Asymptotic Young Tableaux for Discrete Series

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We give an example, how to use Young tableaux for the decomposition of products of *positive discrete series* representations.
Consider $SU(2,1)$ for example. We label its positive discrete series representations $\gamma_{a,b}$ by two integers $a \geq 2$, $b \geq 0$ which give the weight of their *lowest* weight vector. The representations $\gamma_{a,0}$ are on a wall of the dominant Weyl chamber and do not, strictly speaking, fall under the discrete series. They are *ladder representations* which we can include without problems. We have to exclude the other type $\gamma_{1,b}$ of positive ladder representations though, because they do not satisfy the following *asymptotic* properties (for natural reasons):

Multiplicities of $\gamma_{a,b}$: The multiplicities (dimensions) of the weight spaces $V_{c,d}$ of the representation $\gamma_{a,b}$ are the *same* as those of $V_{n-c-d,d}$ in the representation $\Gamma_{n-a-b,b}$ for $n \to \infty$.

Here $\Gamma_{a,b}$ denotes the representation of the compact form $SU(3)$ of $SU(2,1)$ with *highest* weight (a,b). With such a representation we associate a Young tableau whose first row has $a+b$ boxes and whose second row has b boxes. Thus, if we want to decompose $\gamma_{a,b} \otimes \gamma_{c,d}$ say, we choose a large n, decompose $\Gamma_{n-a-b,b} \otimes \Gamma_{n-c-d,d}$ according to Littlewood-Richardson [1] and translate back, ignoring that n is finite. The representation with *highest* highest weight in this product will be $\Gamma_{2n-a-b-c-d,b+d}$ which has to match $\gamma_{a+b,c+d}$, since the *lowest* lowest weight of $\gamma_{a,b} \otimes \gamma_{c,d}$ is of course $(a+b, c+d)$. Thus, on the way back from representations of the compact group to discrete series, we have to choose the obvious matching parameter ($2n$ in this exaple). In practice one would determine the kind of representations occurring in $\Gamma_{n-a-b,b} \otimes \Gamma_{n-c-d,d}$ for $n \to \infty$ starting from a typical large enough n and translate back, ignoring the finiteness of n.

The above simple observation can be used to prove several conjectural formulas in [2]. Its proof and generalisations will be given elsewhere.

References:

[1] e.g.: Fulton and Harris: A first course in Representation Theory. GTM. Springer.

[2] F.Müller: Twistor Diagrams as Projection Operators. DPhil. Thesis. Oxford 1993.