Quantum Electrodynamic Birdtracks

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I. Introduction

This paper provides a diagrammatic version of the Feynman – Dyson derivation of non-commutative electromagnetism from quantum mechanical formalism [1], [2]. Our spin-network formalism lays bare the structure of this result.

Here is a statement of the main result in conventional notation. $X = (X_1, X_2, X_3)$ denotes a vector of non-commutative coordinates, each a differentiable function of time t. Let $\dot{X} = (\dot{X}_1, \dot{X}_2, \dot{X}_3)$ denote the vector of time derivatives of these functions. Let κ be a non-zero scalar. Assume the axioms below:

$$\left\{ \begin{array}{ll} (1) & [X_i, X_j] = 0 \text{ for all } i, j. \\ (2) & [X_i, \dot{X}_j] = \kappa \delta_{ij} \end{array} \right\}$$

Here $[X, Y] = XY - YX$ and

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$$

Then there exist fields E and H such that

(a) $\ddot{X} = E + \dot{X} \times H$

(b) $\nabla \cdot \mathbf{H} = 0$

(c) $\frac{\partial H}{\partial t} + \nabla \times E = 0.$

In fact, we can take $H = \frac{1}{\kappa} \dot{X} \times \dot{X}$ where this denotes the non-commutative vector cross product.

In this paper we give a new proof of this result. These methods apply equally well to the discrete framework employed in [2]. Section II reviews notation and defines the non-commutative vector calculus via abstract tensor diagrams. Section III contains the promised derivation. Section IV discusses problems and questions arising from this work.

II. Vectors, Abstract Tensors and The Epsilon

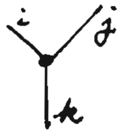
A vector $A = (A_1, A_2, A_3)$ will be indicated by A or A where the arc denotes the index. Multi-indexed objects have multiple arcs. Thus ${}_i\frown_j = \delta_{ij}$ and ${}^{i}/_{j} = \delta^{i}_{j}$. Compare [3].

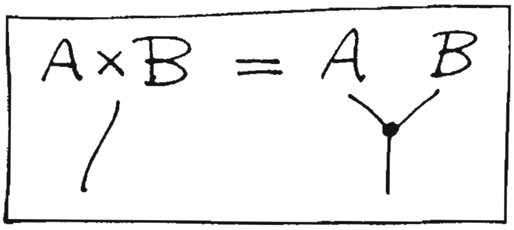
We sum over repeated indices.

Such indices correspond to arcs without free ends.

For example, $\bigcirc = \sum_{i=1}^{3} {}_i\frown_i = \sum_{i=1}^{3} \delta_{ii} = 3.$

$$A \cdot B = \sum_{i=1}^{3} A_i B_i = $$ 

Let  $= \varepsilon_{ijk} = \begin{cases} \mathrm{sgn}(ijk) & \text{if } ijk \text{ distinct} \\ 0 & \text{otherwise} \end{cases}$.

Then  since

$$(A \times B)_k = \sum_{i,j} \varepsilon_{ijk} A_i B_j .$$

Note that in a non-commutative context

$$(A \times A)_k = \sum_{ij} \varepsilon_{ijk} A_i A_j$$

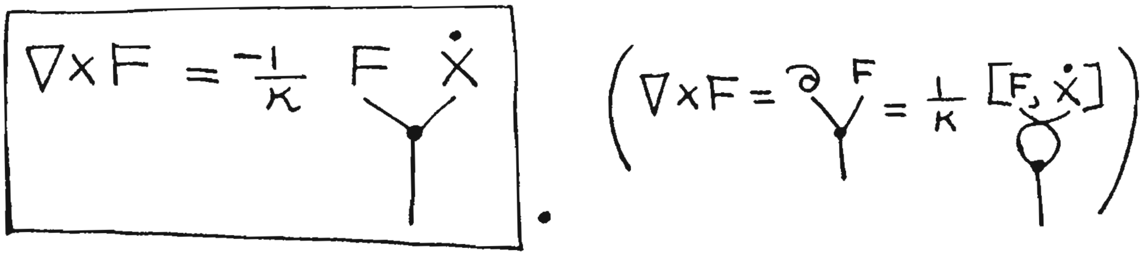
$$(A \times A)_1 = [A_2, A_3]$$
$$(A \times A)_2 = [A_3, A_1]$$
$$(A \times A)_3 = [A_1, A_2].$$

Thus, we can no longer assert $A \times A = 0$, *unless* the coordinates of A commute with one another.

