

$$\bar{\xi}_{(m)} d\xi^{(n)} = \delta^{(n)}_{(m)} \cdot \frac{\bar{\xi}_{(\alpha}\bar{\xi}_{\beta}...\bar{\xi}_{\delta)}}{\sqrt{n!}} \cdot \frac{n}{\sqrt{n!}} d\xi^{(\alpha}\xi^{\beta}\xi^{\gamma}...\xi^{\delta)}$$

$$= \delta^{(n)}_{(m)} \cdot \frac{1}{(n-1)!}(\bar{\xi}_\alpha d\xi^\alpha)\Lambda^{n-1}$$

and similarly

$$\xi^{(n)} d\bar{\xi}_{(m)} = \delta^{(n)}_{(m)} \cdot \frac{1}{(n-1)!}(\xi^\alpha d\bar{\xi}_\alpha)\Lambda^{n-1}$$

for all $n \geq 1$. Now we refer back to our expression for the Fubini-Study line element given in non-homogeneous coordinates derived above. In this expression a vector ξ lies in the Fock space and so is given by the collection $\{\xi^{(n)}\}$ for *all* values of n. Thus to evaluate the line element induced on the coherent state submanifold $\mathcal{C}$ we must sum over all $1 \leq m, n \leq \infty$ in the above identities. In doing this we obtain

$$\sum_{m,n} d\xi^{(n)} d\bar{\xi}_{(m)} = e^\Lambda (d\xi^\alpha d\bar{\xi}_\alpha + |d\xi^\alpha \bar{\xi}_\alpha|^2)$$

and

$$\sum_{m,n} \bar{\xi}_{(m)} d\xi^{(n)} = e^\Lambda \cdot \bar{\xi}_\alpha d\xi^\alpha$$

together with its complex conjugate. Also for the denominator in the Fubini-Study line element we have simply

$$1 + \sum_{m,n} \xi^{(n)}\bar{\xi}_{(m)} = e^\Lambda.$$

Hence our induced line element becomes

$$ds^2 = \frac{4}{e^{2\Lambda}} \cdot [e^{2\Lambda}(d\xi^\alpha d\bar{\xi}_\alpha + |d\xi^\alpha \bar{\xi}_\alpha|^2) - e^{2\Lambda}|\xi^\alpha d\bar{\xi}_\alpha|^2]$$

which is simply

$$ds^2 = 4 d\xi^\alpha d\bar{\xi}_\alpha.$$

This completes the coordinate proof. ∎

Algebraic Proof. This is somewhat more general than the coordinate proof above in that it is valid for any quantum field theory including the possible presence of interactions. For we shall assume only the CCR for the creation and annihilation operators, together with the fact that the coherent states are eigenstates of the annihilation operator, for this defines the coherent states uniquely (Klauder and Skagerstam 1985). For a complicated Lagrangian including interactions the precise details of A^α and C_β of course will change but the fundamental algebraic relation (CCR) between them and the defining properties of the coherent states are always those given above.

We shall adopt the Dirac notation for state vectors according to

$$Z^\alpha \leftrightarrow |\psi\rangle, \quad \bar{Z}_\alpha \leftrightarrow \langle\psi|.$$

In this notation the Fubini-Study line element becomes

$$ds^2_{F.S.} = 4\{\frac{\langle d\psi|d\psi\rangle}{\langle\psi|\psi\rangle} - \frac{\langle\psi|d\psi\rangle\langle d\psi|\psi\rangle}{\langle\psi|\psi\rangle^2}\}.$$

Now we abbreviate so that $|\psi\rangle \in \mathcal{F}$ denotes the state vector coherent to $\psi^\alpha \in \mathcal{H}^1$. Then recall that

$$A^\alpha|\psi\rangle = \psi^\alpha|\psi\rangle \longleftrightarrow \langle\psi|\bar{\psi}_\alpha = \langle\psi|C_\alpha$$

together with

$$C_\alpha|\psi\rangle = \frac{|d\psi\rangle}{d\psi^\alpha} \longleftrightarrow \langle\psi|A^\alpha = \frac{\langle d\psi|}{d\bar{\psi}_\alpha}.$$

