Five things you can do with a surface in $\mathbb{PT}$

Consider Minkowski 4-space $\mathbb{M}$, embedded in its complexification $\mathbb{CM}$. The following five geometrical structures in $\mathbb{M}$, or $\mathbb{CM}$, are essentially equivalent:

(1) a shear-free ray congruence $\mathcal{R}$ in $\mathbb{M}$,

(2) an oriented timelike 2-surface $\mathcal{T}$ in $\mathbb{M}$
(ie an oriented string history);

(3) a complex null hypersurface $\mathcal{N}$ in $\mathbb{CM}$,

(4) a complex null 2-surface $\mathcal{U}$ in $\mathbb{CM}$,

(5) another shear-free ray congruence $\mathcal{R}^*$ in $\mathbb{M}$.
(NB: "ray" is a "null geodesic")

The word "essentially" in the above assertion is a little difficult to be precise about. In essence, what I mean by "essentially" here is a combination of the following three qualifications: (a) "in general", (b) "locally", and (c) "in the analytic case". However, we must be careful about the sense in which (b) is intended: certainly not local to a particular region of $\mathbb{CM}$; local in twistor space is more to the point. Moreover, we must also be a little cautious about the meaning of (a) in conjunction with (c). These matters should become clearer in a moment.

In short, the equivalence between (1), (2), (3), (4), and (5) arises from the fact that each provides a different space-time interpretation (in $\mathbb{M}$ or in $\mathbb{CM}$) of a <u>holomorphic surface</u> $\mathcal{S}$ in $\mathbb{PT}$. Before explaining this, it will clarify matters a little if I give the space-time relations between (1), (2), (3), (4), and (5) directly.

The equivalence between (2) and (3) is explained if we first think of a spacelike 2-surface instead of a timelike one. Such a surface is always (locally) the intersection of two real null hypersurfaces (generated by the rays meeting the 2-surface orthogonally). Complexifying this, we see that a complex (non-null) surface $\mathcal{T}$ is (locally) uniquely the intersection of two complex null hypersurfaces

$\mathcal{N}$ and $\tilde{\mathcal{N}}$. If $\mathcal{T}$ is to be *real*, the unordered pair of hypersurfaces $(\mathcal{N}, \tilde{\mathcal{N}})$ must go to itself under complex conjugation. There are two ways for this to happen: (i) $\mathcal{N} = \bar{\mathcal{N}}$ and $\tilde{\mathcal{N}} = \bar{\tilde{\mathcal{N}}}$ — when $\mathcal{T}$ is spacelike — and (ii) $\tilde{\mathcal{N}} = \bar{\mathcal{N}}$ — when $\mathcal{T}$ is timelike. Thus, a *real* timelike $\mathcal{T}$, being the real intersection of $\mathcal{N}$ with $\bar{\mathcal{N}}$, is simply the *set of real points* of $\mathcal{N}$. The choice of orientation of $\mathcal{T}$ corresponds to the selection of $\mathcal{N}$ rather than $\bar{\mathcal{N}}$; this orientation is reversed when $\mathcal{N}$ is replaced by $\bar{\mathcal{N}}$.

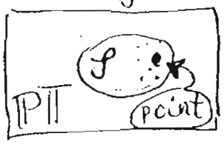
As for the complex 2-surface $\mathcal{U}$ of (4), it may be interpreted as the (primary) *caustic* of $\mathcal{N}$. Thus, $\mathcal{U}$ represents the locus of points of $\mathcal{N}$ at which generators of $\mathcal{N}$ intersect neighbouring generators of $\mathcal{N}$. Starting from the complex null 2-surface $\mathcal{U}$, we can generate $\mathcal{N}$ as the locus of rays which *touch* $\mathcal{U}$. To put this another way, noting that any regular point of $\mathcal{U}$ has a unique null tangent to $\mathcal{U}$ there, we find that (most of) $\mathcal{N}$ is generated by the rays through such points in the direction of these null tangents.

The relation between (1) and (2) is obtained through the fact that the real caustic of a (generic) shear-free ray congruence $\mathcal{R}$ in $\mathbb{M}$ is a timelike 2-surface $\mathcal{T}$. (For example, in the case of the "Kerr black hole" congruence $\mathcal{R}$ in $\mathbb{M}$, $\mathcal{T}$ describes the history of the ring singularity.) In fact, $\mathcal{T}$ is a secondary caustic of $\mathcal{R}$. The primary caustic of $\mathcal{R}$ — or, rather, of its complexification $\mathbb{C}\mathcal{R}$ — is the pair of complex null hypersurfaces $\mathcal{N}, \bar{\mathcal{N}}$, and this largely expresses the relation between (1) and (3). More completely, we can say that $\mathcal{R}$ is the family of real rays lying in those α-planes which contain a generator of $\mathcal{N}$. Equivalently (by complex conjugation), $\mathcal{R}$ consists of real rays in β-planes through generators of $\bar{\mathcal{N}}$. The relation between (1) and (4) can be directly obtained from the fact that $\mathcal{R}$ consists of the real rays in α-planes touching $\mathcal{U}$.

