Formal Adjoints and a Canonical Form for Linear Operators

Suppose $E$ and $F$ are smooth vector bundles on an oriented smooth manifold $M$. Let $\text{vol}$ denote the bundle of volume forms on $M$. The \textit{formal adjoint} of a linear differential operator $L : E \to F$ is the differential operator $L^* : F^* \otimes \text{vol} \to E^* \otimes \text{vol}$ characterised by the equation

$$
\int_M (L^* \sigma, \tau) = \int_M \langle \sigma, L \tau \rangle \quad \text{for} \quad \sigma \in \Gamma(M, F^* \otimes \text{vol}) \text{ and } \tau \in \Gamma(M, E).$

If $F = E^* \otimes \text{vol}$, then $L^* : E \to F$ and there is a canonical decomposition

$$
L = L_+ + L_- = \frac{1}{2}[L + L^*] + \frac{1}{2}[L - L^*]
$$

into self-adjoint and skew-adjoint parts. If $E$ and $F$ are tensor bundles on a Riemannian manifold, then $L$ may be written in terms of the Levi-Civita connection and a formula for its adjoint determined by integration by parts. Suppose, for example, that $E$ and $F$ are both trivial and $L$ is second order. Then we may write

$$
L = S^{ab} \nabla_a \nabla_b + T^b \nabla_b + R,
$$

where the tensor $S^{ab}$ is symmetric. Adopting the convention that $\nabla_a$ acts on everything to its right, we can re-express $L$ in the form

$$
L = \nabla_a S^{ab} \nabla_b + (\bar{T}^b \nabla_b + \nabla_b \bar{T}^b) + \bar{R}
$$

where $\bar{T}^b = \frac{1}{2}(T^b - (\nabla_a S^{ab}))$ and $\bar{R} = R - \nabla_b \bar{T}^b$. This is congenial since clearly

$$
L^* = \nabla_a S^{ab} \nabla_b - (\bar{T}^b \nabla_b + \nabla_b \bar{T}^b) + \bar{R}.
$$

In particular,

$$
L_+ = \nabla_a S^{ab} \nabla_b + \bar{R} \quad \text{and} \quad L_- = \bar{T}^b \nabla_b + \nabla_b \bar{T}^b.
$$

This generalises immediately to give:

**Proposition.** A self-adjoint $k^{th}$ order linear differential operator taking functions on an oriented Riemannian manifold has even order and may be canonically written in the form:

$$
\sum_{i=0}^{k/2} \nabla_a \nabla_b \cdots \nabla_c S^{ab \cdots cef \cdots g}_i \nabla_e \nabla_f \cdots \nabla_g,
$$

for suitable symmetric tensors $S^{ab \cdots cef \cdots g}_i$. A skew-adjoint $k^{th}$ order linear differential operator taking functions on an oriented Riemannian manifold has odd order and may be canonically written in the form:

$$
\sum_{i=0}^{(k-1)/2} \nabla_a \nabla_b \cdots \nabla_c (\nabla_d A^{ab \cdots cdef \cdots g}_i + A^{ab \cdots cdef \cdots g}_i \nabla_d) \nabla_e \nabla_f \cdots \nabla_g,
$$

for suitable symmetric tensors $A^{ab \cdots cdef \cdots g}_i$.

Now suppose $M$ is an even-dimensional oriented conformal manifold. Graham, Jenne, Mason, and Sparling [2] have shown that there is a conformally invariant operator which, with respect to any Riemannian metric in the conformal class, takes the form

$$
L = \Delta^{n/2} + \text{lower order terms}.
$$
Since $L$ takes functions to volume forms, so does its adjoint. The self-adjoint part $L_+^n$ of $L$ is therefore also conformally invariant. As a conformal analogue of $\Delta^{n/2}$, we may as well replace $L$ by $L_+$. Write this differential operator in the form given by our Proposition. The scalar part $S_{[\theta]}$ is a conformal invariant since it may be obtained by applying $L$ to $f \equiv 1$. We may subtract this part and obtain the following result conjectured to us by Tom Branson.

**Theorem.** The operator $\Delta^{n/2}$ admits a self-adjoint conformally invariant modification of the form $f \mapsto \nabla_{\lambda}(Q^{ab}(\nabla_{\lambda}f))$ for a suitable $(n-2)$th order differential operator $Q : \Lambda^1 \to \Lambda^{n-1}$.

His motivation for this conjecture comes from the case of the sphere where the form of the operator may be verified directly. On the sphere, the operator controls the embedding $L^2_{n/2} \hookrightarrow e^L$ (Orlicz class) as a limiting case of the sharp Sobolev embeddings $L^2_r \hookrightarrow L^{2r+2} \hookrightarrow L^r$ for $r < n/2$ (equivalently, comparing an $L^q$ norm with the complementary series norm). See [1] for further discussion.

**References**


**Michael Eastwood**

**Department of Pure Mathematics**

**University of Adelaide**

**South Australia 5005**

meastwood@unimaths.adelaide.edu.au

**Rod Gover**

**School of Mathematics**

**Queensland University of Technology**

**Gardens Point Campus**

**Brisbane**

**Queensland 2434**

**Australia**

r.gover@fsc.qut.edu.au