Integrable flows on moduli of rational curves with normal bundle $\mathcal{O}^A(n)$

Maciej Dunajski Lionel J. Mason

Let $\mathcal{M}$ be a four-dimensional oriented complex manifold with null coordinates $x^{AA'} = (x^A, w^A)$ and let Θ be a complex-valued function on $\mathcal{M}$. In the last TN, [1], we showed how to embed the Plebański's second form of the ASD vacuum equations

$$\frac{\partial^2\Theta}{\partial w^A \partial x_A} + \frac{1}{2}\frac{\partial^2\Theta}{\partial x^B\partial x^A}\frac{\partial^2\Theta}{\partial x_B\partial x_A} = 0 \tag{1}$$

into an infinite system of overdetermined PDEs

$$\partial_{Ai}\partial_{Bj-1}\Theta - \partial_{Bj}\partial_{Ai-1}\Theta + \partial_{Ai-1}\partial^C{}_{0'}\Theta\partial_{C0'}\partial_{Bj-1}\Theta. \tag{2}$$

Here $\Theta = \Theta(x^{AA'_1...A'_n})$ is a function on a $2n+2$ dimensional manifold $\mathcal{N} = \mathcal{M}\times X$, with coordinates $x^{AA'_1...A'_n}$ or, alternatively[1],

$$x^{Ai} = \binom{n}{i} x^{AA'_1A'_2...A'_n} o_{A'_1}...o_{A'_i}\iota_{A_{i+1}}...\iota_{A_n}(-1)^{n-i}$$

where, for $i = 0, 1$, $x^{Ai} = x^{AA'}$ are coordinates on $\mathcal{M}$ and for $i > 1$ $x^{Ai} = t^{Ai}$ are coordinates on the space of parameters, X.

System (2) arises as the compatibility conditions for the $2n$ operators

$$L_{A(A'_2...A'_n)} = \pi^{A'_1} D_{AA'_1(A'_2...A'_n)}, \tag{3}$$

i.e. the condition that they commute. Here $D_{AA'_1(A'_2...A'_n)}$ are $4n$ vector fields on $\mathcal{N}$. Let

$$\nabla_{A(A'_1A'_2...A'_n)} = D_{A(A'_1A'_2...A'_n)}.$$

In the coordinate system above we have

$$D_{A0'(A'_2...A'_n)} = \partial_{A0'A'_2...A'_n}, \quad D_{A1'(A'_2...A'_n)} = \partial_{A1'A'_2...A'_n} + [\partial_{A0'A'_2...A'_n}, V]$$

so that

$$\nabla_{Ai} = \partial_{Ai} + \frac{i}{n}[\partial_{Ai-1}, V], \quad L_{Aj} = -\lambda(\partial_{Aj} + [\partial_{Aj-1}, V]) + \partial_{Aj-1} \tag{4}$$

where $V = \varepsilon_{AB}\partial\Theta/\partial x_A \partial/\partial x_B$. Note that $L_{A1} = L_A$ is the Lax pair for equation (1).

[1] Generally, up to the given combinatorial factor, the index i is related to a totally symmetric primed spinor by contraction of all the primed indices: the first i with $o_{A'}$ and the rest with $\iota_{A'}$.

The corresponding projective twistor space $\mathcal{PT}$ is obtained by factoring the spin bundle $\mathcal{N} \times \mathbb{CP}^1$ by the twistor distibution L_{Ai}. The twistor space is still three-dimensional, however it has a different topology from the one used in the Nonlinear-Graviton construction; the rational curves corresponding to points of $\mathcal{N}$ have normal bundle $\mathcal{O}^A(n) = \mathcal{O}(n) \oplus \mathcal{O}(n)$.

The aim of this note is to start from the natural structures on $\mathcal{PT}$ and show that $\mathcal{N}$, the moduli space of rational curves in $\mathcal{PT}$, is equipped with a function Θ satisfying (2) and with $V_{AA'_2...A'_{n-1}}$, the antisymmetric part of $D_{AA'_1...A'_n}$. This completes the picture of [1], where (2) was derived from space time arguments.

