Coherent Quantum Measurements

Timothy R. Field
D.R.A. Malvern, WR14 3PS, U.K.
email: trfield@dra.hmg.gb

This article continues the discussion of coherent states I give in TN 41 and I shall use the same notation and conventions as I use there in what follows. I apply the basic underlying geometry to motivate a result which concerns the role of coherent states in quantum mechanical measurement. In particular I argue that there exists a critical amplitude associated with a pair of coherent states, above which reduction from their closest quantum mechanical mean to the states themselves is favoured over reduction to their corresponding classical average.

Suppose we take $|\psi_1\rangle$ and $|\psi_2\rangle$ to be coherent state vectors based on ξ^a, $-\xi^a \in \mathcal{H}^1$ respectively according to the exponential map, as detailed in TN 41. The complex projective line joining $P|\psi_1\rangle$ and $P|\psi_2\rangle$ is a topological sphere on which these states are represented as points, and since any pair of coherent states has non-zero overlap, these points are never antipodal. They still define a unique circle C of states equidistant from $P|\psi_{1,2}\rangle$ whose state vectors are given by $\{|\psi_1\rangle + \lambda|\psi_2\rangle\}$ for $|\lambda| = 1$. We the define the abstract Hermitian field operator

$$F^a \equiv F^{\alpha,\alpha'} = A^\alpha \oplus C^{\alpha'}$$

which is the sum of the abstract annihilation and creation operators. The expectation of this operator in the general state $P[|\psi_1\rangle + \lambda|\psi_2\rangle]$ is given by

$$\langle F^a \rangle = \xi^\alpha[1 - |\lambda|^2 + e^{-\Lambda}(\bar{\lambda} - \lambda)] \oplus \bar{\xi}^{\alpha'}[1 - |\lambda|^2 + e^{-\Lambda}(\lambda - \bar{\lambda})]$$

where $\Lambda = \xi^\alpha \bar{\xi}_\alpha$. Thus on the circle C there are precisely two points at which $\langle F \rangle$ vanishes, namely at $\lambda = \pm 1$, and the choice $\lambda = +1$ gives the state closest to $P|\psi_{1,2}\rangle$. We shall focus attention on this case taking $|S\rangle$ to be $\frac{1}{2}(|\psi_1\rangle + |\psi_2\rangle)$ and defining also $P|C\rangle$ to be the coherent vacuum state. From Lemma 1 of TN 41 the state $P|S\rangle$ lies off the coherent state submanifold, and the situation is depicted below.

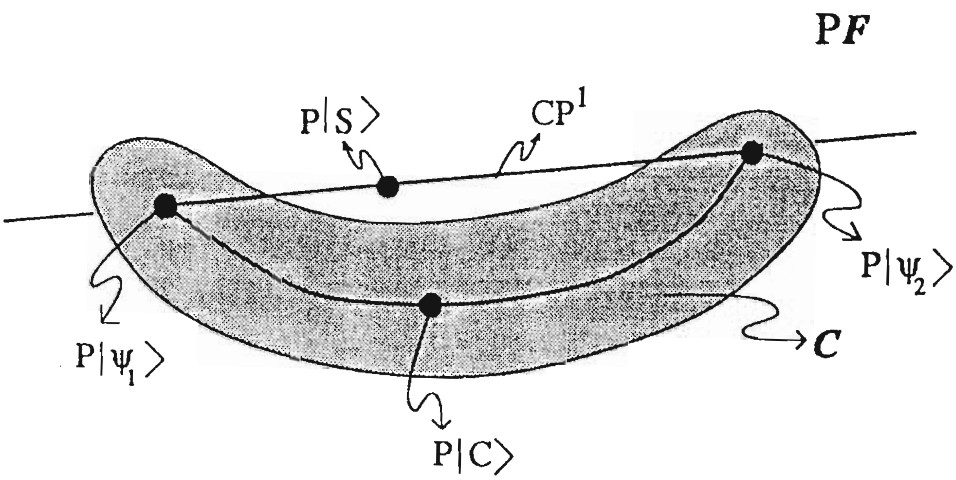

Fig. 1-1. Coherent state submanifold, embedded inside projective Fock space.

We analyze the comparitive distance d in the ambient Fubini-Study geometry, of $P|S\rangle$ from the vacuum and from $P|\psi_{1,2}\rangle$. An intriguing phenomenon arises in this regard. We have, with $\Lambda = \xi^\alpha \bar{\xi}_\alpha$, transition probabilities given by

$$\mathbf{P}[P|S\rangle \mapsto P|0\rangle] = \cos^2(\frac{1}{2}d[P|S\rangle, P|0\rangle]) = \frac{1}{\cosh\Lambda},$$

$$\mathbf{P}[P|S\rangle \mapsto P|\psi_{1,2}\rangle] = \cos^2(\frac{1}{2}d[P|S\rangle, P|\psi_{1,2}\rangle]) = e^{-\Lambda}\cosh\Lambda.$$

Thus for small values of $\Lambda > 0$ the probability of a transition from $P|S\rangle$ to the vacuum is greater than that for the coherent states $P|\psi_{1,2}\rangle$, and correspondingly $d[P|S\rangle, P|0\rangle] < d[P|S\rangle, P|\psi_{1,2}\rangle]$. Moreover for small values of Λ the vacuum state is the closest coherent state to $P|S\rangle$, as one can easily verify. Now if we allow Λ to increase, there exists a critical value Λ_0 for which the above distances are in fact equal, and for $\Lambda > \Lambda_0$ there is a cross-over with $d[P|S\rangle, P|0\rangle] > d[P|S\rangle, P|\psi_{1,2}\rangle]$. The critical value Λ_0 is given by the solution of $e^{\Lambda_0} = \cosh^2\Lambda_0$ which, setting $\Omega = e^{\Lambda_0}$, becomes

$$\Omega^4 - 4\Omega^3 + 2\Omega^2 + 1 \equiv (\Omega - 1)(\Omega^3 - 3\Omega^2 - \Omega - 1) = 0.$$

The solution $\Omega = 1$ clearly corresponds to the trivial case where all three states under consideration coincide at the vacuum. The cubic term above has a minimum at $\Omega = 1 + 2/\sqrt{3}$ and a root $\Omega_0 \in (3, 4)$. Thus the critical value of the amplitude $\Lambda_0 = \log\Omega_0$ lies in the range $1 < \Lambda_0 < \frac{3}{2}$. In physical terms this is telling us that 'collapse of the wavefunction' to $P|\psi_{1,2}\rangle$ is more likely to occur when the expectation of the total number operator $\hat{N}$ exceeds unity. With a photon field (a beam of coherent light) for example, the asymmetric collapse would occur when on average more than a single photon were present in the field. The result is notably independent of the underlying classical field theory or of any dimensional considerations.

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T.R. Field