Formal Adjoints and a Canonical Form for Linear Operators

Suppose E and F are smooth vector bundles on an oriented smooth manifold M. Let *vol* denote the bundle of volume forms on M. The *formal adjoint* of a linear differential operator $L : E \to F$ is the differential operator $L^* : F^* \otimes vol \to E^* \otimes vol$ characterised by the equation

$$\int_M \langle L^*\sigma, \tau\rangle = \int_M \langle \sigma, L\tau\rangle \quad \text{for } \sigma \in \Gamma(M, F^* \otimes vol) \text{ and } \tau \in \Gamma_*(M, E).$$

If $F = E^* \otimes vol$, then $L^* : E \to F$ and there is a canonical decomposition

$$L = L_+ + L_- = \tfrac{1}{2}[L + L^*] + \tfrac{1}{2}[L - L^*]$$

into self-adjoint and skew-adjoint parts. If E and F are tensor bundles on a Riemannian manifold, then L may be written in terms of the Levi-Civita connection and a formula for its adjoint determined by integration by parts. Suppose, for example, that E and F are both trivial and L is second order. Then we may write

$$L = S^{ab}\nabla_a\nabla_b + T^b\nabla_b + R,$$

where the tensor S^{ab} is symmetric. Adopting the convention that ∇_a acts on everything to its right, we can re-express L in the form

$$L = \nabla_a S^{ab}\nabla_b + (\tilde{T}^b\nabla_b + \nabla_b\tilde{T}^b) + \tilde{R}$$

where $\tilde{T}^b = \frac{1}{2}(T^b - (\nabla_a S^{ab}))$ and $\tilde{R} = R - \nabla_b\tilde{T}^b$. This is congenial since clearly

$$L^* = \nabla_a S^{ab}\nabla_b - (\tilde{T}^b\nabla_b + \nabla_b\tilde{T}^b) + \tilde{R}.$$

In particular,

$$L_+ = \nabla_a S^{ab}\nabla_b + \tilde{R} \quad \text{and} \quad L_- = \tilde{T}^b\nabla_b + \nabla_b\tilde{T}^b.$$

This generalises immediately to give:

Proposition . *A self-adjoint k^{th} order linear differential operator taking functions to functions on an oriented Riemannian manifold has even order and may be canonically written in the form:*

$$\sum_{i=0}^{k/2} \underbrace{\nabla_a\nabla_b\cdots\nabla_c}_{i} S_{(i)}^{ab\cdots cef\cdots g} \underbrace{\nabla_e\nabla_f\cdots\nabla_g}_{i},$$

for suitable symmetric tensors $S_{(i)}^{ab\cdots cef\cdots g}$. A skew-adjoint k^{th} order linear differential operator taking functions to functions on an oriented Riemannian manifold has odd order and may be canonically written in the form:

$$\sum_{i=0}^{(k-1)/2} \underbrace{\nabla_a\nabla_b\cdots\nabla_c}_{i}(\nabla_d A_{(i)}^{ab\cdots cdef\cdots g} + A_{(i)}^{ab\cdots cdef\cdots g}\nabla_d)\underbrace{\nabla_e\nabla_f\cdots\nabla_g}_{i},$$

for suitable symmetric tensors $A_{(i)}^{ab\cdots cdef\cdots g}$.

Now suppose M is an even-dimensional oriented conformal manifold. Graham, Jenne, Mason, and Sparling [2] have shown that there is a conformally invariant operator which, with respect to any Riemannian metric in the conformal class, takes the form

$$L = \Delta^{n/2} + \text{lower order terms}.$$

Since L takes functions to volume forms, so does its adjoint. The self-adjoint part L_+ of L is therefore also conformally invariant. As a conformal analogue of $\Delta^{n/2}$, we may as well replace L by L_+. Write this differential operator in the form given by our Proposition. The scalar part $S_{(0)}$ is a conformal invariant since it may be obtained by applying L to $f \equiv 1$. We may subtract this part and obtain the following result conjectured to us by Tom Branson.

Theorem . *The operator $\Delta^{n/2}$ admits a self-adjoint conformally invariant modification of the form $f \mapsto \nabla_a(Q^{ab}(\nabla_b f))$ for a suitable $(n-2)^{nd}$ order differential operator $Q : \Lambda^1 \to \Lambda^{n-1}$.*

His motivation for this conjecture comes from the case of the sphere where the form of the operator may be verified directly. On the sphere, the operator controls the embedding $L^2_{n/2} \hookrightarrow e^L$ (Orlitz class) as a limiting case of the sharp Sobolev embeddings $L^2_r \hookrightarrow L^{\frac{2n}{n-2r}}$ for $r < n/2$ (equivalently, comparing an L^q norm with the complementary series norm). See [1] for further discussion.

References

[1] T.P. Branson, *Sharp inequalities, the functional determinant, and the complementary series*, Trans. A.M.S. **347** (1995), 3671–3742.

[2] C.R. Graham, R. Jenne, L.J. Mason, and G.A.J. Sparling, *Conformally invariant powers of the Laplacian, I: Existence*, Jour. Lond. Math. Soc. **46** (1992), 557–565.

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