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Initial Value Problem of the Colliding Plane Wave

Key-Yong, Tee
Mathematical Institute, Oxford

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In this paper we consider collision of two gravitational plane wave which has the same polarisation. We know that in the region of collision of the colliding plane wave, there admits two commuting killing vectors along the wave fronts, which are preserved from the two initial plane waves. Einsteins Field Equation, when reduced by two commuting and orthogonally transitive Killing vectors, which is the Yang's Equation, indicates a ASDYM gauge field in a corresponding subset **U** in complexified Minkowski Space **M** [1]. With the aid of Ward's Correspondence, a correspondence between the gauge field, and hence the metric function, and the vector bundle E over a reduced twistor space R can be established. In this paper, we would like to look at describing the metric function of the collision region using the initial metric data of the incoming wavefronts. The crucial point is to describe the vector bundle E over R using the metric functions on the initial surface.

1 Background Mathematical Concepts

We begin with the general metric for space-time which admits two orthogonally transitive Killing vectors.

$$ds^2 = -g_{xy}dxdy + \Omega^2 dudv \tag{1}$$

where g_{xy} are 2 by 2 matrix function of u and v, and Ω is a function of u and v. We write the matrix g_{xy} as $g = g(u, v)$. We use the coordinates (u, v, x, y). We have also that $\det g = (u + v)^2$. The Killing vectors are in this case $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$. ([1] and [4])

Using the coordinates above, we arrive at the collision region as shown as the figure (1). This figure obscures two coordinates, as we will find that

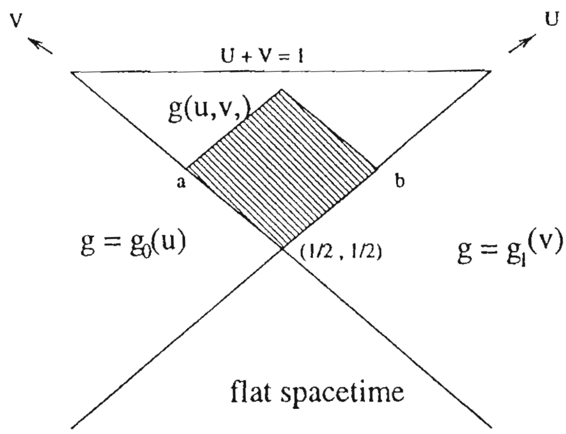

Figure 1: Colliding Gravitational Plane Wave

the essential part of the reduced Einsteins Equation are concern only on that two coordinates. Also, we focus on the highlighted region.

The Reduced Einsteins Field Equations are two equation on the space of orbits of the Killing Vector fields. The essential one is comformally invariantly and hence can be reduced to [1]

$$\partial_u \left(\tau g^{-1}\partial_v g\right) + \partial_v \left(\tau g^{-1}\partial_u g\right) = 0 \tag{2}$$

and the second equation can be solved once this is solved.

Equation (2) is the Yang's Equation. The equation on Complexified Minkowski Space $\mathbf{M}$ is a ASDYM equation on some open set $\mathbf{U} \in \mathbf{M}$ the Minkowski space. Here U is defined by the region of collision outside singularity. One can construct Φ_a the gauge field from matrix $g(u,v)$ which is ASD in $\mathbf{U}$ given equation (2).

One can also get equation (2) by allowing the gauge field Φ_a to be invariant along the two vectors $X = \frac{\partial}{\partial y}$ and $Y = \frac{\partial}{\partial \theta}$ of Minkowski space. One can then reduce the ASDYM equation by quotient with respect to the two commuting vectors, to obtain equation (2).

From $\mathbf{M}$, we define the correspondence space $\mathbf{F} = \mathbf{M} \times S_\zeta$, where $S_\zeta = \mathbf{CP}^1$. We define $F = (u, v, \mu) = M \times S_\mu$. Here we have $\mu = \zeta e^\theta$ and $M = \{(u,v) : (u,v,y,\theta) \in \mathbf{U}\}$. μ is an invariant spectral parameter with respect to the vectors X and Y.

