Twistor Newsletter

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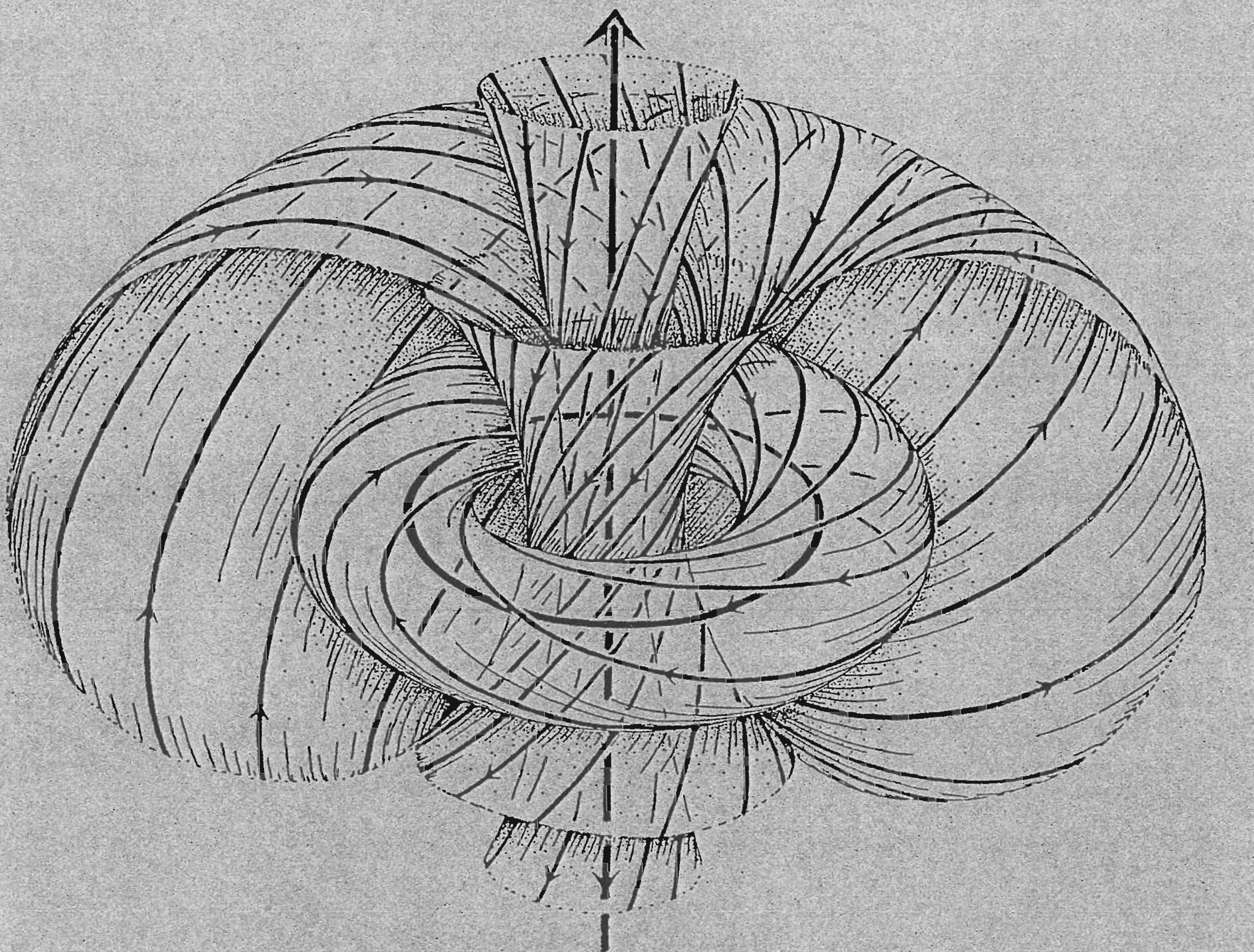

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Five things you can do with a surface in $\mathbb{PT}$

Consider Minkowski 4-space $\mathbb{M}$, embedded in its complexification $\mathbb{CM}$. The following five geometrical structures in $\mathbb{M}$, or $\mathbb{CM}$, are essentially equivalent:

(1) a shear-free ray congruence $\mathcal{R}$ in $\mathbb{M}$,

(2) an oriented timelike 2-surface $\mathcal{T}$ in $\mathbb{M}$ (ie an oriented string history);

(3) a complex null hypersurface $\mathcal{N}$ in $\mathbb{CM}$,

(4) a complex null 2-surface $\mathcal{U}$ in $\mathbb{CM}$,

(5) another shear-free ray congruence $\mathcal{R}^*$ in $\mathbb{M}$.

(NB: "ray" is a "null geodesic")

The word "essentially" in the above assertion is a little difficult to be precise about. In essence, what I mean by "essentially" here is a combination of the following three qualifications: (a) "in general", (b) "locally", and (c) "in the analytic case". However, we must be careful about the sense in which (b) is intended: certainly not local to a particular region of $\mathbb{CM}$; local in twistor space is more to the point. Moreover, we must also be a little cautious about the meaning of (a) in conjunction with (c). These matters should become clearer in a moment.

In short, the equivalence between (1), (2), (3), (4), and (5) arises from the fact that each provides a different space-time interpretation (in $\mathbb{M}$ or in $\mathbb{CM}$) of a holomorphic surface $\mathcal{S}$ in $\mathbb{PT}$. Before explaining this, it will clarify matters a little if I give the space-time relations between (1), (2), (3), (4), and (5) directly.

The equivalence between (2) and (3) is explained if we first think of a spacelike 2-surface instead of a timelike one. Such a surface is always (locally) the intersection of two real null hypersurfaces (generated by the rays meeting the 2-surface orthogonally). Complexifying this, we see that a complex (non-null) surface $\mathcal{T}$ is (locally) uniquely the intersection of two complex null hypersurfaces

$\mathcal{N}$ and $\tilde{\mathcal{N}}$. If $\mathcal{T}$ is to be real, the unordered pair of hypersurfaces $(\mathcal{N}, \tilde{\mathcal{N}})$ must go to itself under complex conjugation. There are two ways for this to happen: (i) $\mathcal{N} = \bar{\mathcal{N}}$ and $\tilde{\mathcal{N}} = \bar{\tilde{\mathcal{N}}}$ — when $\mathcal{T}$ is spacelike — and (ii) $\tilde{\mathcal{N}} = \bar{\mathcal{N}}$ — when $\mathcal{T}$ is timelike. Thus, a real timelike $\mathcal{T}$, being the real intersection of $\mathcal{N}$ with $\bar{\mathcal{N}}$, is simply the set of real points of $\mathcal{N}$. The choice of orientation of $\mathcal{T}$ corresponds to the selection of $\mathcal{N}$ rather than $\bar{\mathcal{N}}$; this orientation is reversed when $\mathcal{N}$ is replaced by $\bar{\mathcal{N}}$.

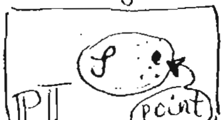
As for the complex 2-surface $\mathcal{U}$ of (4), it may be interpreted as the (primary) caustic of $\mathcal{N}$. Thus, $\mathcal{U}$ represents the locus of points of $\mathcal{N}$ at which generators of $\mathcal{N}$ intersect neighbouring generators of $\mathcal{N}$. Starting from the complex null 2-surface $\mathcal{U}$, we can generate $\mathcal{N}$ as the locus of rays which touch $\mathcal{U}$. To put this another way, noting that any regular point of $\mathcal{U}$ has a unique null tangent to $\mathcal{U}$ there, we find that (most of) $\mathcal{N}$ is generated by the rays through such points in the direction of these null tangents.

