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Where are the Twistors in the Null-Surface Formulation of GR?

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We give a brief review of the null-surface approach to general relativity and then speculate on a possible connection between this approach and twistor theory.

We first briefly review a formulation of GR (the Null-Surface Formulation of GR), whose basic variables are families of surfaces on a four-manifold, M, and a scalar function. The metric appears as a derived concept obtained from the assumption that the surfaces are characteristic surfaces of some conformal metric. The scalar function plays the role of a conformal factor converting the conformal metric into a vacuum metric. No proofs are given, as they have already appeared in the literature. We then speculate on the relationship of this reformulation of GR with twistor theory.

The null-surface approach to general relativity rests essentially on the introduction of two real functions, Z and Ω, living on the bundle of null directions over M; i.e., on $M \times S^2$ with x^a in M, and ζ on S^2. These two functions capture the information contained in an Einstein metric: $Z(x^a, \zeta)$ encodes the conformal structure of the Einstein space-time and singles out a preferred member of the conformal class of metrics, while $\Omega(x^a, \zeta)$ represents the appropriate conformal factor that turns the preferred member into an Einstein metric. In the following, we show how these two functions are introduced.

On a manifold M consider an S^2-family of functions $u = Z(x^a, \zeta) = constant$ such that the equation

$$g^{ab}(x^a) Z_{,a}(x^c, \zeta) Z_{,b}(x^d, \zeta) = 0 \tag{1}$$

can be solved for a metric $g^{ab}(x^a)$ for all values of ζ. The surfaces $Z(x^a, \zeta) = const.$ are then null surfaces of the metric g^{ab}. For every fixed value of ζ, the equation $Z(x^a, \zeta) = u$ represents a null foliation of the spacetime (M, g^{ab}). It is an interesting kinematical problem (also studied by L. Mason [1]) to derive the conditions on the function Z that imply the existence of a g^{ab} that satisfies Eq. (1). We approach this problem by resorting to a special set (families) of null coordinate systems $(u, R, \omega, \bar{\omega}) \equiv \theta^i(x^a)$, $i = 0, 1, +, -$, on M defined by the transformation

$$(u, R, \omega, \bar{\omega}) = (Z, \bar{\eth}\eth Z, \eth Z, \bar{\eth} Z) \tag{2}$$

Eq. (2) should be interpreted as a coordinate transformation $x^a \to \theta^i$ for every fixed value of ζ; i.e., a family of coordinate transformations dependent on ζ. Using the gradient basis $\theta^i{}_{,a}$, the metric g^{ab} can be expressed in the new coordinates as $g^{ij}(\theta^i) = \theta^i{}_{,a}\, \theta^j{}_{,b}\, g^{ab}(x)$.

By repeated differentiation of Eq. (1) with respect to ζ and $\bar{\zeta}$, and using the independence of the metric on the variable ζ, we obtain our two main results [2]. First, the metric components g^{ij} are all expressible in terms of the two quantities, Z and $\Omega \equiv \sqrt{g^{01}}$ and their derivatives; explicitly we have $g^{ij} = \Omega^2 \tilde{g}^{ij}(Z)$. Second, we find that Ω and Z are not arbitrary nor independent of each other. They must satisfy two coupled complex differential equations, referred to as the metricity conditions.

The metricity conditions constitute the requirement on Z in order for a metric g^{ab} to exist and not depend on ζ and such that $Z = const.$ are characteristic surfaces of the metric, for every value of ζ. They leave Ω undetermined up to a factor dependent only on x^a.

