26

Heavenly hierarchies and curved twistor spaces

Maciej Dunajski Lionel J. Mason

The anti-self-dual vacuum equations (ASDVE) share many properties with lower-dimensional integrable systems. It is therefore reasonable to hope that some constructions well known from the theory of such integrable models are also present in the ASDVE. Boyer and Plebański [1] obtained an infinite number of conservation laws for the ASDVE equations and established some connections with the nonlinear graviton construction. Some of their results were later rediscovered and extended in various papers of Strachan and Takasaki. With the appropriate symplectic structure, the sequence of conserved quantities should lead to a hierarchy of evolution equations that are 'hidden symmetries' for the ASDVE equations.

In what follows, the hierarchical structure of the ASDVE in the heavenly form due to Plebański is constructed by looking at ways of generating sequences of solutions to the linearized heavenly equations. This approach is motivated by that in [5] for the treatment of ASDYM hierarchies.

We use the formulation of the ASDVE condition in [4]. All the spinor indices are assumed to be concrete and indices are raised and lowered with $\varepsilon_{AB} = \varepsilon_{[AB]}, \varepsilon_{01} = 1$ according to the usual Penrose and Rindler conventions. We work in the holomorphic category. Let $\mathcal{M}$ be a complex four-manifold equipped with a holomorphic volume form ν. Choose a normalised null tetrad $\nabla_{AA'}$ which consists of four independent and volume preserving vector fields. We will also require that the null tetrad contracted with the volume form yields one. The ASDVE on the metric for which $\nabla_{AA'}$ are a null tetrad arise as a consequence of the integrability of the distribution spanned by the Lax operators $L_A = \pi^{A'}\nabla_{AA'}$

$$[L_A, L_B] = 0 \tag{1}$$

where $\pi^{A'} = (-1, \lambda)$ is a constant spinor; all local solutions of the ASDVE arise in this way. Part of the residual gauge freedom in (1) is fixed by selecting one of Plebański's coordinate systems, $(w, z, \tilde{w}, \tilde{z}) =: (w^A, \tilde{w}^A)$ for the first equation and setting

$$\begin{aligned}\nabla_{AA'} &= \begin{pmatrix} \Omega_{w\tilde{w}}\partial_{\tilde{z}} - \Omega_{w\tilde{z}}\partial_{\tilde{w}} & \partial_w \\ \Omega_{z\tilde{w}}\partial_{\tilde{z}} - \Omega_{z\tilde{z}}\partial_{\tilde{w}} & \partial_z \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial^2\Omega}{\partial w^A \partial \tilde{w}^B}\varepsilon^{BC}\frac{\partial}{\partial \tilde{w}^C} & \frac{\partial}{\partial w^A} \end{pmatrix}\end{aligned}$$

and coordinate system $(w, z, x, y) =: (w^A, x_A)$ for the second where

$$\begin{aligned}\nabla_{AA'} &= \begin{pmatrix} -\partial_y & \partial_w + \Theta_{yy}\partial_x - \Theta_{xy}\partial_y \\ \partial_x & \partial_z - \Theta_{xy}\partial_x + \Theta_{xx}\partial_y \end{pmatrix} \\ &= \begin{pmatrix} \varepsilon_{BA}\frac{\partial}{\partial x_B} & \frac{\partial}{\partial w^A} + \varepsilon_{BC}\frac{\partial^2\Theta}{\partial x^A \partial x_B}\frac{\partial}{\partial x_C} \end{pmatrix}.\end{aligned}$$

These choices lead to heavenly equations.

$$\Omega_{w\tilde{z}}\Omega_{z\tilde{w}} - \Omega_{w\tilde{w}}\Omega_{z\tilde{z}} = 1 \qquad \text{First} \tag{2}$$

$$\Theta_{xw} + \Theta_{yz} + \Theta_{xx}\Theta_{yy} - {\Theta_{xy}}^2 = 0 \quad \text{Second} \tag{3}$$

Both (2) and (3) were originaly derived from the formulation dual to (1). The one forms $e^{AA'}$ dual to the tetrad $\nabla_{AA'}$ are used to construct the two form $\Sigma(\lambda) = \pi_{A'}\pi_{B'}\varepsilon_{AB}e^{AA'} \wedge e^{BB'} = \varepsilon^{AB}\nu(\cdot,\cdot,L_A,L_B) = \pi_{A'}\pi_{B'}\Sigma^{A'B'}$, which can be used as a set of basic variables in GR. Equation (1) becomes

$$d\Sigma(\lambda) = 0, \qquad \Sigma(\lambda) \wedge \Sigma(\lambda) = 0, \tag{4}$$

where in the first equation λ is regarded as a parameter and not differentiated. The closure condition is used to introduce ω^A, canonical coordinates on the the spin bundle, holomorphic around $\lambda = 0$ such that $d\Sigma(\lambda) = d\omega^A \wedge d\omega_A$. The various forms of the heavenly equations can be obtained by adapting coordinates etc. to these forms.