Finally, the ray congruence $\mathcal{R}^*$ of (5) is obtained simply by reversing the orientation of $\mathcal{T}$, i.e. by replacing $\mathcal{N}$ by $\bar{\mathcal{N}}$, so that

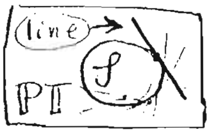
$\mathcal{R}^*$ consists of the real rays in β-planes containing generators of $\mathcal{N}$ — or, equivalently, touching $\mathcal{U}$ — or in α-planes containing generators of $\bar{\mathcal{N}}$. (In the case of a Kerr black hole congruence, $\mathcal{R}^*$ is related to $\mathcal{R}$ simply by having the opposite angular momentum, but in general $\mathcal{R}^*$ will look completely different from $\mathcal{R}$.)

Let us now see how all this relates to a holomorphic surface $\mathcal{S}$ in $\mathbb{PT}$. The well-known Kerr theorem (see <u>Spinors</u> <u>and</u> <u>Space-time</u> Vol. 2, pp. 200-203) asserts that any real (analytic) shear-free ray congruence $\mathcal{R}$ in $\mathbb{M}$ consists of the rays represented by the intersection with $\mathbb{PN}$ (-I) of a holomorphic surface $\mathcal{S} \subset \mathbb{PT}$. Moreover, $\mathcal{S}$ is determined by $\mathcal{R}$. Thus (1) is equivalent to a holomorphic surface $\mathcal{S}$ in $\mathbb{PT}$. Closely related is the representation of the <u>whole</u> of $\mathcal{S}$, in <u>complex</u> space-time terms, as a 2-parameter family of α-planes in $\mathbb{CM}$ (where $\mathcal{R}$ consists of the real rays in these α-planes). This particular (complex) space-time description interprets

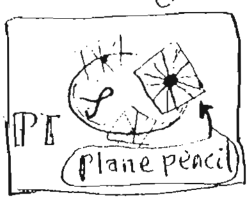


$\mathcal{S}$ as a system of <u>points</u>.

Another way of interpreting $\mathcal{S}$ is in terms of the <u>lines</u> which <u>touch</u> it. This



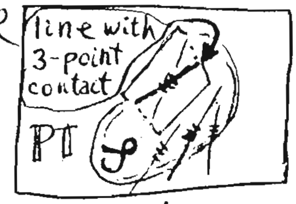
is a 3-parameter family of lines. **These lines** correspond to the points of $\mathbb{CM}$ which lie on some hypersurface — in fact, the null hypersurface $\mathcal{N}$. The generators



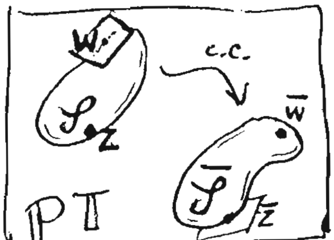
of $\mathcal{N}$ correspond to the <u>plane</u> <u>pencils</u> of lines touching $\mathcal{S}$ at fixed points. Thus, (3) is also equivalent to $\mathcal{S}$. We can also represent $\mathcal{S}$ in terms of the 2-complex parameter family of <u>planes</u> which <u>touch</u> $\mathcal{S}$. This gives a **2**-parameter family of

β-planes in $\mathbb{CM}$, and the intersections of these β-planes with $\mathbb{M}$ provide another shear-free ray congruence in $\mathbb{M}$, namely $\mathcal{R}^*$.

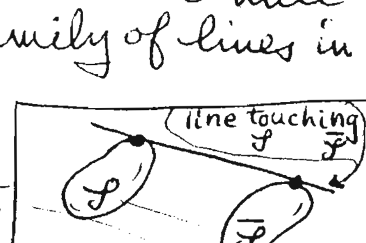
As for the complex null surface $\mathcal{U}$, its points



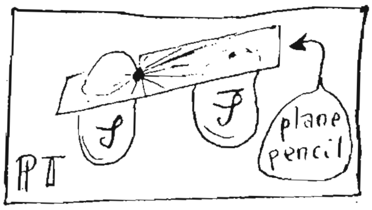
correspond to <u>lines</u> in $\mathbb{P}\mathbb{T}$ which have <u>three-point contact</u> with $\mathcal{S}$ (i.e. they are not simply tangent to $\mathcal{S}$, but touch it to higher order). These lines constitute a 2-parameter family — giving $\mathcal{U}$ as a 2-surface. At a general point of $\mathcal{S}$, there are two such lines in the plane pencil of tangent lines there. Thus, a general generator of $\mathcal{N}$ meets $\mathcal{U}$ in two distinct points. This shows how (4) is equivalent to $\mathcal{S}$.



