Divergence-free geodesic congruences in R^3

Twistor theorists are familiar with the problem of finding shear-free null geodesic congruences in Minkowski space and its solution by the Kerr Theorem. Many will know the related problem of finding shear-free geodesic congruences in the (flat) Euclidean space of three dimensions, R^3, which in turn is solved by a 'mini-Kerr' theorem (see e.g. my §II.1.12 and §II.1.13 in *Further advances in twistor theory vol II.*) Here I want to consider a similar-sounding problem, namely that of finding all divergence-free geodesic congruences in R^3. This problem may be thought of arising from a very degenerate case of the steady Euler equations for an incompressible fluid. In terms of the fluid velocity vector **u** and pressure p these are

$$(\mathbf{u}.\nabla)\mathbf{u} = -\nabla p \quad \text{and} \quad \nabla.\mathbf{u} = 0 \tag{1}$$

The degenerate case I have in mind is p = constant, which is not interesting for a fluid mechanic! Then (1) collapses to the geodesic equation for a divergence-free **u** and **u.u** is constant along the flow. By rescaling **u** we can take it to be a unit vector, then any other solution is obtained by rescaling with a function constant along **u**. The problem is therefore to solve

$$(\mathbf{u}.\nabla)\mathbf{u} = 0; \quad \nabla.\mathbf{u} = 0; \quad \mathbf{u}.\mathbf{u} = 1 \tag{2}$$

I claim the solutions are given by the following proposition:

Proposition

The solutions of (2) fall into two classes:

(a) for one class choose a curve **x**(s) parametrised by path length s and with its Serret-Frenet frame (**t**,**n**,**b**); for each s take the normal plane orthogonal to **t**; in the normal plane take the straight-line congruence parallel to **b**. This is the desired congruence.

(b) in terms of standard Cartesian coordinates (x,y,z) the vector field **u** is given by

$$\mathbf{u} = (\cos(f(z)), \sin(f(z)), 0)$$

for arbitrary f(z).

I have two rather 'bare-handed' ways of proving this. However the result, on the one hand, looks like something which should be in Darboux and, on the other hand, looks as if there should be a mini-twistor-space proof.

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