Quantum Electrodynamic Birdtracks
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I. Introduction

This paper provides a diagrammatic version of the Feynman-Dyson derivation of non-commutative electromagnetism from quantum mechanical formalism [1], [2]. Our spin-network formalism lays bare the structure of this result.

Here is a statement of the main result in conventional notation. $X = (X_1, X_2, X_3)$ denotes a vector of non-commutative coordinates, each a differentiable function of time $t$. Let $\dot{X} = (\dot{X}_1, \dot{X}_2, \dot{X}_3)$ denote the vector of time derivatives of these functions. Let $\kappa$ be a non-zero scalar. Assume the axioms below:

\[
\begin{aligned}
&(1) \quad [X_i, X_j] = 0 \quad \text{for all } i, j. \\
&(2) \quad [X_i, \dot{X}_j] = \kappa \delta_{ij}
\end{aligned}
\]

Here $[X, Y] = XY - YX$ and

$\delta_{ij} = \begin{cases} 
0 & \text{if } i \neq j \\
1 & \text{if } i = j
\end{cases}$

Then there exist fields $E$ and $H$ such that
(a) \( \ddot{X} = E + \dot{X} \times H \)

(b) \( \nabla \cdot H = 0 \)

(c) \( \frac{\partial H}{\partial t} + \nabla \times E = 0. \)

In fact, we can take \( H = \frac{1}{\kappa} \dot{X} \times \dot{X} \) where this denotes the non-commutative vector cross product.

In this paper we give a new proof of this result. These methods apply equally well to the discrete framework employed in [2]. Section II reviews notation and defines the non-commutative vector calculus via abstract tensor diagrams. Section III contains the promised derivation. Section IV discusses problems and questions arising from this work.

II. Vectors, Abstract Tensors, and The Epsilon

A vector \( A = (A_1, A_2, A_3) \) will be indicated by \( \vec{A} \) or \( \circ A \) where the arc denotes the index. Multi-indexed objects have multiple arcs. Thus \( \vec{i} \vec{j} i = \delta \vec{i} \vec{j} \) and \( \dot{i} = \delta \dot{i} \). Compare [3].

We sum over repeated indices. Such indices correspond to arcs without free ends. For example, \( 0 = \sum_{i=1}^{3} \vec{i} = \sum_{i=1}^{3} \dot{i} = 3. \)
\[ A \cdot B = \sum_{i=1}^{3} A_i B_i = A \times B. \]

Let \( Y^i_k = \varepsilon_{i j k} = \begin{cases} \text{sgn}(i j k) & \text{if } i j k \text{ distinct} \\ 0 & \text{otherwise} \end{cases} \)

Then \( A \times B = A \times B \) since \( (A \times B)_k = \sum_{i,j} \varepsilon_{i j k} A_i B_j. \)

Note that in a non-commutative context,
\[
(A \times A)_k = \sum_{i,j} \varepsilon_{i j k} A_i A_j
\]
\[
(A \times A)_1 = [A_2, A_3]
\]
\[
(A \times A)_2 = [A_3, A_1]
\]
\[
(A \times A)_3 = [A_1, A_2].
\]

Thus, we can no longer assert \( A \times A = 0 \), unless the coordinates of \( A \) commute with one another.

We shall use the axioms stated in the introduction. Thus \( X \times X = 0 \) is equivalent to the first axiom. The second axiom states that \( [X_i, X_j] = \lambda j i \). Thus \( \frac{1}{\lambda} [X_i, X_j] = \partial X_i / \partial X_j \). This means that if \( F \) is any function of these non-commuting variables, we can define

\[
\frac{\partial F}{\partial X_i} = \frac{1}{\lambda} [F, X_i].
\]
Thus \[ \nabla \cdot A = \sum_i \frac{\partial A_i}{\partial x_i} = \frac{1}{\kappa} \left[ A_j \cdot \nabla_j \right] \]
and
\[ \nabla \times F = -\frac{1}{\kappa} F \cdot \nabla \]

With these definitions, we can proceed to work out the details on non-commutative vector calculus. The key to calculations is the basic \textit{epsilon} identity:

\[ \sum_i \epsilon_{abi} \epsilon_{cde} = -\delta_d^a \delta_c^b + \delta_c^a \delta_d^b \]

\[ \iff \]

\[ \chi = -\big( + \chi \big) \]

For example,
\[ \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{A} \mathbf{B} \mathbf{C} = -\mathbf{A} \mathbf{B} \mathbf{C} + \mathbf{A} \mathbf{B} \mathbf{C} \]

In the commutative case, this reduces directly to:
\[ \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = -(\mathbf{A} \cdot \mathbf{B}) \mathbf{C} + (\mathbf{A} \cdot \mathbf{C}) \mathbf{B} \]

In the non-commutative case we are left with
\[ \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = -(\mathbf{A} \times \mathbf{B}) \mathbf{C} + \mathbf{A} \mathbf{B} \mathbf{C} \]

The network formalism articulates the non-commutative vector analysis.
III. Non-Commutative Electromagnetism

As explained in sections I and II, we assume coordinates \( \bar{x} \) such that

(1) \[ [\bar{x}, \bar{x}] = 0 \]

(2) \[ [\bar{x}, \dot{\bar{x}}] = \kappa \bar{x} \]

In section II, we showed that (1) \( \Leftrightarrow \bar{x} \times \bar{x} = 0 \), and defined differentiation so that (2) implied

\[ \frac{\partial}{\partial \bar{x}_2} = [\bar{x}, \dot{\bar{x}}]/\kappa. \]

We now define the fields \( H \) and \( E \) by the formulas

\[
\begin{align*}
\{ & H = \frac{1}{\kappa} \dot{\bar{x}} \times \ddot{\bar{x}} = \frac{1}{\kappa} \dot{\bar{x}} \times \bar{x}, \\
& E = \ddot{\bar{x}} - \bar{x} \times H \}
\end{align*}
\]

Our first task is to show that neither \( E \) nor \( H \) have any dependence on \( \bar{x} \). Since (2) (above) holds, this is equivalent to showing that \( [E, \bar{x}] = 0 = [H, \bar{x}] \).