On this kinematical scheme we impose the (trace-free) vacuum Einstein equations by $\theta_1^a \theta_1^b (R_{ab} - \frac{1}{2} g_{ab} R) = 0$. In terms of the variables Z and Ω this equation takes the following form:

$$D^2 \Omega - Q\Omega = 0 \tag{3}$$

where $D \equiv \frac{\partial}{\partial R}$, $\Lambda \equiv \eth^2 Z$ and

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$$Q \equiv -\frac{1}{4q} D^2\Lambda D^2\bar{\Lambda} - \frac{3}{8q^2}(Dq)^2 + \frac{1}{4q}D^2 q \ , \qquad q \equiv 1 - D\Lambda D\bar{\Lambda} \tag{4}$$

The two complex metricity conditions and equation (3) constitute a set of coupled differential equations for the variables Z and Ω that are completely equivalent to the vacuum Einstein equations.

Though any vacuum Einstein space-time can be investigated in this manner we make the specialization, here, to asymptotically flat vacuum space-times. In this case the geometrical meanings to the various quantities become more focused and clearer and the differential equations become easier to handle. We begin with the fact that now null infinity, $\mathcal{I}^+$, exists. It can be coordinatized with a Bondi coordinate system,

$$(u, \zeta, \bar{\zeta}) \tag{5}$$

with $u \in R$, the Bondi retarded time, and $(\zeta, \bar{\zeta}) \in S^2$ labeling the null generators of $\mathcal{I}^+$. With this notation we can give a precise meaning to the null surfaces described by $u = Z(x^a, \zeta)$; they are taken to be the past null cones of the points $(u, \zeta, \bar{\zeta})$ of $\mathcal{I}^+$. In addition to this meaning of Z, there is a dual meaning, namely, if the space-time point x^a is held constant but the $(\zeta, \bar{\zeta})$ is varied over the S^2, we obtain a two-surface on $\mathcal{I}^+$, the so-called light-cone cut of $\mathcal{I}^+$. It consists of all points of $\mathcal{I}^+$ reached by null-geodesics from x^a. Z is then referred to as the light-cone cut function. We now have a geometric interpretation, not only of $Z(x^a, \zeta)$, but also of both, $\omega = \eth Z(x^a, \zeta)$ and $R = \eth\bar{\eth} Z(x^a, \zeta)$. ω is the "stereographic angles" that the light-cone cuts make with the Bondi $u = const.$ cuts (i.e., it labels the backward direction of the null geodesics from $\mathcal{I}^+$ to x^a, and R is a measure of the curvature of the cut and, thus, a measure of the "distance" from $\mathcal{I}^+$ to x^a along the null geodesic. Any null geodesic can be labeled by the five parameters, its intersection point with $\mathcal{I}^+$, i.e., $(u, \zeta, \bar{\zeta})$, and its "null angle", $(\omega, \bar{\omega})$.

In the special case of asymptotic flatness, we obtain a considerable simplification in the two metricity conditions; by differential and algebraic manipulation, they can be expressed as a single equation of the form

$$\eth^2\bar{\eth}^2 Z = \bar{\eth}^2\sigma(Z, \zeta) + \eth^2\bar{\sigma}(Z, \zeta) + \mathcal{D}(\Omega, \Lambda) \tag{6}$$

where σ is a free function of (u, ζ) (the Bondi shear, as characteristic data) on $R \times S^2$. $\mathcal{D}$ is an (explicitly known) non-linear polynomial in Λ and its derivatives and linear in derivatives of $\ln\Omega$. A paper on this is in preparation. Eqs. (3) and (6) are the Einstein equations (including the free data) for asymptotically flat-space-times.

We point out several items of potential interest.

1. There is a very straightforward perturbation scheme (off flat space) for these equations [3].

2. There is a canonical choice of interior space-time coordinates x^a, (essentially the coefficients of the first four spherical harmonics in the expansion of Z) so that there is no gauge freedom [4].