Equations (2), (3) admit Lagrangian formulations

$$\mathcal{L}_\Omega = \Omega(\nu - \frac{1}{3}(\partial\tilde{\partial}\Omega)^2),$$

where $\partial = e^{A1'}\otimes\nabla_{A1'} = dz\otimes\partial_z + dw\otimes\partial_w$, $\tilde{\partial} = e^{B0'}\otimes\nabla_{B0'} = d\tilde{z}\otimes\partial_{\tilde{z}} + d\tilde{w}\otimes\partial_{\tilde{w}}$ and forms are multiplied by exterior multiplication and

$$\mathcal{L}_\Theta = \frac{2}{3}\Theta(\partial_2\Theta)^2 - \frac{1}{2}(\partial\Theta)\wedge(\partial_2\Theta)\wedge e^{A0'}\wedge e^{0'}_A$$

where $\partial_2 = e^{B1'}\nabla_{B0'} = dz\otimes\partial_x - dw\otimes\partial_y$. Note that $e^{A0'}\wedge e^{0'}_A$ can be replaced by $dx\wedge dy$ in the second Lagrangian as it is multiplied by $dw\wedge dz$.

A symplectic form on the space of solutions can be derived from the boundary term in the variational principle and is given by

$$\Omega(\delta_1\Omega, \delta_2\Omega) = \frac{2}{3}\int_{\delta\mathcal{M}} e^{1'}{}_B \wedge e^{B0'} \wedge (e^{A0'}\delta_2\Omega\nabla_{A0'}\delta_1\Omega - e^{A1'}\delta_1\Omega\nabla_{A1'}\delta_2\Omega),$$

and similarly for the second equation. They coincide with the symplectic form on the solution space to the wave equation on the ASD background.

Recursion relations

First we observe that the linearized solutions to (2) and (3) satisfy the wave equation on the ASD background given by Ω and Θ respectively.

$$\nabla_{A1'}\nabla^A{}_{0'}\delta\Omega = \nabla_{A1'}\nabla^A{}_{0'}\delta\Theta = 0. \tag{5}$$

From now on we identify tangent spaces to the moduli spaces of solutions to (2, 3) with the space of solutions to the curved background wave equation, $\mathcal{W}_g$. The linearised vacuum metrics corresponding to $\delta\Omega$ and $\delta\Theta$ are

$$h^I{}_{AA'BB'} = \iota_{(A'}o_{B')}\nabla_{(A1'}\nabla_{B)0'}\delta\Omega, \qquad h^{II}{}_{AA'BB'} = o_{A'}o_{B'}\nabla_{A0'}\nabla_{B0'}\delta\Theta.$$

We are now able to generate new linearized solutions from old ones. Given $\phi \in \mathcal{W}_g$ we use the first of these equations to find h^I. If we put the perturbation obtained in this way on the LHS of the second equation and add an appropriate gauge term then we get ϕ' - the new element of $\mathcal{W}_g$ that provides the $\delta\Theta$ which gives rise to $h^{II}_{ab} = h^I_{ab} + \nabla_{(a}V_{b)}$. This reduces to

$$\nabla_{A1'}\nabla_{B1'}\phi = \nabla_{A0'}\nabla_{B0'}\phi'. \tag{6}$$

Define a recursion operator $R : \mathcal{W}_g \longrightarrow \mathcal{W}_g$ by

$$\nabla_{A1'}\phi = \nabla_{A0'}R\phi \tag{7}$$

so formally $R = (\nabla_{A0'})^{-1} \circ \nabla_{A1'}$. From this definition and from (1) it follows that if ϕ belongs to $\mathcal{W}_g$ then so does $R\phi$. Because $[R,\ \nabla_{B1'}] = 0$ on $\mathcal{W}_g$, we have $R^2\delta\Omega = \delta\Theta$.