Proposition 1 *Let $\mathcal{PT}$ be a 3 dimensional complex manifold with the following structures*

1) a projection $\mu : \mathcal{PT} \longrightarrow \mathbb{C}P^1$,

2) a $2(n+1)$-dimensional family of sections with normal bundle $\mathcal{O}^A(n)$,

3) a nondegenerate 2-form Σ on the fibres of μ, with values in the pullback from $\mathbb{CP}^1$ of $\mathcal{O}(2n)$

and let $\mathcal{N}$ be the moduli space of sections from (2). Then

a) There exists a function $\Theta : \mathcal{N} \longrightarrow \mathbb{C}$ such that (with the appropriate coordinatisation) equation (2) is satisfied

b) The moduli space $\mathcal{N}$ of sections is equipped with the following structures

- *factorisation of the tangent bundle*

$$T\mathcal{N} = S^A \otimes \odot^n S^{A'},$$

- *a $2n$-dimensional distribution on the 'spin bundle' $D \subset T(\mathcal{N} \times \mathbb{CP}^1)$ that is tangent to the fibres of* **r** *over $\mathbb{CP}^1$ and, as a bundle on $\mathcal{N} \times \mathbb{CP}^1$ has an identification with $\mathcal{O}(-1) \otimes S_{AA'...A'_{n-1}}$ so that the linear system can be written as in equation (3).*

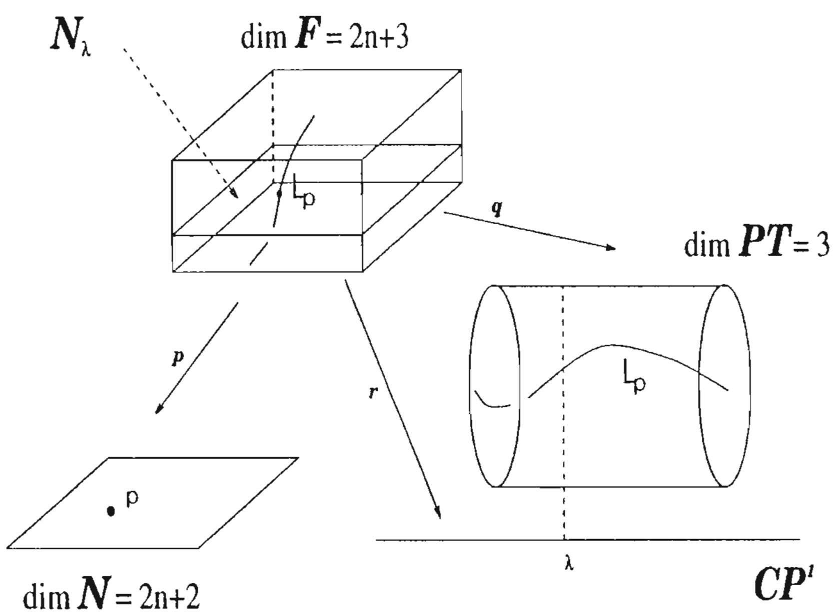
Proof of a). We use a double fibration picture (Figure 1) . First define homogeneous coordinates on $\mathcal{PT}$. These are coordinates on $\mathcal{T}$, the pullback from $\mathbb{CP}^1$ of the tautological line bundle $\mathcal{O}(-1)$. Let $\pi_{A'}$ be homogeneous coordinates on $\mathbb{CP}^1$ pulled back to $\mathcal{T}$ and let ω^A be local coordinates on $\mathcal{T}$ chosen on a neighbourhood of $\mu^{-1}\{\pi_{0'} = 0\}$ that are homogeneous of degree n and canonical so that $\Sigma = \varepsilon_{AB} d\omega^A \wedge d\omega^B$. We also use $\lambda = \pi_{0'}/\pi_{1'}$ as an affine coordinate on $\mathbb{CP}^1$. Let L_p be the line in $\mathcal{PT}$ that corresponds to $p \in \mathcal{N}$ and let $Z \in \mathcal{PT}$ lie on L_p. We denote by $\mathcal{F}$ the correspondence space $\mathcal{PT} \times \mathcal{N}|_{Z \in L_p} = \mathcal{N} \times \mathbb{CP}^1$. We can pull back the twistor coordinates to $\mathcal{F}$ and choose $2(n+1)$ coordinates on $\mathcal{N}$ by

$$x^{A(A'_1 A'_2 ... A'_n)} := \frac{\partial^n \omega^A}{\partial \pi_{A'_1} \partial \pi_{A'_2} ... \partial \pi_{A'_n}} \Big|_{\pi_{A'_i} = o_{A'}}.$$

where the derivative is along the fibres of $\mathcal{F}$ over $\mathcal{N}$. This can alternatively be expressed by expanding the coordinates ω^A pulled back to $\mathcal{F}$ in λ as follows

$$\omega^A = \sum_{i=0}^{n} x^{Ai} \lambda^{n-i} + \lambda^{n+1} \sum_{i=0}^{n-1} s_i^A \lambda^i + \dots \ , \tag{5}$$