The relation between (1) and (2) is obtained through the fact that the real caustic of a (generic) shear-free ray congruence $\mathcal{R}$ in $\mathbb{M}$ is a timelike 2-surface $\mathcal{T}$. (For example, in the case of the "Kerr-black-hole" congruence $\mathcal{R}$ in $\mathbb{M}$, $\mathcal{T}$ describes the history of the ring singularity.) In fact, $\mathcal{T}$ is a secondary caustic of $\mathcal{R}$. The primary caustic of $\mathcal{R}$ — or, rather, of its complexification $\mathbb{C}\mathcal{R}$ — is the pair of complex null hypersurfaces $\mathcal{N}, \bar{\mathcal{N}}$, and this largely expresses the relation between (1) and (3). More completely, we can say that $\mathcal{R}$ is the family of real rays lying in those α-planes which contain a generator of $\mathcal{N}$. Equivalently (by complex conjugation), $\mathcal{R}$ consists of real rays in β-planes through generators of $\bar{\mathcal{N}}$. The relation between (1) and (4) can be directly obtained from the fact that $\mathcal{R}$ consists of the real rays in α-planes touching $\mathcal{U}$.

Finally, the ray congruence $\mathcal{R}^*$ of (5) is obtained simply by reversing the orientation of $\mathcal{T}$, i.e. by replacing $\mathcal{N}$ by $\bar{\mathcal{N}}$, so that

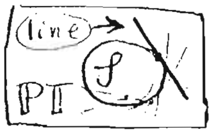
$\mathcal{R}^*$ consists of the real rays in β-planes containing generators of $\mathcal{N}$ — or, equivalently, touching $\mathcal{U}$ — or in α-planes containing generators of $\bar{\mathcal{N}}$. (In the case of a Kerr black hole congruence, $\mathcal{R}^*$ is related to $\mathcal{R}$ simply by having the opposite angular momentum, but in general $\mathcal{R}^*$ will look completely different from $\mathcal{R}$.)

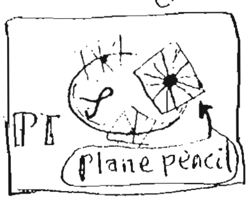
Let us now see how all this relates to a holomorphic surface $\mathcal{S}$ in $\mathbb{PT}$. The well-known Kerr theorem (see <u>Spinors</u> <u>and</u> <u>Space-time</u> Vol. 2, pp. 200-203) asserts that any real (analytic) shear-free ray congruence $\mathcal{R}$ in $\mathbb{M}$ consists of the rays represented by the intersection with $\mathbb{PN}$ (-I) of a holomorphic surface $\mathcal{S} \subset \mathbb{PT}$. Moreover, $\mathcal{S}$ is determined by $\mathcal{R}$. Thus (1) is equivalent to a holomorphic surface $\mathcal{S}$ in $\mathbb{PT}$. Closely related is the representation of the <u>whole</u> of $\mathcal{S}$, in <u>complex</u> space-time terms, as a 2-parameter family of α-planes in $\mathbb{CM}$ (where $\mathcal{R}$ consists of the real rays in these α-planes). This particular (complex) space-time description interprets



$\mathcal{S}$ as a system of <u>points</u>.

Another way of interpreting $\mathcal{S}$ is in terms of the <u>lines</u> which <u>touch</u> it. This is a 3-parameter family of lines. **These lines** correspond to the points of $\mathbb{CM}$ which lie on some hypersurface — in fact, the null hypersurface $\mathcal{N}$. The generators

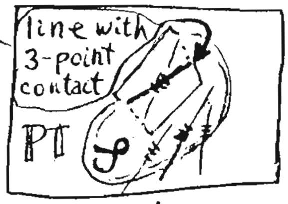




of $\mathcal{N}$ correspond to the <u>plane</u> <u>pencils</u> of lines touching $\mathcal{S}$ at fixed points. Thus, (3) is also equivalent to $\mathcal{S}$. We can also represent $\mathcal{S}$ in terms of the 2-complex parameter family of <u>planes</u> which <u>touch</u> $\mathcal{S}$. This gives a **2**-parameter family of

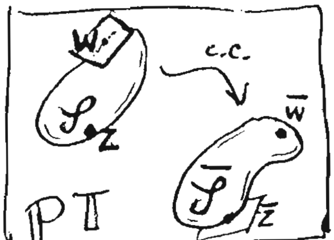
β-planes in $\mathbb{CM}$, and the intersections of these β-planes with $\mathbb{M}$ provide another shear-free ray congruence in $\mathbb{M}$, namely $\mathcal{R}^*$.

As for the complex null surface $\mathcal{U}$, its points correspond to <u>lines</u> in $\mathbb{PT}$ which have <u>three-point contact</u> with $\mathcal{S}$ (i.e. they are not simply tangent to $\mathcal{S}$, but touch it to higher order).



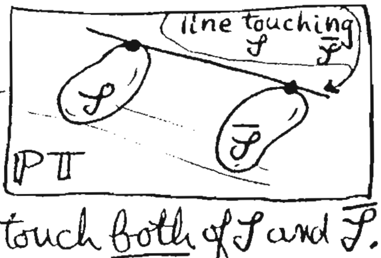
These lines constitute a 2-parameter family — giving $\mathcal{U}$ as a 2-surface. At a general point of $\mathcal{S}$, there are two such lines in the plane pencil of tangent lines there. Thus, a general generator of $\mathcal{N}$ meets $\mathcal{U}$ in two distinct points. This shows how (4) is equivalent to $\mathcal{S}$.

Let us return to $\mathcal{R}^*$. We can think of it as arising in another way. The 2-parameter family of planes touching $\mathcal{S}$ can be represented as a surface in $\mathbb{PT}^*$ which, by complex conjugation, provides us with a surface $\bar{\mathcal{S}}$ in $\mathbb{PT}$.

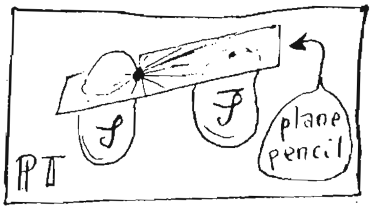


Thus, whenever the plane W touches $\mathcal{S}$, the point $\bar{W}$ lies on $\bar{\mathcal{S}}$. It follows that whenever a point Z lies on $\mathcal{S}$ then the plane $\bar{Z}$ touches $\bar{\mathcal{S}}$. We can call $\mathcal{S}$ and $\bar{\mathcal{S}}$ <u>reciprocal</u> surfaces in $\mathbb{PT}$. (Note that $\bar{\mathcal{S}}$ is actually a <u>holomorphic</u> surface in $\mathbb{PT}$, though its relationship to $\mathcal{S}$ is anti-holomorphic. If $\mathcal{S}$ is given by the relation $f(Z^\alpha)=0$, with f homogeneous and holomorphic, then $\bar{f}(W_\alpha)=0$ is a holomorphic equation for the envelope of $\bar{\mathcal{S}}$. The function $\bar{f}$ is, of course, holomorphic because $\bar{f}(\bar{Z}_\alpha) = \overline{f(Z^\alpha)}$ is anti-holomorphic in Z^α and therefore holomorphic in $\bar{Z}_\alpha$.) The surface $\bar{\mathcal{S}}$ gives rise to the congruence $\mathcal{R}^*$ in $\mathbb{M}$ in exactly the same way as $\mathcal{S}$ gives rise to $\mathcal{R}$. Moreover the null hypersurface $\bar{\mathcal{N}}$ corresponds to the family of lines in $\mathbb{PT}$ which touch $\bar{\mathcal{S}}$.