It follows that

$$|d\psi\rangle = C_\alpha|\psi\rangle d\phi^\alpha$$

and correspondingly

$$\langle d\psi| = d\bar{\phi}_\beta\langle\psi|A^\beta.$$

Now using the CCR

$$[A^\alpha, C_\beta] = \delta^\alpha_\beta$$

we calculate

$$\begin{aligned}\langle d\psi|d\psi\rangle &= d\bar{\phi}_\beta d\phi^\alpha\langle\psi|A^\beta C_\alpha|\psi\rangle \\ &= d\bar{\phi}_\beta d\phi^\alpha\langle\psi|[A^\beta, C_\alpha] + C_\alpha A^\beta|\psi\rangle \\ &= [d\phi^\alpha d\bar{\phi}_\alpha + (\phi^\beta d\bar{\phi}_\beta)(\bar{\phi}_\alpha d\phi^\alpha)]\langle\psi|\psi\rangle.\end{aligned}$$

Similarly we calculate

$$\begin{aligned}\langle\psi|d\psi\rangle &= \langle\psi|C_\alpha|\psi\rangle d\phi^\alpha = \langle\psi|\bar{\phi}_\alpha|\psi\rangle d\phi^\alpha \\ &= \langle\psi|\psi\rangle(\bar{\phi}_\alpha d\phi^\alpha).\end{aligned}$$

Therefore the line element induced on $\mathcal{C}$ reduces simply to

$$ds^2_{F.S.} = 4(d\phi^\alpha d\bar{\phi}_\alpha)^2$$

as required. ∎

Kählerian Proof. We recall the Kähler scalar function for CP^n where now we take n to be countable infinity, $\aleph_0$. For $\mathcal{A} \subset \mathcal{F}$ defined in section 3 we have

$$K = 4\log(1 + |\xi^{(1)}|^2 + \ldots + |\xi^{(j)}|^2 + \ldots \text{ ad inf.})$$

where $\xi^{(j)} = \xi^{\alpha_1\ldots\alpha_j}$ as before. Now for $\mathcal{C}$ we have for the coherent state vector associated to $\psi^\alpha \in \mathcal{H}^1$

$$\psi^{(j)} = \frac{(\psi^\alpha)^{\otimes^j_S}}{\sqrt{j!}}$$

and therefore

$$|\psi^{(j)}|^2 = \frac{\Lambda^j}{j!}$$

where as before $\Lambda = \psi^\alpha \bar{\psi}_\alpha$. Summing over j all the way to infinity (to sum to infinity is in fact necessary for flatness) we obtain the remarkably simple relation

$$K|_C = 4\Lambda$$

and then the induced metric on C is given by

$$h_{\alpha\beta'} = 4\partial_\alpha \bar{\partial}_{\beta'} (\psi^\gamma \bar{\psi}_\gamma) = 4\delta_{\alpha\beta'}$$

as required. ∎

We have remarked earlier that the coherent state submanifold C is non-linear, in the sense that the complex projective line joining two distinct coherent states lies entirely in the complement of C except at its two intersection points which are the coherent states themselves. This is an algebraic result whose proof relies upon the uniqueness of decomposition of any given state into coherent states. It suggests the following geometrical property of the coherent state submanifold.

Proposition 1. *Given any two distinct coherent states the complex projective line joining them intersects C transversally, that is to say, the line joining the two coherent states does not lie in the tangent space to C at either intersection point.*

Proof. We make essential use of the main theorem. Since C is homogeneous we may assume that one of the coherent states is the vacuum state, that is $P|0\rangle$ where $|0\rangle$ is the element of Fock space which is the exponential of the origin in the vector space $\mathcal{H}^1$. Then from the main theorem the *intrinsic* geodesic distance s from $P|0\rangle$ to $P|\xi_c\rangle$, $\xi \neq 0$ is given by

$$s = 2\Lambda^{1/2}.$$

Now recall (cf. Hughston 1996) that the geodesic distance θ between the two states in $P\mathcal{F}$ with respect to the ambient Fubini-Study metric on $P\mathcal{F}$ is determined by

$$\frac{1}{2}(1 + \cos\theta) = \frac{\langle\xi_c|0\rangle\langle 0|\xi_c\rangle}{\langle 0|0\rangle\langle\xi_c|\xi_c\rangle}$$

where we take θ to be the principal value determined from the above equation,

$$0 \leq \theta \leq \pi$$

so that the cross ratio expression above fixes θ uniquely. Now clearly

$$\langle 0|0\rangle = \langle 0|\xi_c\rangle = \langle\xi_c|0\rangle = 1$$

and therefore

$$\theta = \cos^{-1}(2e^{-\Lambda} - 1), \quad 0 \leq \theta \leq \pi$$

from which it follows that

$$\frac{d\theta}{d\Lambda} = (e^\Lambda - 1)^{-1/2}.$$

Hence we obtain

$$\frac{d\theta}{ds} = \left(\frac{\Lambda}{e^\Lambda - 1}\right)^{1/2}.$$

Thus clearly $d\theta/ds$ is a monotone decreasing function beginning at $d\theta/ds = 1$, where $\Lambda = 0$, and decaying to zero as Λ tends to infinity. Now tangency at $P|\xi_c\rangle$ would require $d\theta/ds = 1$ for some $\Lambda \neq 0$ and this is clearly not possible from the form of the function $d\theta/ds$. This completes our proof.