Figure 1: Double fibration.



where s_i^A and terms of order higher than $2n$ can be expressed as functions of $x^{AA'\dots A'_n}$. The symplectic 2-form Σ on the fibres of μ, when pulled back to the spin bundle, has expansion in powers of λ that truncates at order $2n+1$ by globality and homogeneity so that

$$\Sigma = d\omega_A \wedge d\omega^A = \pi_{A'_1}\dots\pi_{A'_n}\pi_{B'_1}\dots\pi_{B'_n}\Sigma^{A'_1\dots A'_n B'_1\dots B'_n}$$

for some spinor indexed 2-form $\Sigma^{A'_1\dots A'_n B'_1\dots B'_n}$. We have

$$\Sigma(\lambda)\wedge\Sigma(\lambda) = 0, \qquad d\Sigma(\lambda) = 0. \tag{6}$$

If we express the forms in terms of the x^{Ai} and the s_i^A, the closure condition is satisfied identically, whereas the algebraic relations and truncation condition will give rise to integrability conditions on the s_i^A allowing one to express them in terms of a function $\Theta(x^{AA'\dots A'_n})$ and to field equations on Θ. Set

$$\Sigma^i = \binom{2n}{i}\Sigma^{A'_1\dots A'_n B'_1\dots B'_n} o_{A'_1}\dots o_{A'_i}\iota_{A_{i+1}}\dots\iota_{B_{2n}}(-1)^{2n-i},$$

so that

$$\Sigma = \sum_{i=0}^{2n}\lambda^{2n-i}\Sigma^i$$

where

$$\begin{aligned}\Sigma^j &= \sum_{i=j-n}^{n}\varepsilon_{AB}dx^{Ai}\wedge dx^{Bj-i} \quad \text{for } j \geq n\\ &= \sum_{i=0}^{j}\varepsilon_{AB}dx^{Ai}\wedge dx^{Bj-i} + \sum_{i=j+1}^{n}\varepsilon_{AB}dx^{Ai}\wedge ds^B_{i+j-1} \quad \text{for } j < n\,.\end{aligned}$$

The algebraic condition in (6) implies

$$\sum_{i=\max\{0,k-2n\}}^{\min\{2n,k\}} \Sigma^i \wedge \Sigma^{k-i} = 0 \qquad \text{for } 0 \leq k \leq 4n. \tag{7}$$

To deduce the existence of $\Theta(x^{AA'_1\cdots A'_n})$ equate the λ^{2n+1} coefficient in $d\omega^A \wedge d\omega_A$ to 0 (vanishing of higher terms gives recursion relations). This yields

$$\sum_{j=0}^{n}\sum_{i=0}^{n-1} \frac{\partial s_{Ai}}{\partial x^{Bj}} dx^{Bj} \wedge dx^{Ai} = 0 \longrightarrow s_{Ai} = \frac{\partial \Theta}{\partial x^{Ai}}. \tag{8}$$

To derive the hierarchy of flows consider the identity $\Sigma^0 \wedge \Sigma^0 = 0$ which is a particular case of (7).

$$\begin{aligned} 0 &= 2\sum_{j=1}^{n}\sum_{i=1}^{n} \varepsilon_{AB}\varepsilon_{CD} dx^C \wedge dx^D \wedge dx^{Ai} \wedge dx^{Bj} \frac{\partial^2\Theta}{\partial x^{Ai-1}\partial x^{Bj}} \\ &+ \sum_{i=1}^{n}\sum_{k=0}^{n}\sum_{j=1}^{n}\sum_{l=0}^{n} dx^{Ai} \wedge dx^{Ck} \wedge dx^{Bj} \wedge dx^{Dl} \frac{\partial^2\Theta}{\partial x^{Ai-1}\partial x^{Ck}} \frac{\partial^2\Theta}{\partial x^{Bj-1}\partial x^{Dl}}. \end{aligned}$$

Equate the coefficient of $\varepsilon_{AB} dx_C \wedge dx^C \wedge dx^{Ai} \wedge dx^{Bj}$ to 0 (i.e. the term with $k = l = 0$). This yields (2). One can show that all the other equations in (7) are satisfied trivialy.