Define, for $i \in \mathbf{Z}$, a hierarchy of linear fields, $\phi_i \equiv R^i\phi_0$. Put $\Phi = \sum_{-\infty}^{\infty}\phi_i\lambda^i$ and observe that the recursion equations are equivalent to $L_A\Phi = 0$. Thus Φ is a function on $\mathcal{PT}$. Conversely every solution of $L_A\Phi = 0$ defined on a neighbourhood of $|\lambda| = 1$ can be expanded in a Laurent series in λ with coefficients being a series of elements of $\mathcal{W}_g$ related by the recursion operator. The function Φ can be thought of as a Cech representative of the element of $H^1(\mathcal{PT}, \mathcal{O}(-2))$ that corresponds to the solution of the wave equation ϕ (here $\mathcal{PT}$ is the twistor space of $\mathcal{M}$).

It is clear that a series corresponding to $R\phi$ is the function $\lambda^{-1}\Phi$. Note that R is not completely well defined when acting on $\mathcal{W}_g$ because of the ambiguity in the inversion of $\nabla_{A0'}$. This means that if one treats $\Phi(\lambda)$ as a twistor function on $\mathcal{PT}$, pure gauge elements of the first sheaf cohomology group $H^1(\mathcal{PT}, \mathcal{O}(-2))$ of the twistor space corresponding to $\mathcal{M}$ are mapped to nontrivial terms. Note, however, that the action of R is well defined on twistor functions. By iterating R we generate an infinite sequence of elements of $H^1(\mathcal{PT}, \mathcal{O}(-2))$ belonging to different classes.

By a formal application of Stokes' theorem

$$\Omega(R\phi,\ \phi') = \Omega(\phi,\ R\phi'). \tag{8}$$

We can construct an infinite sequence of symplectic forms $\Omega^k(\phi,\ \phi') \equiv \Omega(R^k\phi,\ \phi')$ which play a role in the bihamiltonian formulation.

Twistor surfaces

We can use R to build a family of foliations by twistor surfaces starting from a given one. Put $\omega_0^A = w^A = (w, z)$; the surfaces of constant ω_0^A are twistor surfaces. We have that $\nabla^A{}_{0'}\omega_0^B = 0$ so that in particular $\nabla_{A1'}\nabla^A{}_{0'}\omega_0^B = 0$ and if we define $\omega_i^A = R^i\omega_0^A$ then we can choose $\omega_i^A = 0$ for negative i. We define

$$\omega^A = \omega_0^A + \sum_{i=1}^{\infty} \omega_i^A \lambda^i. \tag{9}$$

We can similarly define $\tilde{\omega}^A$ by $\tilde{\omega}_0^A = \tilde{w}^A$ and choose $\tilde{\omega}_i^A = 0$ for $i > 0$. Note that ω^A and $\tilde{\omega}^A$ are solutions of L_A holomorphic around $\lambda = 0$ and $\lambda = \infty$ respectively and they can be chosen so that they extend to a neighbourhood of the unit disc and a neighbourhood of the complement of the unit disc. We have that $\mathcal{PT}$ can be covered by two sets, U and $\tilde{U}$ with $|\lambda| < 1 + \epsilon$ on U and $|\lambda| > 1 - \epsilon$ on $\tilde{U}$ with (ω^A, λ) coordinates on U and $(\tilde{\omega}^A, \lambda^{-1})$ on $\tilde{U}$. $\mathcal{PT}$ is then determined by the transition function $\tilde{\omega}^B = f^B(\omega^A, \pi_{A'})$ on $U \cap \tilde{U}$.

Newman et. al. [6] make equation (2) λ-dependent and show that ω^A may be found by integrating the hamiltonian system which has Ω as its hamiltonian. In their treatment λ plays the role of time. We give an analogous interpretation of the 2nd equation.

Choose a spinor say $\kappa_{A'} = (0,1)$, in the base space and parametrize a curve by the coordinates

$$x^{AA'} = \frac{\partial \omega^A}{\partial \pi_{A'}}\Big|_{\pi_{A'}=\kappa_{A'}}, \; x^{A1'} = \omega_0^A = (w,z), \; x^{A0'} = x^A = (-y,x)$$

where $x^{A1'}$ gives the initial point on the curve, while $x^{A0'}$ is a tangent vector to the curve. To procede further, ie to find higher terms in (9) we do one of the following (all give the same answer).