3. All conformal information of the space-time is contained in knowledge of the function $Z(x^a, \zeta, [data])$. The four functions $\theta^i(x^a, \zeta, [data])$, which are defined geometrically on $\mathcal{I}^+$, and describe the interior of the space-time are obtained from derivatives of Z. In turn, they can, in principle, be inverted leading to

$$x^a = x^a(\theta^i, \zeta, [data]), \tag{7}$$

   the location of space-time points in terms of information on $\mathcal{I}^+$. If Eq. (7) is written as

$$x^a = x^a(R, u, \omega, \zeta, [data]), \tag{8}$$

   we have the explicit form of all null geodesics; fixed values of (u, ω, ζ) in the parameter space of null geodesics picks out the geodesic while R is the geodesic parameter, which is affine length for our special metric $\tilde{g}^{ij}$ in the conformal class.

4. The work described here is the (in no way obvious) generalization of the study of self-dual space-times via the good cut equation, $\eth^2 Z = \sigma(Z, \zeta)$ which in turn led to the first insights into asymptotic twistor theory [5].

It is this last remark that raises the issue of what relationship does the present work have with twistor theory? The answer is that we simply do not know, though there are a few suggestions, which we now discuss, of a possible connection.

a. Flat and asymptotically flat twistor theory is clearly intimately concerned with null geodesics and the associated self- (or anti-self-) dual blades; we have, in our generalization, via the function $Z(x^a, \zeta, \bar{\zeta})$, a complete description of all null geodesics with the facility to manipulate them and to add and study any further conformally invariant structures associated with them, e.g., geodesic deviation vectors between different members. In particular, we could have and study deviation vectors at null infinity that make self- (or anti-self-) dual blades with the tangent vectors at null infinity.

b. The light-cone cuts of $\mathcal{I}^+$ for the flat and asymptotically flat self-dual space-times led directly to a definition of asymptotically flat twistors, namely the curves on complexified $\mathcal{I}^+$ defined by $u = Z(x^a, \zeta, \bar{\zeta})$ holding x^a and ζ fixed were the twistor curves. We do not know how this can be related or extended to the general asymptotically flat space-times.

c. In the case of asymptotically flat self-dual space-times, though there was a complete description of the light-cone cuts of interior space-time points (and thus the description of the associated twistor lines) on $\mathcal{I}^+$, for a variety of reasons (e.g., the googly problem) it was of great interest to study the behaviour of the light-cone cuts (and twistor lines) in the limit as the interior points approached $\mathcal{I}^+$. This limit is singular and its study was very difficult and inconclusive. Part of the difficulty was that we did not know how to introduce (interior) Bondi coordinates in place of the x^a so that the limit could be easily achieved. We have, in both the self-dual and the full vacuum case, recently seen how the Bondi coordinates could be introduced and, thus, we have the real hope of being able to study the structure of the light-cone cuts in any of the cases as the interior points x^a approach $\mathcal{I}^+$. This transformation to Bondi coordinates is given in the following fashion. If the $Z(x^a, \zeta, \bar{\zeta})$ is a known function of the local coordinates x^a, then the Bondi coordinates, $y^a = (u_B, r_B, \zeta_B)$ are given implicitly in terms of the x^a by the equations

$$\begin{aligned} u_B &= Z(x^a, \zeta_B, \bar{\zeta}_B) \\ 0 &= \eth Z(x^a, \zeta_B, \bar{\zeta}_B), \qquad \text{and c.c.} \\ r_B &= \eth\bar{\eth} Z(x^a, \zeta_B, \bar{\zeta}_B). \end{aligned} \tag{9}$$

Using these relationships and the known asymptotic form [6,7] of the metric we expect that we will be able to write the Z asymptotically (but explicitly) in terms of inverse powers of the r_B for space-time points near $\mathcal{I}^+$. This, then, would hopefully allow us to see how the limit behaves. Again, though we do not know what relationship this might have with twistor theory, we feel that there is considerable hope from this approach.

d. There is a version of our Eq. (6) that has a chiral appearance, so that it formally resembles the self-dual case. It should be looked at carefully, although we suspect that its resemblance to the self-dual case is purely notational and that it may have no deeper significance.

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