□

Proof of b). The isomorphism $T\mathcal{N} = S^A \otimes \odot^n S^{A'}$ follows simply from the structure of the normal bundle; from Kodaira theory, since the appropriate obstruction groups vanish, we have

$$T_p\mathcal{N} = \Gamma(N_p, L_p) = S_p^A \otimes \odot^n S_p^{A'}$$

where N_p is the normal bundle to the rational curve L_p in $\mathcal{PT}$ corresponding to the point $p \in \mathcal{N}$. The bundle S^A on space-time is the Ward transform of $\mathcal{O}(-n) \otimes T_V\mathcal{PT}$ where the subscript V denotes the subbundle of the tangent bundle consisting of vectors up the fibres of the projection to $\mathbb{CP}^1$, so that $S_p^A = \Gamma(\mathcal{O}(-n) \otimes T_V\mathcal{PT}, L_p)$. The bundle $S_p^{A'} = \Gamma(\mathcal{O}, L_p)$ is canonically trivial. Let $\nabla_{AA'_1\cdots A'_n} = \nabla_{A(A'_1\cdots A'_n)}$ be the indexed vector field that establishes the above isomorphism and let $e^{AA'_1\cdots A'_n} = e^{A(A'_1\cdots A'_n)} \in \Omega^1 \otimes S^A \otimes \odot^n S^{A'}$ be the dual (inverse) map.

We now wish to derive the form of the linear system, equation (3). For each fixed $\pi_{A'} = (\lambda, 1) \in \mathbb{C}P^1$ we have a copy of a space-time $\mathcal{N}_\lambda$. A vertical (with respect to the projection r) subspace of $T_p(\mathcal{N}_\lambda)$ at a point p is spanned by $\nabla_{A(A'\ldots A'_n)}$. A normal bundle to the corresponding line L_p consists of vectors tangent to p modulo the twistor distribution. Therefore we have a sequence of sheaves over $\mathbb{C}P^1$

$$0 \longrightarrow D \longrightarrow T\mathcal{N} \xrightarrow{e^A} \mathcal{O}^A(n) \longrightarrow 0.$$

The map $T\mathcal{N} \longrightarrow \mathcal{O}^A(n)$ is given by the contraction of $T\mathcal{N}$ with $e^A = e^{A(A'\ldots A'_n)}\pi_{A'}\ldots\pi_{A'_n}$ since e^A annihilates all L_{Bi}s in D. Consider the dual sequence

$$0 \longrightarrow \mathcal{O}_A(-n) \longrightarrow T^*\mathcal{N} \longrightarrow D^* \longrightarrow 0$$

and tensor it with $\mathcal{O}(-1)$ to obtain

$$0 \longrightarrow \mathcal{O}_A(-n-1) \longrightarrow T^*\mathcal{N}(-1) \longrightarrow D^*(-1) \longrightarrow 0. \tag{9}$$

From here we would like to extract the Lax distribution

$$L_{AA'_2...A'_n} = \pi^{A'_1} D_{AA'_1A'_2...A'_n} \in S_{AA'_1...A'_n} \otimes \mathcal{O}(1) \otimes D.$$

This can be achieved by globalising (9) in $\pi^{A'}$. The corresponding long exact sequence of cohomology groups yields

$$0 \longrightarrow \Gamma(\mathcal{O}_A(-n-1)) \longrightarrow \Gamma(T^*\mathcal{N}(-1)) \longrightarrow \Gamma(D^*(-1)) \stackrel{\delta}{\longrightarrow} H^1(\mathcal{O}_A(-n-1))$$
$$\longrightarrow H^1(T^*\mathcal{N}(-1)) \longrightarrow ...$$

which (because $T^*\mathcal{N}$ is a trivial bundle so that $\mathcal{O}(-1) \otimes T^*\mathcal{N}$ has no sections or cohomology) reduces to

$$0 \longrightarrow \Gamma(D^*(-1)) \stackrel{\delta}{\longrightarrow} H^1(\mathcal{O}_A(-n-1)) \longrightarrow 0. \tag{10}$$