a) Insert the 2nd heavenly tetrad into the recursion relations and solve for ω_3^A

$$\omega^A = x^{A1'} + \lambda x^{A0'} + \lambda^2 \varepsilon^{BA}\frac{\partial\Theta}{\partial x^{B0'}} + \lambda^3 \varepsilon^{BA}\frac{\partial\Theta}{\partial x^{B1'}} + \ldots \; . \tag{10}$$

Note that (7) is used to find the fourth term in the series, since the third one is the definition of Θ.

b) Use the globality and the degree two homogeneity of $\Sigma(\lambda)$

$$o_{A_1'}o_{A_2'}\ldots o_{A_k'}\frac{\partial^k d\omega^A \wedge d\omega_A}{\partial\pi_{A_1'}\partial\pi_{A_2'}\ldots\partial\pi_{A_k'}}\Big|_{\pi_{A_i'}=o_{A_i'}}$$

$$= \sum_{i=0}^{k} \varepsilon_{AB} d\omega_i^A \wedge d\omega_{k-i}^B = \begin{cases} 0 & \text{for } k > 2 \\ -\Sigma^{0'0'} & \text{for } k = 2\;. \end{cases} \tag{11}$$

c) Make the 2nd equation $\pi_{A'}$, i.e. λ-dependent. Define $X^{AA'} = \partial\omega^A/\partial\pi_{A'}$. Continue the curve to another order in λ so that to order λ^2

$$X^{A1'} = x^{A1'} + \lambda x^{A0}, \qquad X^{A0'} = x^{A0'} + \lambda\varepsilon^{BA}\frac{\partial\Theta}{\partial x^{B0'}}.$$

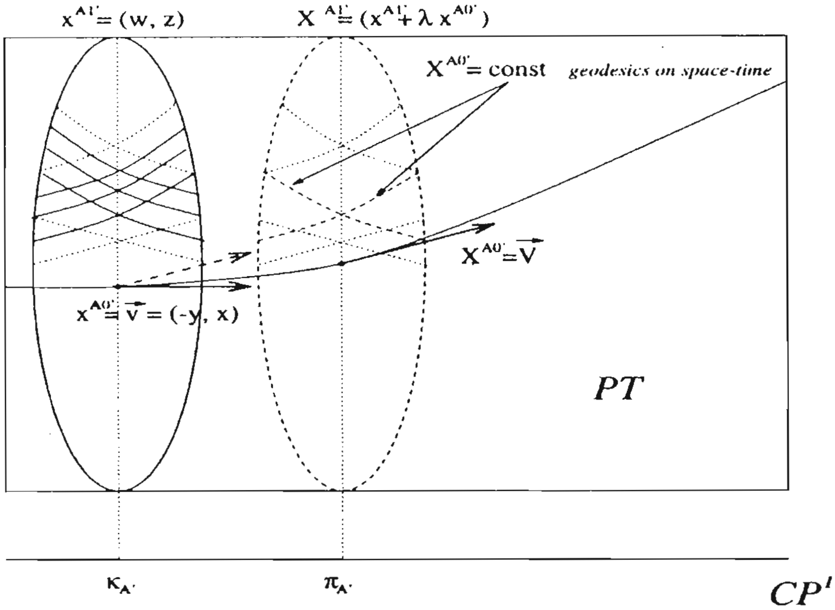
We then put the space-time metric into a standard, 2nd heavenly form with respect to the coordinates $X^{AA'}$

$$ds^2 = \varepsilon_{AB} dX^{A1'} dX^{B0'} + \frac{\partial^2\Theta'}{\partial X^{A0'}\partial X^{B0'}} dX^{A1'} dX^{B1'}$$

which forces us to introduce Θ', differing from Θ by terms of order λ

$$\Theta'(X^{AA'}, \pi_{A'}) = \Theta(x^{AA'}) + \lambda\tau(x^{AA'}).$$

We find Θ' and can then iterate the process to obtain the subsequent orders in $X^{AA'}$. The parameter λ plays the role of time and Θ plays the



role of a time dependent hamiltonian. In homogeneous coordinates, Θ is homogeneous of degree -4 in $\pi_{A'}$. The construction may be summarized by the following

$$\dot{X}^{A1'} = X^{A0'}, \quad \dot{X}^{A0'} = \varepsilon^{BA}\frac{\partial\Theta'}{\partial X^{B0'}}, \quad \dot{\Theta}' = \tau, \quad \frac{\partial\tau}{\partial X^{A0'}} = \frac{\partial\Theta'}{\partial X^{A1'}}. \tag{12}$$

Dot means differentiation with respect to λ. The last equation (which gives the recursion relations) is valid up to the addition of $f(X^{A1'})$. The first two equations can (for those familiar with *edth*= $\eth$) be written as

$$\eth^2\omega^A = \pi^{A'}\nabla^A{}_{A'}\Theta, \quad \text{or} \quad \eth X^{AA'} = \{X^{AA'}, \Theta\}_\Pi,$$

where $\Pi = \pi^{A'}\pi^{B'}\nabla_{AA'} \wedge \nabla^A{}_{B'}$ is a (homogeneous) Poisson structure defined on the spin bundle tangent to the α-planes. Note that it projects down to zero by the twistor fibration.