We shall use the axioms stated in the introduction. Thus $X \times X = 0$ is <u>equivalent</u> to the first axiom. The second axiom states that $[X_i, \dot{X}_j] = \kappa \delta_{ij}$. Thus $\frac{1}{\kappa}[X_i, \dot{X}_j] = \partial X_i / \partial X_j$. This means that if F is any function of these non-commuting variables, we can define

$$\boxed{\frac{\partial F}{\partial X_i} = \frac{1}{\kappa}[F, \dot{X}_i]}.$$

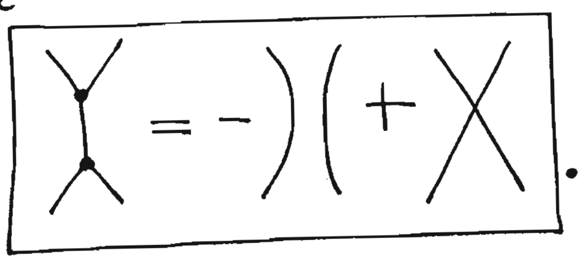
Thus $\boxed{\nabla\cdot A = \sum_i \frac{\partial A_i}{\partial X_i} = \frac{1}{\kappa}[A,\dot{X}]}$ and



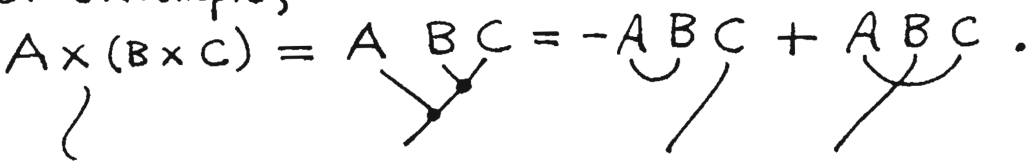
With these definitions, we can proceed to work out the details on non-commutative vector calculus. The key to calculations is the basic epsilon identity:

$$\sum_i \varepsilon_{abi}\,\varepsilon_{icd} = -\delta^a_d\,\delta^b_c + \delta^a_c\,\delta^b_d$$

$\Leftrightarrow$



For example,



In the commutative case, this reduces directly to: $A\times(B\times C) = -(A\cdot B)C + (A\cdot C)B$.

In the non-commutative case we are left with $\boxed{A\times(B\times C) = -(A\cdot B)C + A\,B\,C}$.

The network formalism articulates the non-commutative vector analysis.

III. Non-Commutative Electromagnetism

As explained in sections I and II, we assume coordinates X such that

$$(1)\quad [X, X] = 0$$

$$(2)\quad [X, \dot{X}] = \kappa \frown .$$

In section II we showed that (1) $\Leftrightarrow X \times X = 0$, and defined differentiation so that (2) implied $\frac{\partial}{\partial X_i} = [\ \ , \dot{X}_i]/\kappa$. We now define the fields H and E by the formulas

$$\left\{ \begin{aligned} H &= \frac{1}{\kappa} \dot{X}\ \dot{X} = \frac{1}{\kappa} \dot{X} \times \dot{X}, \\ E &= \ddot{X} - \dot{X} \times H \end{aligned} \right\}.$$

Our first task is to show that neither E nor H have any dependence on $\dot{X}$. Since (2) (above) holds, this is equivalent to showing that $[E, X] = 0 = [H, X]$.

<u>Lemma 1</u>. $\kappa\, H = [X, \ddot{X}]$

<u>Proof</u>. $H = \frac{1}{\kappa} \dot{X}\ \dot{X} = -\frac{1}{\kappa} \dot{X}\ \dot{X} + \frac{1}{\kappa} \dot{X}\ \dot{X}$

$$= -\frac{1}{\kappa}[\dot{X}, \dot{X}]$$

$$= \frac{1}{\kappa}[X, \ddot{X}] \qquad \left([X, \dot{X}]^{\bullet} = 0\right) \quad //$$

<u>Lemma 2</u>. $[E, X] = [H, X] = 0$.