Let us return to $\mathcal{R}^*$. We can think of it as arising in another way. The 2-parameter family of planes touching $\mathcal{S}$ can be represented as a surface in $\mathbb{P}\mathbb{T}^*$ which, by complex conjugation, provides us with a surface $\bar{\mathcal{S}}$ in $\mathbb{P}\mathbb{T}$. Thus, whenever the plane W touches $\mathcal{S}$, the point $\bar{W}$ lies on $\bar{\mathcal{S}}$. It follows that whenever a point Z lies on $\mathcal{S}$ then the plane $\bar{Z}$ touches $\bar{\mathcal{S}}$. We can call $\mathcal{S}$ and $\bar{\mathcal{S}}$ <u>reciprocal</u> surfaces in $\mathbb{P}\mathbb{T}$. (Note that $\bar{\mathcal{S}}$ is actually a <u>holomorphic</u> surface in $\mathbb{P}\mathbb{T}$, though its relationship to $\mathcal{S}$ is anti-holomorphic. If $\mathcal{S}$ is given by the relation $f(Z^\alpha)=0$, with f homogeneous and holomorphic, then $\bar{f}(W_\alpha)=0$ is a holomorphic equation for the envelope of $\bar{\mathcal{S}}$. The function $\bar{f}$ is, of course, holomorphic because $\bar{f}(\bar{Z}_\alpha) = \overline{f(Z^\alpha)}$ is anti-holomorphic in Z^α and therefore holomorphic in $\bar{Z}_\alpha$.) The surface $\bar{\mathcal{S}}$ gives rise to the congruence $\mathcal{R}^*$ in $\mathbb{M}$ in exactly the same way as $\mathcal{S}$ gives rise to $\mathcal{R}$. Moreover the null hypersurface $\bar{\mathcal{N}}$ corresponds to the family of lines in $\mathbb{P}\mathbb{T}$ which touch $\bar{\mathcal{S}}$.



We can now see how the timelike surface $\mathcal{T}$ arises from $\mathcal{S}$. We recall that $\mathcal{T} = \mathcal{N} \cap \bar{\mathcal{N}} \cap \mathbb{M}$ $(= \mathcal{N} \cap \mathbb{M})$, so the points of $\mathcal{T}$ correspond to the <u>lines</u> in $\mathbb{P}\mathbb{N}$ which touch <u>both</u> of $\mathcal{S}$ and $\bar{\mathcal{S}}$. Moreover, the <u>pair</u> of surfaces $(\mathcal{S}, \bar{\mathcal{S}})$ provides a better insight into $\mathcal{R}$ (and also $\mathcal{R}^*$) than either does individually, especially if one is concerned with the <u>complexification</u> $\mathbb{C}\mathcal{R}$ (or $\mathbb{C}\bar{\mathcal{R}}$).



A complex ray belonging to $\mathbb{C}\mathcal{R}$ is represented in $\mathbb{P}\mathbb{T}$ as a <u>plane pencil</u> whose vertex lies on $\mathcal{S}$ and whose plane touches $\mathcal{F}$. (In the case of $\mathbb{C}\mathcal{R}^*$, the vertex lies on $\mathcal{F}$ and the plane touches $\mathcal{S}$.)

Many of these considerations have some sort of analogue in curved (analytic) space-time $\mathcal{M}$, but where some of the constructions have to be understood in terms of an analytic (preferably spacelike) hypersurface $\mathcal{H} \subset \mathcal{M}$ and the corresponding hypersurface twistor space $\mathbb{P}\mathcal{T}(\mathcal{H})$. The notions of a complex null hypersurface $\mathcal{N}$, of its (primary) caustic which is a complex null 2-surface $\mathcal{U}$, of its complex conjugate $\overline{\mathcal{N}}$, and the timelike 2-surface $\mathcal{T}$ of real points of $\mathcal{N}$ all carry through without change. However, the notion of shear-freeness for a ray congruence must be taken relative to $\mathcal{H}$ (i.e. at the intersections of the rays with $\mathcal{H}$). With this proviso, all the equivalences (1), (2), (3), (4), and (5) are still maintained and each of these structures can, in a certain sense, be thought of as representing a holomorphic 2-surface $\mathcal{S}(\mathcal{H})$ in $\mathbb{P}\mathcal{T}(\mathcal{H})$. If we consider the time-evolution of $\mathcal{H}$ to some later hypersurface $\mathcal{H}'$, so that the twistor space changes to $\mathbb{P}\mathcal{T}(\mathcal{H}')$, we find that the ray congruence $\mathcal{R}$ is not preserved in general. However, the complex null hypersurface $\mathcal{N}$ remains unchanged in the evolution, and this fixes the precise way in which $\mathcal{R}$ changes (observation due to George Sparling).

The fact that there is a clear-cut identification of a surface $\mathcal{S}(\mathcal{H}')$ in $\mathbb{P}\mathcal{T}(\mathcal{H}')$ with the original surface $\mathcal{S}(\mathcal{H})$ in $\mathbb{P}\mathcal{T}(\mathcal{H})$ is closely related to the fact that the cotangent bundle of $\mathbb{P}\mathcal{T}$ (essentially ambitwistor space) is preserved under evolution, $\mathcal{S}$ having a well-defined lift into the bundle.

Most of the details of the above account (though not its particular emphasis) are to be found in my article Twistor geometry of light rays, <u>Class. Quantum Grav.</u> 14 (1997) A299-A323, in honour of Andrzej Trautman. ~ Roger Penrose