We can now see how the timelike surface $\mathcal{T}$ arises from $\mathcal{S}$. We recall that $\mathcal{T} = \mathcal{N} \cap \bar{\mathcal{N}} \cap \mathbb{M}$ $(= \mathcal{N} \cap \mathbb{M})$, so the points of $\mathcal{T}$ correspond to the <u>lines</u> in $\mathbb{PN}$ which touch <u>both</u> of $\mathcal{S}$ and $\bar{\mathcal{S}}$.



Moreover, the <u>pair</u> of surfaces $(\mathcal{S}, \bar{\mathcal{S}})$ provides a better insight into $\mathcal{R}$ (and also $\mathcal{R}^*$) than either does individually, especially if one is concerned with the <u>complexification</u> $\mathbb{C}\mathcal{R}$ (or $\mathbb{C}\bar{\mathcal{R}}$).



A complex ray belonging to $\mathbb{C}\mathscr{R}$ is represented in $\mathbb{PT}$ as a <u>plane pencil</u> whose vertex lies on $\mathscr{S}$ and whose plane touches $\mathscr{F}$. (In the case of $\mathbb{C}\mathscr{R}^*$, the vertex lies on $\mathscr{F}$ and the plane touches $\mathscr{S}$.)

Many of these considerations have some sort of analogue in curved (analytic) space-time $\mathcal{M}$, but where some of the constructions have to be understood in terms of an analytic (preferably spacelike) hypersurface $\mathscr{H} \subset \mathcal{M}$ and the corresponding hypersurface twistor space $\mathbb{P}\mathscr{T}(\mathscr{H})$. The notions of a complex null hypersurface $\mathcal{N}$, of its (primary) caustic which is a complex null 2-surface $\mathcal{U}$, of its complex conjugate $\overline{\mathcal{N}}$, and the timelike 2-surface $\mathcal{T}$ of real points of $\mathcal{N}$ all carry through without change. However, the notion of shear-freeness for a ray congruence must be taken relative to $\mathscr{H}$ (i.e. at the intersections of the rays with $\mathscr{H}$). With this proviso, all the equivalences (1), (2), (3), (4), and (5) are still maintained and each of these structures can, in a certain sense, be thought of as representing a holomorphic 2-surface $\mathscr{S}(\mathscr{H})$ in $\mathbb{P}\mathscr{T}(\mathscr{H})$. If we consider the time-evolution of $\mathscr{H}$ to some later hypersurface $\mathscr{H}'$, so that the twistor space changes to $\mathbb{P}\mathscr{T}(\mathscr{H}')$, we find that the ray congruence $\mathscr{R}$ is not preserved in general. However, the complex null hypersurface $\mathcal{N}$ remains unchanged in the evolution, and this fixes the precise way in which $\mathscr{R}$ changes (observation due to George Sparling).

The fact that there is a clear-cut identification of a surface $\mathscr{S}(\mathscr{H}')$ in $\mathbb{P}\mathscr{T}(\mathscr{H}')$ with the original surface $\mathscr{S}(\mathscr{H})$ in $\mathbb{P}\mathscr{T}(\mathscr{H})$ is closely related to the fact that the cotangent bundle of $\mathbb{P}\mathscr{T}$ (essentially ambitwistor space) is preserved under evolution, $\mathscr{S}$ having a well-defined lift into the bundle.

Most of the details of the above account (though not its particular emphasis) are to be found in my article Twistor geometry of light rays, <u>Class. Quantum Grav.</u> 14 (1997) A299-A323, in honour of Andrzej Trautman. ~ Roger Penrose

6

Initial Value Problem of the Colliding Plane Wave

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In this paper we consider collision of two gravitational plane wave which has the same polarisation. We know that in the region of collision of the colliding plane wave, there admits two commuting killing vectors along the wave fronts, which are preserved from the two initial plane waves. Einsteins Field Equation, when reduced by two commuting and orthogonally transitive Killing vectors, which is the Yang's Equation, indicates a ASDYM gauge field in a corresponding subset **U** in complexified Minkowski Space **M** [1]. With the aid of Ward's Correspondence, a correspondence between the gauge field, and hence the metric function, and the vector bundle E over a reduced twistor space R can be established. In this paper, we would like to look at describing the metric function of the collision region using the initial metric data of the incoming wavefronts. The crucial point is to describe the vector bundle E over R using the metric functions on the initial surface.

1 Background Mathematical Concepts

We begin with the general metric for space-time which admits two orthogonally transitive Killing vectors.

$$ds^2 = -g_{xy}dxdy + \Omega^2 dudv \tag{1}$$

where g_{xy} are 2 by 2 matrix function of u and v, and Ω is a function of u and v. We write the matrix g_{xy} as $g = g(u, v)$. We use the coordinates (u, v, x, y). We have also that $\det g = (u + v)^2$. The Killing vectors are in this case $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$. ([1] and [4])

Using the coordinates above, we arrive at the collision region as shown as the figure (1). This figure obscures two coordinates, as we will find that

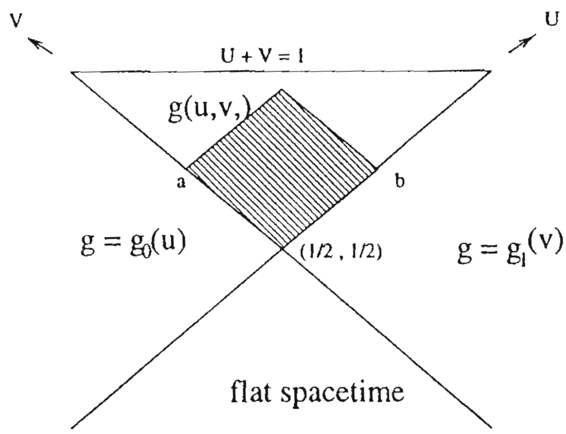


Figure 1: Colliding Gravitational Plane Wave

the essential part of the reduced Einsteins Equation are concern only on that two coordinates. Also, we focus on the highlighted region.

The Reduced Einsteins Field Equations are two equation on the space of orbits of the Killing Vector fields. The essential one is comformally invariantly and hence can be reduced to [1]

$$\partial_u \left(\tau g^{-1}\partial_v g\right) + \partial_v \left(\tau g^{-1}\partial_u g\right) = 0 \tag{2}$$

and the second equation can be solved once this is solved.

Equation (2) is the Yang's Equation. The equation on Complexified Minkowski Space $\mathbf{M}$ is a ASDYM equation on some open set $\mathbf{U} \in \mathbf{M}$ the Minkowski space. Here U is defined by the region of collision outside singularity. One can construct Φ_a the gauge field from matrix $g(u,v)$ which is ASD in $\mathbf{U}$ given equation (2).

One can also get equation (2) by allowing the gauge field Φ_a to be invariant along the two vectors $X = \frac{\partial}{\partial y}$ and $Y = \frac{\partial}{\partial \theta}$ of Minkowski space. One can then reduce the ASDYM equation by quotient with respect to the two commuting vectors, to obtain equation (2).