Hierarchies

Finally we embed (1) in an infinite system of overdetermined PDEs. The associated linear system is

$$L_{Ai}s = (\lambda\nabla_{Ai} - D_{Ai-1})s = 0, \tag{13}$$

where

1. $s = s(x^{AA'\ldots A'_k}, \pi_{A'})$ is a function on a spin bundle over $\mathcal{M} \times X$, where $x^{AA'\ldots A'_k} = x^{A(A'\ldots A'_k)}$ are coordinates on the $\mathcal{M} \times X$ and $X \equiv \mathbb{C}^{2(n-1)}$ is a space of parameters ("times"), and

$$x^{Ai} = o_{A'}o_{A'_2}\ldots o_{A'_i}\iota_{A'_{i+1}}\ldots\iota_{A'_n}x^{AA'A'_2\ldots A'_n}$$

for $i = 0, 1$ are coordinates on $\mathcal{M}$ and for $i > 1$ are coordinates on X.

2. $\nabla_{AA'i} = \iota^{A'_1}\ldots\iota^{A'_{i-1}}o^{A'_i}\ldots o^{A'_n}\nabla_{AA'(A'_2\ldots A'_n)}$, $\nabla_{A1'i} = \nabla_{Ai}$, $\nabla_{A0'i} = D_{Ai-1}$.

3. $L_{A(A'_2\ldots A'_n)} = \pi^{A'}\nabla_{AA'(A'_2\ldots A'_n)}$, $L_{Ai} = \pi^{A'}\nabla_{AA'i}$.

Compatibility conditions for (13) yield

$$[\nabla_{Ai}, \nabla_{Bj}] = 0, \tag{14}$$

$$[D_{Ai-1}, D_{Bj-1}] = 0, \tag{15}$$

$$[\nabla_{Ai}, D_{Bj-1}] - [\nabla_{Bj}, D_{Ai-1}] = 0. \tag{16}$$

In what follows we shall give the appropriate generalisation of the second heavenly formulation; the first hierarchy is a consequence of a different gauge choice. (15) is used to fix $D_{Ai-1} = \partial_{Ai-1}$. Equation (16) implies that we can put $\nabla_{Ai} = \partial_{Ai} + [\partial_{Ai-1}, V]$ for some vector field V. In the basic tetrad for the second heavenly equation we see that $V = \varepsilon_{AB}\partial\Theta/\partial x_A \partial/\partial x_B$, i.e. V is the hamiltonian vector field of Θ with respect to the Poisson structure $\varepsilon_{AB}\partial/\partial x_A \wedge \partial/\partial x_B = \partial_x \wedge \partial_y$. The higher flows of the *second heavenly hierarchy* are, after redefinition of Θ, given by reducing (14) to give

$$\partial_{Ai}\partial_{Bj-1}\Theta - \partial_{Bj}\partial_{Ai-1}\Theta + \{\partial_{Ai-1}\Theta, \partial_{Bj-1}\Theta\}_{yx} = 0. \tag{17}$$

The sub-hierarchy $[L_A, L_{B_i}] = 0$ gives the recursion relations (7), which now generalize to

$$\nabla_{Ai}\partial_{Bj-1}\Theta = D_{Ai-1}\partial_{Bj}\Theta = D_{Ai-1}R\partial_{Bj-1}\Theta. \tag{18}$$

Let $e^{AA'_2...A'_n}$ be the set of one forms dual to $\nabla_{AA'A'_2...A'_n}$. In the adopted gauge

$$e^{A1'i} = dx^{Ai}, \quad e^{A0'i} = dx^{Ai-1} + \varepsilon^{AC} dx^{Di} \frac{\partial^2 \Theta}{\partial x^{Di-1} \partial x^{C0'}}.$$

An analogue of the formulation (4) may be achieved by introducing a two form homogeneous of degree $2n$ in $\pi_{A'}$

$$\Sigma_{(n)}(\lambda) = \varepsilon_{AB} \pi_{A'} ... \pi_{A'_n} \pi_{B'} ... \pi_{B'_n} e^{AA'...A'_n} \wedge e^{BB'...B'_n} \tag{19}$$

which satisfies $d\Sigma(\lambda) = 0, \;\; \Sigma(\lambda) \wedge \Sigma(\lambda) = 0$. The second equation follows from the choice of gauge while the closure condition is equivalent to (17).