Reduced twistor Space R is the quotient of $\mathbf{PT}$ with respect to the two vector fields in $\mathbf{PT}$, which we called X' and Y'. They are representation of the Lie Algebra of X and Y in $\mathbf{PT}$. It turns out that we can form R by the foliation of F by the two vector fields, V_0 and V_1. The constant of foliation

w satisfies

$$V_0 w = \left(2\partial_u + \frac{1}{\mu\tau}\left(\mu^2 - 1\right)\partial_\mu\right) w = 0 \tag{3}$$

$$V_1 w = \left(2\partial_v + \frac{\mu}{\tau}\left(\mu^2 - 1\right)\partial_\mu\right) w = 0 \tag{4}$$

solving yields

$$w = \frac{u + v\mu^2}{1 - \mu^2} \tag{5}$$

We can see that $\{w\} = \mathbf{CP}^1$. Details of this foliation can be found in [2].

Our assumption is that $\mathbf{U}$, and hence M is connected, as shown in figure (1). w can therefore either represent one point in R or two. Consider the region of interest from figure(1). We complexify the coordinates u and v. We make $M = U \times V$, where U and V are open, and $\left(a, \frac{1}{2}\right] \subset U$ and $\left(b, \frac{1}{2}\right] \subset b$. So w represent one point if

$$w = 2u \tag{6}$$

where $u \in U$, or

$$w = -2v \tag{7}$$

where $v \in V$. Since $a + b < 1$, we have that the two sets B_u and B_v, where $B_u = \{w : w = 2u, u \in U\}$ and $B_v = \{w : w = -2v, v \in V\}$, are mutually exclusive. Hence explain the non-Hausdorff structure of R. [2]

2 Mechanism of the Initial Value Problem

We define the incoming wave metric function as g_0 and g_1, such that

$$g\left(u, \frac{1}{2}\right) = g_0(u) \tag{8}$$

$$g\left(\frac{1}{2}, v\right) = g_1(v) \tag{9}$$

These are the metric functions before collision. (refer to figure (1). And we label the functions $B_0(u, w)$ and $B_1(v, w)$ as

$$B_0(u, w) = \frac{1}{2}\sqrt{\frac{w + 1}{w - 2u}} g_0^{-1} \frac{\partial g_0}{\partial u} \tag{10}$$

$$B_1(v, w) = \frac{1}{2}\sqrt{\frac{w - 1}{w + 2v}} g_1^{-1} \frac{\partial g_1}{\partial v} \tag{11}$$

Notice that the terms $\sqrt{\frac{w+1}{w-2u}}$ and $\sqrt{\frac{w-1}{w+2v}}$ are just $\frac{1}{\mu}$ and μ respectively when $v = \frac{1}{2}$ and $u = \frac{1}{2}$. We shall call them $a(w,u)$ and $b(w,v)$. These functions are are doubled valued, and contains a choice of branches.

Choose two base points $(u_0, \frac{1}{2})$ and $(\frac{1}{2}, v_0)$ in $U \times V$. These are points on one of each initial surfaces. We write down three matrices $P_1(w)$, $P_2(w)$ and $Q(w)$. The $P_i(w)$ matrices are of the form.

$$Pexp_{v=\frac{1}{2}}\left[-\int_{u_0\to\frac{1}{2}} B_0 du\right] Pexp_{u=\frac{1}{2}}\left[-\int_{\frac{1}{2}\to v_0} B_1 dv\right] \tag{12}$$

In this case, there are four choices considering the branches of $a(w,u)$ and $b(w,v)$ in the equations. Out of the four, they form pair of twos which are the same, due to the integrability condition. We label the two as $P_+(w)$ and $P_-(w)$. We put $Q(w)$ as

$$Pexp_{v=v_0}\left[-\int_{\gamma_1} B_0 du\right] \tag{13}$$

or

$$Pexp_{u=u_0}\left[-\int_{\gamma_1} B_1 dv\right] \tag{14}$$

The two integrals are the same, by the integrability condition.

We now put $w = \frac{u+v\mu^2}{1-\mu^2}$ into the matrices $P_+(w)$, $P_-(w)$ and $Q(w)$ to form $F_+(\mu)$, $F_-(\mu)$ and $G(\mu)$. Note that functions F_+ F_- and G are well-defined functions of μ.

The final three matrices are important. We would later carry out a Riemann-Hilbert Splitting to get the solution to Einsteins Field Equation.