The method above also proves another result which one expects intuitively from the non-linear picture of C.

Corollary. *The geodesic distance along the projective line in $P\mathcal{F}$ joining two distinct coherent states is always strictly greater than the intrinsic geodesic distance within $\mathcal{C}$.*

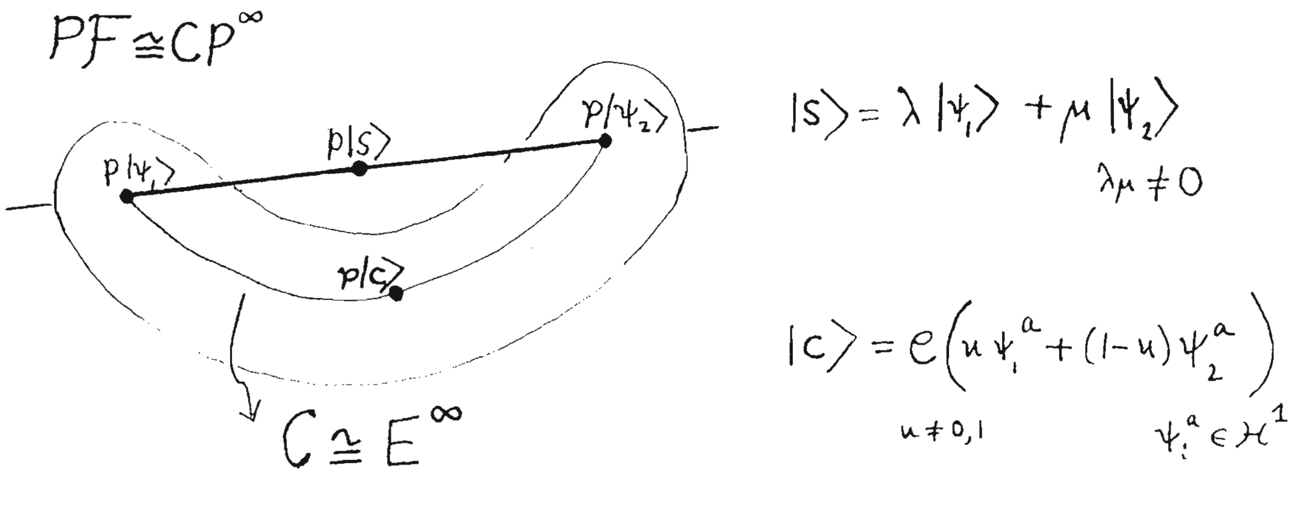
The theorem also proves the following simple geometrical properties of $\mathcal{C}$.

Lemma. *The intrinsic $\mathcal{C}$ geodesic distance between two coherent states is given by the Hilbert space norm of the difference field of the two corresponding vectors in $\mathcal{H}^1$. Thus the $\mathcal{C}$ distance of a coherent state from the vacuum state is equal to its Hilbert space norm.*

Lemma. *The overlap $\langle \xi_c | \psi_c \rangle$ of two normalized coherent state vectors is equal to the cosine of the angle these states subtend at the vacuum state.*

Discussion

These results illustrate the geometrical character of two emergent linear structures in quantum theory. One is able to simply add solutions of a classical linear field equation to obtain a new classical field, and as we have seen the geometry of the associated coherent states is Euclidean, with the vector solutions in fact serving as Euclidean coordinates. On the other hand one can take any two distinct coherent states and superpose them in the quantum mechanical sense of joining them with the unique complex projective line in the ambient Fubini-Study geometry of the underlying state space. Then we have seen that this superposition is necessarily non-coherent, or in familiar language non-classical. Such features as these are notably fully present even in a linear theory of gravity, and this is in accordance with common experience, since one does not observe classical superpositions of weak gravitational fields. Thus a preferred basis of states is given, namely the coherent states, together with a unique probability distribution for the associated resolution of unity, and this in turn gives rise to unique coherence transition probabilities. The geometry clearly illustrates how a quantum superposition of distinct classical geometries is outside the classical domain.



An interesting application of this geometry to a problem in stochastic state vector reduction is currently being pursued in collaboration with L. P. Hughston, to whom I express thanks for valuable discussions in respect of the work presented here.

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