The corresponding twistor space is obtained by factorizing the spin bundle $\mathcal{M} \times X \times \mathbb{CP}^1$ by the twistor distibution L_{Ai}. The resulting twistor space is still three-dimensional however has it a different topology as the holomorphic curves corresponding to points of $\mathcal{M} \times X$ have normal bundle $\mathcal{O}^A(n)$.

Define a conjugate recursion operator $R^*_{(i)} = D_{Ai-1} R_{(i)} {D_{Ai-1}}^{-1}$ and rewrite (17) in the bihamiltonian form

$$\partial_{Bj}(\partial_{Ai-1}\Theta) = D_{Ai-1}\partial_{Bj}\Theta = \nabla_{Ai}\partial_{Bj-1}\Theta =$$

$$\nabla_{Ai} R^{j-1} \partial_{B0'}\Theta = R^{*j-1} \nabla_{Ai} \partial_{B0'}\Theta.$$

The comparison with the KdV hierarchy

$$u_{t_j} = \mathcal{E}\frac{\delta h_{j-1}}{\delta u} = \mathcal{D}\frac{\delta h_j}{\delta u}$$

suggests that D_{Ai-1} and ∇_{Ai} play the role of first and second Poisson operators, while the solutions of the wave equation correspond to (functional derivatives of) different hamiltonians.

Observe that $\sum_{m=0}^{j} \{\partial_{Ai+j-m-1}\Theta, \partial_{Bm-1}\Theta\}_{yx} = \partial_{Bj}\partial_{Ai-1}\Theta$. This, with the definition $\omega_{Bj}(\lambda) = \sum_{m=0}^{j} \partial_{Bm-1}\Theta\lambda^{m+1}$, gives another form of the 2nd hierarchy

$$\partial_{Bj}\, \omega^A(\lambda) = \{\omega^A(\lambda), \lambda^{-j-1}\omega_{Bj}(\lambda)\}_{yx}. \tag{20}$$

Finally we would like to make a few remarks about where the above ideas could be applied.

- WDVV-the scaling reduction of the equation of associativity in the theory of Frobenius manifolds [2] coincides with Painleve VI for $n = 3$. It seems likely that a general case (which may be viewed as a higher order analogue of the Painleve equations) can be obtained as a reduction of higher flows in (17). Another possibility is that the Lax representation of the N-wave system lifted to the spin bundle will give rise to that of WDVV.

- Closely related to the previous is Krichever's work on Witham hierarchies in the context of equations of hydrodynamic type. In his approach [3], there is a closed two-form of an arbitrary homogeneity associated to each flow. This suggests an analogy with (19).

- In the last **TN** MD showed how $\nabla_{AA'}$ can be constructed from a solution of the Sine-Gordon equation. The same construction applies to KdV and NlS. It should be possible to obtain a recursion operator for these soliton equations from (7). In [5] such a reduction was implemented from the ASDYM recursion operator.

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References

[1] Boyer, C.P. & Plebański, J.F.(1985) An infinite hierarchy of conservation laws and nonlinear superposition principles for self-dual Einstein spaces, J. Math. Phys. **26**, 229-234.

[2] Dubrovin, B. (1992) Integrable systems in topological field theory, Nucl. Phys. **B 379**, 627-689.

[3] Krichever, I.M. (1988) The averaging method for two-dimensional integrable equations, Funct. Anal. Appl **3**, 200-213.

[4] Mason, L.J. & Newman, E.T. (1989) A connection between the Einstein and Yang-Mills equations, Comm. Math. Phys., **121**, 659-668.

[5] Mason, L.J. & Woodhouse, N.M.J. (1996) *Integrability, Self-Duality, and Twistor Theory*, L.M.S. Monographs New Series, **15**, OUP.

[6] Newman, E.T., Porter, J.R. & Tod, K.P. (1978) Twistor Surfaces and Right-Flat Spaces, Gen. Rel. Grav. **12** 1129-1142.