From $\mathbf{M}$, we define the correspondence space $\mathbf{F} = \mathbf{M} \times S_\zeta$, where $S_\zeta = \mathbf{CP}^1$. We define $F = (u, v, \mu) = M \times S_\mu$. Here we have $\mu = \zeta e^\theta$ and $M = \{(u,v) : (u,v,y,\theta) \in \mathbf{U}\}$. μ is an invariant spectral parameter with respect to the vectors X and Y.

Reduced twistor Space R is the quotient of $\mathbf{PT}$ with respect to the two vector fields in $\mathbf{PT}$, which we called X' and Y'. They are representation of the Lie Algebra of X and Y in $\mathbf{PT}$. It turns out that we can form R by the foliation of F by the two vector fields, V_0 and V_1. The constant of foliation

w satisfies

$$V_0 w = \left(2\partial_u + \frac{1}{\mu T}\left(\mu^2 - 1\right)\partial_\mu\right) w = 0 \quad (3)$$

$$V_1 w = \left(2\partial_v + \frac{\mu}{T}\left(\mu^2 - 1\right)\partial_\mu\right) w = 0 \quad (4)$$

solving yields

$$w = \frac{u + v\mu^2}{1 - \mu^2} \quad (5)$$

We can see that $\{w\} = \mathbf{CP}^1$. Details of this foliation can be found in [2].

Our assumption is that $\mathbf{U}$, and hence M is connected, as shown in figure (1). w can therefore either represent one point in R or two. Consider the region of interest from figure(1). We complexify the coordinates u and v. We make $M = U \times V$, where U and V are open, and $\left(a, \frac{1}{2}\right] \subset U$ and $\left(b, \frac{1}{2}\right] \subset b$. So w represent one point if

$$w = 2u \quad (6)$$

where $u \in U$, or

$$w = -2v \quad (7)$$

where $v \in V$. Since $a + b < 1$, we have that the two sets B_u and B_v, where $B_u = \{w : w = 2u, u \in U\}$ and $B_v = \{w : w = -2v, v \in V\}$, are mutually exclusive. Hence explain the non-Hausdorff structure of R. [2]

2 Mechanism of the Initial Value Problem

We define the incoming wave metric function as g_0 and g_1, such that

$$g\left(u, \frac{1}{2}\right) = g_0(u) \quad (8)$$

$$g\left(\frac{1}{2}, v\right) = g_1(v) \quad (9)$$

These are the metric functions before collision. (refer to figure (1). And we label the functions $B_0(u, w)$ and $B_1(v, w)$ as

$$B_0(u, w) = \frac{1}{2}\sqrt{\frac{w+1}{w-2u}}\, g_0^{-1}\frac{\partial g_0}{\partial u} \quad (10)$$

$$B_1(v, w) = \frac{1}{2}\sqrt{\frac{w-1}{w+2v}}\, g_1^{-1}\frac{\partial g_1}{\partial v} \quad (11)$$

Notice that the terms $\sqrt{\frac{w+1}{w-2u}}$ and $\sqrt{\frac{w-1}{w+2v}}$ are just $\frac{1}{\mu}$ and μ respectively when $v=\frac{1}{2}$ and $u=\frac{1}{2}$. We shall call them $a(w,u)$ and $b(w,v)$. These functions are are doubled valued, and contains a choice of branches.

Choose two base points $(u_0,\frac{1}{2})$ and $(\frac{1}{2},v_0)$ in $U\times V$. These are points on one of each initial surfaces. We write down three matrices $P_1(w)$, $P_2(w)$ and $Q(w)$. The $P_i(w)$ matrices are of the form.

$$Pexp_{v=\frac{1}{2}}\left[-\int_{u_0\to\frac{1}{2}}B_0du\right]Pexp_{u=\frac{1}{2}}\left[-\int_{\frac{1}{2}\to v_0}B_1dv\right] \qquad (12)$$

In this case, there are four choices considering the branches of $a(w,u)$ and $b(w,v)$ in the equations. Out of the four, they form pair of twos which are the same, due to the integrability condition. We label the two as $P_+(w)$ and $P_-(w)$. We put $Q(w)$ as

$$Pexp_{v=v_0}\left[-\int_{\gamma_1}B_0du\right] \qquad (13)$$

or

$$Pexp_{u=u_0}\left[-\int_{\gamma_1}B_1dv\right] \qquad (14)$$

The two integrals are the same, by the integrability condition.

We now put $w=\frac{u+v\mu^2}{1-\mu^2}$ into the matrices $P_+(w)$, $P_-(w)$ and $Q(w)$ to form $F_+(\mu)$, $F_-(\mu)$ and $G(\mu)$. Note that functions F_+ F_- and G are well-defined functions of μ.

The final three matrices are important. We would later carry out a Riemann-Hilbert Splitting to get the solution to Einsteins Field Equation.

3 Vector Bundle E over R

Ward's Correspondence, when reduced by the two vectors X and Y, leads to a correspondence between a vector bundle E over R and the solution of $g(u,v)$ on M. [2]. The definition of the vector bundle E is defined as the solution space of the Lax Pair, which are the pair of equations

$$L_0s = \left\{2\partial_u+\frac{1}{\mu T}(\mu^2-1)\partial_\mu+(1-\frac{1}{\mu})g^{-1}g_u\right\}s=0 \qquad (15)$$

$$L_1s = \left\{2\partial_v+\frac{\mu}{T}(\mu^2-1)\partial_\mu+(1-\mu)g^{-1}g_v\right\}s=0 \qquad (16)$$

over each foliation of constant w. The integrability of the Lax Pair is provided by the equation (2). Using the transformation $\mu\to w$, the Lax Pair becomes

$$L_0s = \left\{\partial_u+\frac{1}{2}\left(1-\frac{1}{\mu}\right)g^{-1}g_u\right\}s=0 \qquad (17)$$

$$L_1 s \;=\; \left\{\partial_v + \frac{1}{2}(1-\mu)\, g^{-1} g_v\right\} s = 0 \tag{18}$$

with the double value function $\mu = \sqrt{\frac{w-2u}{w+2v}}$.

Local trivialisations of E over R can be formed by the following way:

Take the solution of the Lax Pair on two points, one on each of the initial surfaces, namely $(u_0, \frac{1}{2})$ and $(\frac{1}{2}, v_0)$. These choice can defined consistently a local trivialisation for a pair of open sets of R, which we called $X_\pm$ and $Y_\pm$ respectively. They can be represented by w in sets X and Y respectively, which are open subsets of w-$\mathbf{CP}^1 = S_w$. We have

$$X = S_w - \Gamma_X \tag{19}$$

$$Y = S_w - \Gamma_Y \tag{20}$$

Γ_X is a cut on from $w = 2u_0$ to $w = -1$ and Γ_Y is a cut on from $w = 1$ to $w = 2v_0$ inclusive. We choose the cuts so that Γ_X and Γ_Y do not cross. Note that $X \subset S_w$ represents two sets $X_\pm \subset S_\mu$, and similarly $Y \in S_w$ represents two sets $Y_\pm \in R$.