3 Vector Bundle E over R

Ward's Correspondence, when reduced by the two vectors X and Y, leads to a correspondence between a vector bundle E over R and the solution of $g(u,v)$ on M. [2]. The definition of the vector bundle E is defined as the solution space of the Lax Pair, which are the pair of equations

$$L_0 s = \left\{2\partial_u + \frac{1}{\mu T}(\mu^2-1)\partial_\mu + (1-\frac{1}{\mu})g^{-1}g_u\right\} s = 0 \tag{15}$$

$$L_1 s = \left\{2\partial_v + \frac{\mu}{T}(\mu^2-1)\partial_\mu + (1-\mu)g^{-1}g_v\right\} s = 0 \tag{16}$$

over each foliation of constant w. The integrability of the Lax Pair is provided by the equation (2). Using the transformation $\mu \to w$, the Lax Pair becomes

$$L_0 s = \left\{\partial_u + \frac{1}{2}\left(1-\frac{1}{\mu}\right)g^{-1}g_u\right\} s = 0 \tag{17}$$

$$L_1 s \;=\; \left\{\partial_v + \frac{1}{2}(1-\mu)\, g^{-1} g_v\right\} s = 0 \tag{18}$$

with the double value function $\mu = \sqrt{\frac{w-2u}{w+2v}}$.

Local trivialisations of E over R can be formed by the following way:

Take the solution of the Lax Pair on two points, one on each of the initial surfaces, namely $(u_0, \frac{1}{2})$ and $(\frac{1}{2}, v_0)$. These choice can defined consistently a local trivialisation for a pair of open sets of R, which we called $X_\pm$ and $Y_\pm$ respectively. They can be represented by w in sets X and Y respectively, which are open subsets of w-$\mathbf{CP}^1 = S_w$. We have

$$X = S_w - \Gamma_X \tag{19}$$

$$Y = S_w - \Gamma_Y \tag{20}$$

Γ_X is a cut on from $w = 2u_0$ to $w = -1$ and Γ_Y is a cut on from $w = 1$ to $w = 2v_0$ inclusive. We choose the cuts so that Γ_X and Γ_Y do not cross. Note that $X \subset S_w$ represents two sets $X_\pm \subset S_\mu$, and similarly $Y \in S_w$ represents two sets $Y_\pm \in R$.

Transition functions are calculated by parallel propagating the solution vectors of the Lax Pair from one point to another along the leaf of constant w. Using equation (17) and (18), one can arrive at transition matrices by using path-ordered integrals. Integrability condition given by equation (2) further splits the rank two path-order integration further into a product of two path-order integral or rank one, one in the u complex plane and one in the v complex plane. Hence we get expressions of (12), (13) and (14). It can be seen that there are only three essential transition matrices to describe the bundle E over R with respect to the local trivialisations. They are $P_+(w)$, $P_-(w)$ and $Q(w)$ from the above.

The equation $w = \frac{u+v\mu^2}{1-\mu^2}$ induces a map $\pi : R \to S_\mu$. By the Ward's Correspondence, we know that the pull back bundle of E, $\pi^*(E)$ is trivial. Therefore, we have Riemann-Hilbert problem on the transition functions $F_\pm$ and G of the pull-back bundle $\pi^*(E)$ over S_μ.

4 Riemann-Hilbert Problem

Let $T_\pm = \pi^{-1}(X_\pm)$ and $W_\pm = \pi^{-1}(Y_\pm)$. We know that on on $T_+ \cap W_+$, $F_+(\mu)$ splits like

$$F_+(\mu) = K_W(\mu)^{-1} K_T(\mu) \tag{21}$$

where $K_W(\mu)$ is holomorphic in W_+ and $K_T(\mu)$ is holomorphic in T_+. Also on $T_- \cap W_-$, $F_-(\mu)$ splits like

$$F_-(\mu) = L_W(\mu)^{-1} L_T(\mu) \tag{22}$$

where $L_W(\mu)$ is holomorphic in W_- and $L_T(\mu)$ is holomorphic in T_-.

Now the transition function $M(\mu)$ on $(T_+ \cup W_+) \cap (T_- \cup W_-)$ can be constructed, which splits as

$$M(\mu) = L_T(\mu)G(\mu)K_T^{-1}(\mu) = H_B^{-1}(\mu)H_A(\mu) \tag{23}$$

Finally, we put

$$g(u,v) = \left[H_B^{-1}(-1)H_A(+1)\right]\big|_{\sigma=(u,v)} \tag{24}$$

It can be checked that $g(u,v)$ satisfies equation (2), and hence is a solution of the reduced Einsteins Field Equation.

The whole procedure shows that, in the case of colliding plane wave, it is theoretically possible to derived from the data of the two incoming wave. The metric of the collision region shown in figure (1) is derived from the initial data from metric $g_0(u)$ and $g_1(v)$, which are taken from the initial surfaces of collision.

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