**Lemma 1.** \( \bar{x} \times H = [\bar{x}, \ddot{\bar{x}}] \)

**Proof.**

\[
\begin{align*}
H &= \frac{1}{\kappa} \dot{\bar{x}} \times \ddot{\bar{x}} = -\frac{1}{\kappa} \dot{\bar{x}} \times \dot{\bar{x}} + \frac{1}{\kappa} \bar{x} \times \bar{x} \\
&= -\frac{1}{\kappa} [\dot{\bar{x}}, \ddot{\bar{x}}] \\
&= \frac{1}{\kappa} \left[ \bar{x}, \dot{\bar{x}} \right] \quad ([\bar{x}, \dot{\bar{x}}] = 0)
\end{align*}
\]
Lemma. \([E, X] = [H, X] = 0\).

Proof. \([H, X] = \frac{1}{\kappa} \ddot{X} X X - \frac{1}{\kappa} \ddot{X} X X \]
\[\quad = \frac{1}{\kappa} \ddot{X} [\dot{X}, X] - \frac{1}{\kappa} [\dot{X}, \dot{X}] X \]
\[\quad = -\ddot{X} Y - \dddot{X} = 0 / / \]

\([X, E] = [X, \dot{X}] - \dot{X} \dot{X} H + \dot{X} H \dot{X} \]
\[\quad = k H - [\dot{X}, \dot{X}] H + \dot{X} [H, X] \]
\[\quad = k H - k Y^H + 0 \]
\[\quad = 0 / / \]

Remark. Lemma \(\text{II}\) implies that \(E\) and \(H\) are functions only of \(X_1, X_2, X_3\). Hence \(E \times E = H \times H = 0\), since \(X \times X \times X = 0\).

If \(F\) is a function only of \(X_1, X_2, X_3\), then
\[
\ddot{F} = \frac{\partial F}{\partial X} + \sum_{i} x_i \frac{\partial F}{\partial x_i} = \frac{\partial F}{\partial X} + \frac{1}{\kappa} \dddot{X} [F, X].
\]

Thus
\[\partial_t H = \dot{H} + \frac{1}{\kappa} \dddot{X} [\dot{X}, \dot{X}],\]
Lemma 3. $\nabla \cdot H = 0$.

Proof. $\nabla \cdot H = \frac{1}{k} \left[ H, \dot{x} \right]$

$= \frac{1}{k} \dot{x} H - \frac{1}{k} \dot{H} \dot{x}$

$= \frac{1}{k^2} \ddot{x} \ddot{x} x - \frac{1}{k^2} \ddot{x} \ddot{x} \dot{x}$

$= 0$ //

Lemma 4. $\partial_t H + \nabla \times E = 0$.

Proof. $\partial_t H = \dot{H} + \frac{1}{k} \dot{x} \left[ \dot{x}, H \right]$

$\dot{H} = \frac{1}{ak} \left[ \dot{x}, \dot{x} \right] = \frac{1}{k} \left[ \ddot{x}, \dot{x} \right] = \frac{1}{k} \left[ E, \dot{x} \right] + \frac{1}{k} \left[ x \times H, \dot{x} \right]$

$= -\nabla \times E + \frac{1}{k} \left[ \dot{x}, H, \dot{x} \right]$

$= -\nabla \times E - \frac{1}{k} \left[ \dot{x}, H, \dot{x} \right] + \frac{1}{k} \left[ \dot{x}, H, \dot{x} \right]$

$= -\nabla \times E - \frac{1}{k} \dot{x} H \dot{x} + \frac{1}{k} \ddot{x} H \ddot{x} + \frac{1}{k} \dddot{x} H \dddot{x} - \frac{1}{k} \dddot{x} \dddot{x} H$

$\dot{H} = -\nabla \times E + \frac{1}{k} \dot{x} \left[ H, \dot{x} \right] - \frac{1}{k} \dot{x} \left[ H, \dot{x} \right] + \frac{1}{k} \left[ \dddot{x}, \dot{x} \right] H$

The third term vanishes because $\nabla \cdot H = 0$. 
Hence \( \partial_x H + \nabla x E = \frac{1}{\kappa} \left[ \dot{x}, \dot{x} \right] H \)

\[
= \frac{1}{\kappa} \left\{ \dot{x} \dot{x} H - \dot{x} \dot{x} H \right\} 
\]

\[
= \frac{1}{\kappa} \dot{x} \dot{x} H
= H \dot{H} = H \times H = 0
\]

### IV. Discussion

In this entire derivation the only appearance of non-algebraic calculus is in the time derivative \( \dot{x} \). This can also be made algebraic by choosing a formal discrete time-evolution operator \( J \) so that \( x' = Jx J \) denotes the next (discrete) time value of \( x \). Then we can take \( \dot{x} = J(x' - x) = \left[ x, J \right] \) and proceed as before. This is a capsule summary of the approach taken by the author and Pierre Noyes in [2].

### References