Transition functions are calculated by parallel propagating the solution vectors of the Lax Pair from one point to another along the leaf of constant w. Using equation (17) and (18), one can arrive at transition matrices by using path-ordered integrals. Integrability condition given by equation (2) further splits the rank two path-order integration further into a product of two path-order integral or rank one, one in the u complex plane and one in the v complex plane. Hence we get expressions of (12), (13) and (14). It can be seen that there are only three essential transition matrices to describe the bundle E over R with respect to the local trivialisations. They are $P_+(w)$, $P_-(w)$ and $Q(w)$ from the above.

The equation $w = \frac{u+v\mu^2}{1-\mu^2}$ induces a map $\pi : R \to S_\mu$. By the Ward's Correspondence, we know that the pull back bundle of E, $\pi^*(E)$ is trivial. Therefore, we have Riemann-Hilbert problem on the transition functions $F_\pm$ and G of the pull-back bundle $\pi^*(E)$ over S_μ.

4 Riemann-Hilbert Problem

Let $T_\pm = \pi^{-1}(X_\pm)$ and $W_\pm = \pi^{-1}(Y_\pm)$. We know that on on $T_+ \cap W_+$, $F_+(\mu)$ splits like

$$F_+(\mu) = K_W(\mu)^{-1} K_T(\mu) \tag{21}$$

where $K_W(\mu)$ is holomorphic in W_+ and $K_T(\mu)$ is holomorphic in T_+. Also on $T_- \cap W_-$, $F_-(\mu)$ splits like

$$F_-(\mu) = L_W(\mu)^{-1} L_T(\mu) \tag{22}$$

where $L_W(\mu)$ is holomorphic in W_- and $L_T(\mu)$ is holomorphic in T_-.

Now the transition function $M(\mu)$ on $(T_+ \cup W_+) \cap (T_- \cup W_-)$ can be constructed, which splits as

$$M(\mu) = L_T(\mu)G(\mu)K_T^{-1}(\mu) = H_B^{-1}(\mu)H_A(\mu) \tag{23}$$

Finally, we put

$$g(u,v) = \left[H_B^{-1}(-1)H_A(+1)\right]|_{\sigma=(u,v)} \tag{24}$$

It can be checked that $g(u,v)$ satisfies equation (2), and hence is a solution of the reduced Einsteins Field Equation.

The whole procedure shows that, in the case of colliding plane wave, it is theoretically possible to derived from the data of the two incoming wave. The metric of the collision region shown in figure (1) is derived from the initial data from metric $g_0(u)$ and $g_1(v)$, which are taken from the initial surfaces of collision.

Acknowledgement

I would like to thank Dr Lionel Mason for the lecture notes that he had given on this topic, and the useful discussions that I had with him. Furthermore, I would like to thank Dr Nicholas Woodhouse for his guidance into this area of research and the many fruitful discussions and suggestions.

References

[1] J. Fletcher and N. M. J. Woodhouse. *Twistors in Mathematics and Physics*, chapter 14. Editors T. N. Bailey and R. J. Baston. Twistor Characterisation of

[2] N. M. J. Woodhouse and L. J. Mason. *The Geroch Group and Non-Hausdorff Twistor Space*. Non-Linearity, 1(1) Page 73-114. *1988*.

[3] Richard S. Ward and Raymond O. Wells Jr. *Twistor Geometry and Field Theory*, Chapter 8.1. Cambridge University Press, Cambridge. *1990*.

[4] R. M. Wald *General Relativity*. University of Chicago Press, Chicago, *1984*.

Divergence-free geodesic congruences in R^3

Twistor theorists are familiar with the problem of finding shear-free null geodesic congruences in Minkowski space and its solution by the Kerr Theorem. Many will know the related problem of finding shear-free geodesic congruences in the (flat) Euclidean space of three dimensions, R^3, which in turn is solved by a 'mini-Kerr' theorem (see e.g. my §II.1.12 and §II.1.13 in *Further advances in twistor theory vol II.*) Here I want to consider a similar-sounding problem, namely that of finding all divergence-free geodesic congruences in R^3. This problem may be thought of arising from a very degenerate case of the steady Euler equations for an incompressible fluid. In terms of the fluid velocity vector **u** and pressure p these are

$$(\mathbf{u}.\nabla)\mathbf{u} = -\nabla p \quad \text{and} \quad \nabla.\mathbf{u} = 0 \qquad (1)$$

The degenerate case I have in mind is p = constant, which is not interesting for a fluid mechanic! Then (1) collapses to the geodesic equation for a divergence-free **u** and **u.u** is constant along the flow. By rescaling **u** we can take it to be a unit vector, then any other solution is obtained by rescaling with a function constant along **u**. The problem is therefore to solve

$$(\mathbf{u}.\nabla)\mathbf{u} = 0; \ \nabla.\mathbf{u} = 0; \ \mathbf{u}.\mathbf{u} = 1 \qquad (2)$$

I claim the solutions are given by the following proposition:

Proposition

The solutions of (2) fall into two classes:

(a) for one class choose a curve x(s) parametrised by path length s and with its Serret-Frenet frame (**t**,**n**,**b**); for each s take the normal plane orthogonal to **t**; in the normal plane take the straight-line congruence parallel to **b**. This is the desired congruence.

(b) in terms of standard Cartesian coordinates (x,y,z) the vector field **u** is given by

$$\mathbf{u} = (\cos(f(z)), \sin(f(z)), 0)$$

for arbitrary f(z).

I have two rather 'bare-handed' ways of proving this. However the result, on the one hand, looks like something which should be in Darboux and, on the other hand, looks as if there should be a mini-twistor-space proof.

Paul Tod

'Special' Einstein-Weyl spaces

In this note, I will describe a class of 'special' three-dimensional Einstein-Weyl spaces which turned up in joint work with Chave and Valent [1] and Gauduchon [2]. They can be thought of as coming from hyper-Kähler metrics with a tri-holomorphic *homothetic* vector field (recall this is a vector field such that the Lie derivative of the metric along it is a constant but non-zero multiple of the metric). Determination of them reduces to solving a single, second-order, non-linear PDE, equation (11) below, which must, for the usual reasons, be completely integrable.

We start by considering a system of equations: suppose e_1, e_2, e_3 and a are 1-forms on R^3 and K is a function on R^3 and consider the system:

$$\begin{aligned} de_1 &= a \wedge e_1 + Ke_2 \wedge e_3 \\ de_2 &= a \wedge e_2 + Ke_3 \wedge e_1 \\ de_3 &= a \wedge e_3 + Ke_1 \wedge e_2 \end{aligned} \qquad 1$$

I claim that, given a solution of this system, the metric h defined by

$$h = (e_1)^2 + (e_2)^2 + (e_3)^2, \qquad 2$$

together with the 1-form a, defines an Einstein-Weyl structure on R^3 (recall this is a pair consisting of a conformal metric, here h, and a torsion-free connection which preserves the conformal metric; here a defines the difference between this connection, call it the Weyl connection, and the Levi-Civita connection of h; finally the Ricci tensor of the Weyl connection is proportional to h).

Furthermore, for this EW space, the scalar curvature of the Weyl connection is

$$S = 3K^2/2 \qquad 3$$

This is not too difficult to see. Also it draws attention to a conformal rescaling invariance in the system (1).