From the standard formulae

$$H^0(\mathbb{C}P^1, \mathcal{O}(k)) = \begin{cases} 0 & \text{for } k < 0 \\ \mathbb{C}^{k+1} & \text{for } k \geq 0. \end{cases}$$

$$H^1(\mathbb{C}P^1, \mathcal{O}(-k-2) = H^0(\mathbb{C}P^1, \mathcal{O}(k))^*$$

we conclude, since D has rank $2n$, that the connecting map δ is an isomorphism

$$\delta : \Gamma(D^*(-1)) \longrightarrow S_{AA'...A'_{n-1}}.$$

Therefore

$$\delta \in \Gamma(D \otimes \mathcal{O}(1) \otimes S_{AA'...A'_{n-1}}) \tag{11}$$

is a canonically defined object anihilating ω^A given by (5). A coordinate calculation shows that δ can be put into the form (3).

□

Remarks

(i) By examining the relevant sheaf cohomology groups and using Kodaira deformation theory, we can show that, analogous to the four-dimensional case, flat $\mathcal{PT}$ (ie $\Theta = 0$) admits complex deformations preserving (1)-(3).

(ii) For n odd $T\mathcal{N}$ is equipped with a metric with holonomy SL$(2,\mathbb{C})$. For n even, $T\mathcal{N}$ is endowed with a skew form. They are both given by

$$G(U,V) = \varepsilon_{AB}\varepsilon_{A'_1B'_1}...\varepsilon_{A'_nB'_n}U^{AA'_1...A'_n}V^{BB'_1...B'_n}. \tag{12}$$

(iii) If one considers $\mathcal{N} = \mathcal{M} \times \mathcal{X}$ as being foliated by four dimensional slices $t^{Ai} = const$ then structures (1)-(3) on $\mathcal{PT}$ induce an anti-self-dual vacuum metrics on the leaves of the foliation. Consider $\Theta(x^{AA'}, \mathbf{t})$ where $\mathbf{t} = \{t^{Ai}, i = 2...n\}$. For each fixed $\mathbf{t}$ the function Θ satisfies the 2nd heavenly equation. The ASD metric on a corresponding four-dimensional slice $\mathcal{N}_{\mathbf{t}=\mathbf{t}_0}$ is given by

$$ds^2 = \varepsilon_{AB} dx^{A1'} dx^{B0'} + \frac{\partial^2 \Theta}{\partial x^{A0'} \partial x^{B0'}} dx^{A1'} dx^{B1'}.$$

One would like to determine this metric from the structure of the $\mathcal{O}^A(n)$ twistor space.

If we fix $2n-2$ parameters in the expansion (5) then the normal vector $V = V^A \partial / \partial \omega^A$ is given by

$$V^A = \delta\omega^A = \lambda^{n-1} V^{A1'} + \lambda^n V^{A0'} + \lambda^{n+1} \frac{\partial \delta\Theta}{\partial x^{0'}_A} + ...$$

where $\delta\Theta = V^{AA'} \partial\Theta / \partial x^{AA'}$. The metric is

$$g(U, V) = \frac{\alpha^{C'} \beta_{C'}}{(\pi_{0'})^{2n-2} \alpha^{A'} \pi_{A'} \beta^{B'} \pi_{B'}} \Sigma(U(\pi^{D'}), V(\pi^{D'})). \qquad (13)$$

The last formula follows also from (12) if one puts

$$V^{AA'_1 ... A'_n} = V^{A(A'_1} o^{A'_2} .. o^{A'_n)}$$

for V tangent to $t^{Ai} = const.$ Note that it is sufficient to consider the slice $\mathbf{t} = 0$. This is because an appropriate (canonical) coordinate transformation of $\mathcal{PT}$ induces the transformation of parameters

$$\omega^A \to \hat{\omega}^A(\omega^B, \lambda) \quad \text{yields} \quad \{\mathbf{t} = \mathbf{t}_0\} \to \{\hat{\mathbf{t}} = 0\}.$$

(iv) From Merkulov's work [2] it follows that $\mathcal{N}$ is equipped with an affine connection. It would be interesting to relate his approach to ours and in particular to express the connection on $\mathcal{N}$ as a function of Θ.

Thanks to Paul Tod.

References

[1] Dunajski, M. & Mason, L.J. (1996) Heavenly Hierarchies and Curved Twistor Spaces, Twistor Newsletter 41.

[2] Merkulov, S (1995) Relative deformation theory and differential geometry, in *Twistor theory*, ed. S. Huggett, Lecture notes in pure and applied mathematics, **169**, Marcel Dekker.