<u>Proof</u>. $[H, X] = \frac{1}{\kappa}\, \dot{X}\, \dot{X}\, X - \frac{1}{\kappa}\, X\, \dot{X}\, \dot{X}$

$$= \frac{1}{\kappa}\, \dot{X}[\dot{X}, X] - \frac{1}{\kappa}\, [X, \dot{X}]\dot{X}$$

$$= -\dot{X} - \dot{X} = 0 \;//$$

$$[X, E] = [X, \ddot{X}] - X\, \dot{X}\, H + \dot{X}\, H\, X$$

$$= \kappa H - [X, \dot{X}]\, H + \dot{X}\, [H, X]$$

$$= \kappa H - \kappa H + 0$$

$$= 0 \;//$$

<u>Remark</u>. Lemma 2 implies that E and H are functions only of X_1, X_2, X_3. Hence $E \times E = H \times H = 0$, since $X \times X = 0$. If F is a function only of X_1, X_2, X_3, then

$$\dot{F} = \frac{\partial F}{\partial t} + \sum_i \dot{X}_i \frac{\partial F}{\partial X_i} = \frac{\partial F}{\partial t} + \frac{1}{\kappa}\, \dot{X}\, [F, \dot{X}].$$

Thus $\boxed{\partial_t H = \dot{H} + \frac{1}{\kappa}\, \dot{X}[\dot{X}, H]}$.

Lemma 3. $\nabla \bullet H = 0$.

Proof.
$$\nabla \bullet H = \frac{1}{\kappa}[H, \dot{X}]$$
$$= \frac{1}{\kappa} H\dot{X} - \frac{1}{\kappa}\dot{X}H$$
$$= \frac{1}{\kappa^2}\dot{X}\dot{X}\dot{X} - \frac{1}{\kappa^2}\dot{X}\dot{X}\dot{X}$$
$$= 0 \quad //$$

Lemma 4. $\partial_t H + \nabla \times E = 0$.

Proof. $\partial_t H = \dot{H} + \frac{1}{\kappa}\dot{X}[\dot{X}, H]$

$$\dot{H} = \frac{1}{2\kappa}[\dot{X}, \dot{X}]^{\bullet} = \frac{1}{\kappa}[\ddot{X}, \dot{X}] = \frac{1}{\kappa}[E, \dot{X}] + \frac{1}{\kappa}[\dot{X}\times H, \dot{X}]$$
$$= -\nabla \times E + \frac{1}{\kappa}[\dot{X} H, \dot{X}]$$
$$= -\nabla \times E - \frac{1}{\kappa}[\dot{X} H, \dot{X}] + \frac{1}{\kappa}[\dot{X} H, \dot{X}]$$
$$= -\nabla \times E - \frac{1}{\kappa}\dot{X}H\dot{X} + \frac{1}{\kappa}\dot{X}\dot{X}H + \frac{1}{\kappa}\dot{X}H\dot{X} - \frac{1}{\kappa}\dot{X}\dot{X}H$$
$$\dot{H} = -\nabla \times E + \frac{1}{\kappa}\dot{X}[H, \dot{X}] - \frac{1}{\kappa}\dot{X}[H, \dot{X}] + \frac{1}{\kappa}[\dot{X}, \dot{X}]H$$

The third term vanishes because $\nabla \bullet H = 0$.

Hence $\partial_t H + \nabla \times E = \frac{1}{\kappa}[\dot{X}, \dot{X}]H$

$$= \frac{1}{\kappa}\left\{ \dot{X}\dot{X}H - \dot{X}\dot{X}H \right\}$$

$$= \frac{1}{\kappa}\left\{ \dot{X}\dot{X}\ H \right\} = \mathbf{H}\ H = H \times H = 0 \ //$$

IV. Discussion

In this entire derivation the only appearance of non-algebraic calculus is in the time derivative $\dot{X}$. This can also be made algebraic by choosing a formal discrete time-evolution operator J so that $X' = J^{-1}XJ$ denotes the next (discrete) time value of X. Then we can take $\dot{X} = J(X' - X) = [X, J]$ and proceed as before. This is a capsule summary of the approach taken by the author and Pierre Noyes in [2].

References

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3. R. Penrose, Applications of negative dimensional tensors. "Combinatorial Mathematics and Its Applications" edited by D. J. A. Welsh, Academic Press (1971), p. 221–244.