Now it is natural to ask whether every (three-dimensional) EW space can be written like this. Certainly we must have the scalar curvature positive by (3), but there are more conditions which follow as integrability conditions for (1). Differentiating again in (1), we find a necessary condition which is the vanishing of the quantity Q defined by

$$Q = dK + aK + {*}da \qquad 4$$

where the $*$ is dual w.r.t. the volume defined by h. I have written the condition in this way because it might appear that more conditions can arise by differentiating Q. This isn't the case because of the Bianchi identity for the Weyl connection which here, and with (3), is just

$$dQ - a \wedge Q - K{*}Q = 0 \qquad 5$$

Thus the system (1) defines EW spaces with an extra condition on the curvature embodied in (4). This is like a harmonic condition on K which is $S^{1/2}$ (up to factors) but rather than complicate nomenclature (?harmonic curvature ?harmonic square-root-of-curvature) I will just call these 'special' Einstein-Weyl spaces. They turned up in work I did with Thierry Chave and Galliano Valent [1] and with Paul Gauduchon [2] and more about them can be found in those references (in particular, the only compact example is the Berger sphere as an EW space [2]). They come from 4-dimensional hyper-Kähler metrics with a homothetic vector field which preserves all the complex structures. In terms of the objects above, the 4-metric is

$$e^{\hat{t}}[Kh + K^{-1}(d\hat{t} + a)^2] \qquad 6$$

where $\hat{t}$ is the group parameter.

By the usual folk-lore, the system (1) must be 'integrable' and the field equations for the metric (6) should reduce to an integrable equation (much as the hyper-Kähler metric with non-triholomorphic Killing vector reduces to the SU(∞)-Toda field equation). I have a way of reducing it to a single equation which I shall now explain, I haven't seen the resulting equation before and I would be glad to hear from anyone who recognises it.

To start the reduction of (1), define

$$e = e_2 + ie_3$$

then the second and third of (1) may be written

$$de = (a + iKe_1) \wedge e \qquad 7$$

This implies e = WdZ for complex functions W and Z and then

$$a + iKe_1 = d\log W + fdZ \qquad 8$$

for some complex f. Take real and imaginary parts:

$$W = re^{it}; \quad f = p + iq \;; \quad Z = x + iy$$

and then (8) implies

$$a = d\log r + p\,dx - q\,dy\ ;\ Ke_1 = dt + q\,dx + p\,dy \qquad 9$$

The metric h has become

$$h = K^{-2}(dt + q\,dx + p\,dy)^2 + r^2(dx^2 + dy^2) \qquad 10$$

and the remaining equation is the first of the system (1). This is simplified by introducing the variables V (real) and S (complex) by

$$V = (rK)^{-1}\ ;\ S = V(p + iq)$$

when it becomes the pair of equations

$$S_t + iS - 2iV_z = 0;\ 1 + S\bar{S} - V(S_{\bar{z}} + \bar{S}_z) = 0 \qquad 11$$

This is the final result: since we can use the second of (11) to eliminate V from the first, this is effectively a single equation for the complex function S. It is second order in x and y and first order in t, and naively the data is S at t=0, so that special EW spaces depend on 2 free functions of 2 variables (a general EW depends on 4 such functions). The t-independent solution turns out to be the Berger sphere.

So ... what is equation (11)?

acknowledgement

This note is based on work done in Paris in March 1996 and I am grateful to the Université de Paris-VII for support and hospitality and to my co-workers.

references

[1] *(4,0) and (4,4) sigma models with a tri-holomorphic Killing vector* T.Chave, G.Valent and K.P.Tod Phys.Lett. B383 (1996) 262-270

[2] *Hyperhermitian metrics with symmetry* P. Gauduchon and K.P.Tod (1996 preprint)

Paul Tod

Integrable flows on moduli of rational curves with normal bundle $\mathcal{O}^A(n)$

Maciej Dunajski Lionel J. Mason

Let $\mathcal{M}$ be a four-dimensional oriented complex manifold with null coordinates $x^{AA'} = (x^A, w^A)$ and let Θ be a complex-valued function on $\mathcal{M}$. In the last TN, [1], we showed how to embed the Plebański's second form of the ASD vacuum equations

$$\frac{\partial^2\Theta}{\partial w^A \partial x_A} + \frac{1}{2}\frac{\partial^2\Theta}{\partial x^B \partial x^A}\frac{\partial^2\Theta}{\partial x_B \partial x_A} = 0 \tag{1}$$

into an infinite system of overdetermined PDEs

$$\partial_{Ai}\partial_{Bj-1}\Theta - \partial_{Bj}\partial_{Ai-1}\Theta + \partial_{Ai-1}\partial^C{}_{0'}\Theta\partial_{C0'}\partial_{Bj-1}\Theta. \tag{2}$$

Here $\Theta = \Theta(x^{AA'_1\dots A'_n})$ is a function on a $2n+2$ dimensional manifold $\mathcal{N} = \mathcal{M} \times X$, with coordinates $x^{AA'_1\dots A'_n}$ or, alternatively[1],

$$x^{Ai} = \binom{n}{i} x^{AA'_1A'_2\dots A'_n} o_{A'_1}\dots o_{A'_i}\iota_{A_{i+1}}\dots\iota_{A_n}(-1)^{n-i}$$

where, for $i = 0, 1$, $x^{Ai} = x^{AA'}$ are coordinates on $\mathcal{M}$ and for $i > 1$ $x^{Ai} = t^{Ai}$ are coordinates on the space of parameters, X.

System (2) arises as the compatibility conditions for the $2n$ operators

$$L_{A(A'_2\dots A'_n)} = \pi^{A'_1} D_{AA'_1(A'_2\dots A'_n)}, \tag{3}$$

i.e. the condition that they commute. Here $D_{AA'_1(A'_2\dots A'_n)}$ are $4n$ vector fields on $\mathcal{N}$. Let

$$\nabla_{A(A'_1A'_2\dots A'_n)} = D_{A(A'_1A'_2\dots A'_n)}.$$

In the coordinate system above we have

$$D_{A0'(A'_2\dots A'_n)} = \partial_{A0'A'_2\dots A'_n}, \quad D_{A1'(A'_2\dots A'_n)} = \partial_{A1'A'_2\dots A'_n} + [\partial_{A0'A'_2\dots A'_n}, V]$$

so that

$$\nabla_{Ai} = \partial_{Ai} + \frac{i}{n}[\partial_{Ai-1}, V], \quad L_{Aj} = -\lambda(\partial_{Aj} + [\partial_{Aj-1}, V]) + \partial_{Aj-1} \tag{4}$$

where $V = \varepsilon_{AB}\partial\Theta/\partial x_A \partial/\partial x_B$. Note that $L_{A1} = L_A$ is the Lax pair for equation (1).

[1] Generally, up to the given combinatorial factor, the index i is related to a totally symmetric primed spinor by contraction of all the primed indices: the first i with $o_{A'}$ and the rest with $\iota_{A'}$.

The corresponding projective twistor space $\mathcal{PT}$ is obtained by factoring the spin bundle $\mathcal{N} \times \mathbb{CP}^1$ by the twistor distibution L_{Ai}. The twistor space is still three-dimensional, however it has a different topology from the one used in the Nonlinear-Graviton construction; the rational curves corresponding to points of $\mathcal{N}$ have normal bundle $\mathcal{O}^A(n) = \mathcal{O}(n) \oplus \mathcal{O}(n)$.

The aim of this note is to start from the natural structures on $\mathcal{PT}$ and show that $\mathcal{N}$, the moduli space of rational curves in $\mathcal{PT}$, is equipped with a function Θ satisfying (2) and with $V_{AA'_2 \dots A'_{n-1}}$, the antisymmetric part of $D_{AA'_1 \dots A'_n}$. This completes the picture of [1], where (2) was derived from space time arguments.

