'Special' Einstein-Weyl spaces

In this note, I will describe a class of 'special' three-dimensional Einstein-Weyl spaces which turned up in joint work with Chave and Valent [1] and Gauduchon [2]. They can be thought of as coming from hyper-Kähler metrics with a tri-holomorphic *homothetic* vector field (recall this is a vector field such that the Lie derivative of the metric along it is a constant but non-zero multiple of the metric). Determination of them reduces to solving a single, second-order, non-linear PDE, equation (11) below, which must, for the usual reasons, be completely integrable.

We start by considering a system of equations: suppose e_1, e_2, e_3 and a are 1-forms on R^3 and K is a function on R^3 and consider the system:

$$\begin{aligned} de_1 &= a \wedge e_1 + Ke_2 \wedge e_3 \\ de_2 &= a \wedge e_2 + Ke_3 \wedge e_1 \\ de_3 &= a \wedge e_3 + Ke_1 \wedge e_2 \end{aligned} \qquad 1$$

I claim that, given a solution of this system, the metric h defined by

$$h = (e_1)^2 + (e_2)^2 + (e_3)^2, \qquad 2$$

together with the 1-form a, defines an Einstein-Weyl structure on R^3 (recall this is a pair consisting of a conformal metric, here h, and a torsion-free connection which preserves the conformal metric; here a defines the difference between this connection, call it the Weyl connection, and the Levi-Civita connection of h; finally the Ricci tensor of the Weyl connection is proportional to h).

Furthermore, for this EW space, the scalar curvature of the Weyl connection is

$$S = 3K^2/2 \qquad 3$$

This is not too difficult to see. Also it draws attention to a conformal rescaling invariance in the system (1).

Now it is natural to ask whether every (three-dimensional) EW space can be written like this. Certainly we must have the scalar curvature positive by (3), but there are more conditions which follow as integrability conditions for (1). Differentiating again in (1), we find a necessary condition which is the vanishing of the quantity Q defined by

$$Q = dK + aK + {*}da \qquad 4$$

where the $*$ is dual w.r.t. the volume defined by h. I have written the condition in this way because it might appear that more conditions can arise by differentiating Q. This isn't the case because of the Bianchi identity for the Weyl connection which here, and with (3), is just

$$dQ - a \wedge Q - K{*}Q = 0 \qquad 5$$

Thus the system (1) defines EW spaces with an extra condition on the curvature embodied in (4). This is like a harmonic condition on K which is $S^{1/2}$ (up to factors) but rather than complicate nomenclature (?harmonic curvature ?harmonic square-root-of-curvature) I will just call these 'special' Einstein-Weyl spaces. They turned up in work I did with Thierry Chave and Galliano Valent [1] and with Paul Gauduchon [2] and more about them can be found in those references (in particular, the only compact example is the Berger sphere as an EW space [2]). They come from 4-dimensional hyper-Kähler metrics with a homothetic vector field which preserves all the complex structures. In terms of the objects above, the 4-metric is

$$e^{\hat{t}}[Kh + K^{-1}(d\hat{t} + a)^2] \qquad 6$$

where $\hat{t}$ is the group parameter.

By the usual folk-lore, the system (1) must be 'integrable' and the field equations for the metric (6) should reduce to an integrable equation (much as the hyper-Kähler metric with non-triholomorphic Killing vector reduces to the SU(∞)-Toda field equation). I have a way of reducing it to a single equation which I shall now explain, I haven't seen the resulting equation before and I would be glad to hear from anyone who recognises it.

To start the reduction of (1), define

$$e = e_2 + ie_3$$

then the second and third of (1) may be written

$$de = (a + iKe_1) \wedge e \qquad 7$$

This implies $e = WdZ$ for complex functions W and Z and then

$$a + iKe_1 = d\log W + fdZ \qquad 8$$

for some complex f. Take real and imaginary parts:

$$W = re^{it}; \quad f = p + iq \; ; \; Z = x + iy$$

and then (8) implies

$$a = d\log r + p\,dx - q\,dy \; ; \; Ke_1 = dt + q\,dx + p\,dy \qquad 9$$

The metric h has become

$$h = K^{-2}(dt + q\,dx + p\,dy)^2 + r^2(dx^2 + dy^2) \qquad 10$$

and the remaining equation is the first of the system (1). This is simplified by introducing the variables V (real) and S (complex) by

$$V = (rK)^{-1} \; ; \; S = V(p + iq)$$

when it becomes the pair of equations

$$S_t + iS - 2iV_z = 0; \; 1 + S\bar{S} - V(S_{\bar{z}} + \bar{S}_z) = 0 \qquad 11$$

This is the final result: since we can use the second of (11) to eliminate V from the first, this is effectively a single equation for the complex function S. It is second order in x and y and first order in t, and naively the data is S at t=0, so that special EW spaces depend on 2 free functions of 2 variables (a general EW depends on 4 such functions). The t-independent solution turns out to be the Berger sphere.

So ... what is equation (11)?

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references

[1] *(4,0) and (4,4) sigma models with a tri-holomorphic Killing vector* T.Chave, G.Valent and K.P.Tod Phys.Lett. B383 (1996) 262-270

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