Proposition 1 *Let $\mathcal{PT}$ be a 3 dimensional complex manifold with the following structures*

1) a projection $\mu : \mathcal{PT} \longrightarrow \mathbb{CP}^1$,

2) a $2(n+1)$-dimensional family of sections with normal bundle $\mathcal{O}^A(n)$,

3) a nondegenerate 2-form Σ on the fibres of μ, with values in the pullback from $\mathbb{CP}^1$ of $\mathcal{O}(2n)$

and let $\mathcal{N}$ be the moduli space of sections from (2). Then

a) There exists a function $\Theta : \mathcal{N} \longrightarrow \mathbb{C}$ such that (with the appropriate coordinatisation) equation (2) is satisfied

b) The moduli space $\mathcal{N}$ of sections is equipped with the following structures

- *factorisation of the tangent bundle*

$$T\mathcal{N} = S^A \otimes \odot^n S^{A'},$$

- *a $2n$-dimensional distribution on the 'spin bundle' $D \subset T(\mathcal{N} \times \mathbb{CP}^1)$ that is tangent to the fibres of* **r** *over $\mathbb{CP}^1$ and, as a bundle on $\mathcal{N} \times \mathbb{CP}^1$ has an identification with $\mathcal{O}(-1) \otimes S_{AA' \dots A'_{n-1}}$ so that the linear system can be written as in equation (3).*

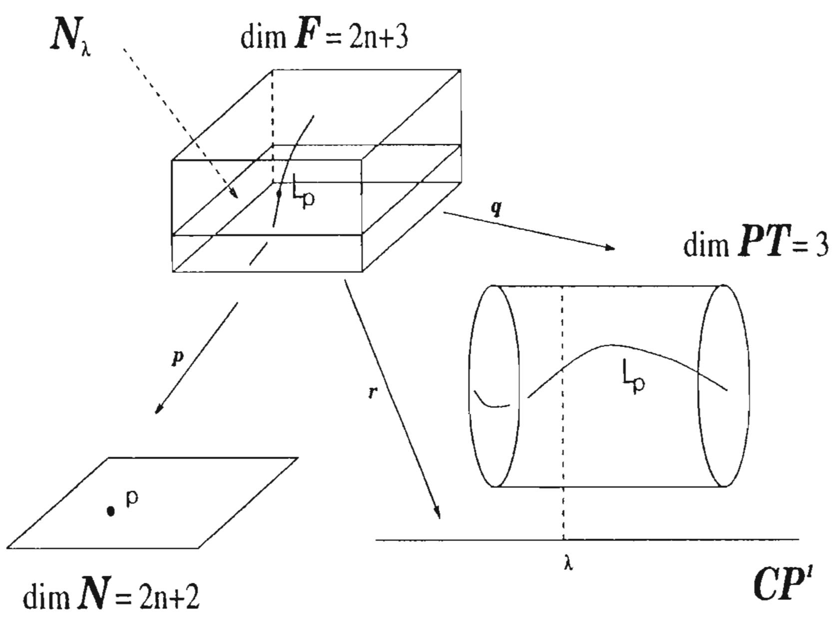
Proof of a). We use a double fibration picture (Figure 1) . First define homogeneous coordinates on $\mathcal{PT}$. These are coordinates on $\mathcal{T}$, the pullback from $\mathbb{CP}^1$ of the tautological line bundle $\mathcal{O}(-1)$. Let $\pi_{A'}$ be homogeneous coordinates on $\mathbb{CP}^1$ pulled back to $\mathcal{T}$ and let ω^A be local coordinates on $\mathcal{T}$ chosen on a neighbourhood of $\mu^{-1}\{\pi_{0'} = 0\}$ that are homogeneous of degree n and canonical so that $\Sigma = \varepsilon_{AB} d\omega^A \wedge d\omega^B$. We also use $\lambda = \pi_{0'}/\pi_{1'}$ as an affine coordinate on $\mathbb{CP}^1$. Let L_p be the line in $\mathcal{PT}$ that corresponds to $p \in \mathcal{N}$ and let $Z \in \mathcal{PT}$ lie on L_p. We denote by $\mathcal{F}$ the correspondence space $\mathcal{PT} \times \mathcal{N}|_{Z \in L_p} = \mathcal{N} \times \mathbb{CP}^1$. We can pull back the twistor coordinates to $\mathcal{F}$ and choose $2(n+1)$ coordinates on $\mathcal{N}$ by

$$x^{A(A'_1 A'_2 \dots A'_n)} := \frac{\partial^n \omega^A}{\partial \pi_{A'_1} \partial \pi_{A'_2} \dots \partial \pi_{A'_n}}\Big|_{\pi_{A'_i} = o_{A'}}.$$

where the derivative is along the fibres of $\mathcal{F}$ over $\mathcal{N}$. This can alternatively be expressed by expanding the coordinates ω^A pulled back to $\mathcal{F}$ in λ as follows

$$\omega^A = \sum_{i=0}^{n} x^{Ai} \lambda^{n-i} + \lambda^{n+1} \sum_{i=0}^{n-1} s_i^A \lambda^i + \dots \, , \qquad (5)$$

Figure 1: Double fibration.



where s_i^A and terms of order higher than $2n$ can be expressed as functions of $x^{AA'\dots A'_n}$. The symplectic 2-form Σ on the fibres of μ, when pulled back to the spin bundle, has expansion in powers of λ that truncates at order $2n+1$ by globality and homogeneity so that

$$\Sigma = d\omega_A \wedge d\omega^A = \pi_{A'_1}\dots\pi_{A'_n}\pi_{B'_1}\dots\pi_{B'_n}\Sigma^{A'_1\dots A'_n B'_1\dots B'_n}$$

for some spinor indexed 2-form $\Sigma^{A'_1\dots A'_n B'_1\dots B'_n}$. We have

$$\Sigma(\lambda)\wedge\Sigma(\lambda) = 0, \qquad d\Sigma(\lambda) = 0. \tag{6}$$

If we express the forms in terms of the x^{Ai} and the s_i^A, the closure condition is satisfied identically, whereas the algebraic relations and truncation condition will give rise to integrability conditions on the s_i^A allowing one to express them in terms of a function $\Theta(x^{AA'\dots A'_n})$ and to field equations on Θ. Set

$$\Sigma^i = \binom{2n}{i}\Sigma^{A'_1\dots A'_n B'_1\dots B'_n} o_{A'_1}\dots o_{A'_i}\iota_{A_{i+1}}\dots\iota_{B_{2n}}(-1)^{2n-i},$$

so that

$$\Sigma = \sum_{i=0}^{2n}\lambda^{2n-i}\Sigma^i$$

where

$$\begin{aligned}
\Sigma^j &= \sum_{i=j-n}^{n}\varepsilon_{AB}dx^{Ai}\wedge dx^{Bj-i} \quad \text{for } j\ge n\\
&= \sum_{i=0}^{j}\varepsilon_{AB}dx^{Ai}\wedge dx^{Bj-i} + \sum_{i=j+1}^{n}\varepsilon_{AB}dx^{Ai}\wedge ds^B_{i+j-1} \quad \text{for } j<n\,.
\end{aligned}$$

The algebraic condition in (6) implies

$$\sum_{i=\max\{0,k-2n\}}^{\min\{2n,k\}} \Sigma^i \wedge \Sigma^{k-i} = 0 \qquad \text{for } 0 \le k \le 4n. \tag{7}$$

To deduce the existence of $\Theta(x^{AA'_1\cdots A'_n})$ equate the λ^{2n+1} coefficient in $d\omega^A \wedge d\omega_A$ to 0 (vanishing of higher terms gives recursion relations). This yields

$$\sum_{j=0}^{n}\sum_{i=0}^{n-1} \frac{\partial s_{Ai}}{\partial x^{Bj}} dx^{Bj} \wedge dx^{Ai} = 0 \longrightarrow s_{Ai} = \frac{\partial \Theta}{\partial x^{Ai}}. \tag{8}$$

To derive the hierarchy of flows consider the identity $\Sigma^0 \wedge \Sigma^0 = 0$ which is a particular case of (7).

$$\begin{aligned} 0 &= 2\sum_{j=1}^{n}\sum_{i=1}^{n} \varepsilon_{AB}\varepsilon_{CD} dx^C \wedge dx^D \wedge dx^{Ai} \wedge dx^{Bj} \frac{\partial^2 \Theta}{\partial x^{Ai-1}\partial x^{Bj}} \\ &+ \sum_{i=1}^{n}\sum_{k=0}^{n}\sum_{j=1}^{n}\sum_{l=0}^{n} dx^{Ai} \wedge dx^{Ck} \wedge dx^{Bj} \wedge dx^{Dl} \frac{\partial^2 \Theta}{\partial x^{Ai-1}\partial x^{Ck}} \frac{\partial^2 \Theta}{\partial x^{Bj-1}\partial x^{Dl}}. \end{aligned}$$

Equate the coefficient of $\varepsilon_{AB} dx_C \wedge dx^C \wedge dx^{Ai} \wedge dx^{Bj}$ to 0 (i.e. the term with $k = l = 0$). This yields (2). One can show that all the other equations in (7) are satisfied trivialy.

□

Proof of b). The isomorphism $T\mathcal{N} = S^A \otimes \odot^n S^{A'}$ follows simply from the structure of the normal bundle; from Kodaira theory, since the appropriate obstruction groups vanish, we have

$$T_p\mathcal{N} = \Gamma(N_p, L_p) = S_p^A \otimes \odot^n S_p^{A'}$$

where N_p is the normal bundle to the rational curve L_p in $\mathcal{PT}$ corresponding to the point $p \in \mathcal{N}$. The bundle S^A on space-time is the Ward transform of $\mathcal{O}(-n) \otimes T_V\mathcal{PT}$ where the subscript V denotes the subbundle of the tangent bundle consisting of vectors up the fibres of the projection to $\mathbb{CP}^1$, so that $S_p^A = \Gamma(\mathcal{O}(-n) \otimes T_V\mathcal{PT}, L_p)$. The bundle $S_p^{A'} = \Gamma(\mathcal{O}, L_p)$ is canonically trivial. Let $\nabla_{AA'_1\cdots A'_n} = \nabla_{A(A'_1\cdots A'_n)}$ be the indexed vector field that establishes the above isomorphism and let $e^{AA'_1\cdots A'_n} = e^{A(A'_1\cdots A'_n)} \in \Omega^1 \otimes S^A \otimes \odot^n S^{A'}$ be the dual (inverse) map.

We now wish to derive the form of the linear system, equation (3). For each fixed $\pi_{A'} = (\lambda, 1) \in \mathbb{C}P^1$ we have a copy of a space-time $\mathcal{N}_\lambda$. A vertical (with respect to the projection r) subspace of $T_p(\mathcal{N}_\lambda)$ at a point p is spanned by $\nabla_{A(A'\ldots A'_n)}$. A normal bundle to the corresponding line L_p consists of vectors tangent to p modulo the twistor distribution. Therefore we have a sequence of sheaves over $\mathbb{C}P^1$

$$0 \longrightarrow D \longrightarrow T\mathcal{N} \xrightarrow{e^A} \mathcal{O}^A(n) \longrightarrow 0.$$

The map $T\mathcal{N} \longrightarrow \mathcal{O}^A(n)$ is given by the contraction of $T\mathcal{N}$ with $e^A = e^{A(A'\ldots A'_n)}\pi_{A'}\ldots\pi_{A'_n}$ since e^A annihilates all L_{Bi}s in D. Consider the dual sequence

$$0 \longrightarrow \mathcal{O}_A(-n) \longrightarrow T^*\mathcal{N} \longrightarrow D^* \longrightarrow 0$$

and tensor it with $\mathcal{O}(-1)$ to obtain

$$0 \longrightarrow \mathcal{O}_A(-n-1) \longrightarrow T^*\mathcal{N}(-1) \longrightarrow D^*(-1) \longrightarrow 0. \tag{9}$$

From here we would like to extract the Lax distribution

$$L_{AA'_2 \dots A'_n} = \pi^{A'_1} D_{AA'_1 A'_2 \dots A'_n} \in S_{AA'_1 \dots A'_n} \otimes \mathcal{O}(1) \otimes D.$$

This can be achieved by globalising (9) in $\pi^{A'}$. The corresponding long exact sequence of cohomology groups yields

$$0 \longrightarrow \Gamma(\mathcal{O}_A(-n-1)) \longrightarrow \Gamma(T^*\mathcal{N}(-1)) \longrightarrow \Gamma(D^*(-1)) \stackrel{\delta}{\longrightarrow} H^1(\mathcal{O}_A(-n-1))$$
$$\longrightarrow H^1(T^*\mathcal{N}(-1)) \longrightarrow \dots$$

which (because $T^*\mathcal{N}$ is a trivial bundle so that $\mathcal{O}(-1) \otimes T^*\mathcal{N}$ has no sections or cohomology) reduces to

$$0 \longrightarrow \Gamma(D^*(-1)) \stackrel{\delta}{\longrightarrow} H^1(\mathcal{O}_A(-n-1)) \longrightarrow 0. \tag{10}$$

From the standard formulae

$$H^0(\mathbb{C}P^1, \mathcal{O}(k)) = \begin{cases} 0 & \text{for } k < 0 \\ \mathbb{C}^{k+1} & \text{for } k \geq 0. \end{cases}$$

$$H^1(\mathbb{C}P^1, \mathcal{O}(-k-2) = H^0(\mathbb{C}P^1, \mathcal{O}(k))^*$$

we conclude, since D has rank $2n$, that the connecting map δ is an isomorphism

$$\delta : \Gamma(D^*(-1)) \longrightarrow S_{AA' \dots A'_{n-1}}.$$

Therefore

$$\delta \in \Gamma(D \otimes \mathcal{O}(1) \otimes S_{AA' \dots A'_{n-1}}) \tag{11}$$

is a canonically defined object anihilating ω^A given by (5). A coordinate calculation shows that δ can be put into the form (3).

□

Remarks

(i) By examining the relevant sheaf cohomology groups and using Kodaira deformation theory, we can show that, analogous to the four-dimensional case, flat $\mathcal{PT}$ (ie $\Theta = 0$) admits complex deformations preserving (1)-(3).

(ii) For n odd $T\mathcal{N}$ is equipped with a metric with holonomy SL$(2, \mathbb{C})$. For n even, $T\mathcal{N}$ is endowed with a skew form. They are both given by

$$G(U, V) = \varepsilon_{AB} \varepsilon_{A'_1 B'_1} \dots \varepsilon_{A'_n B'_n} U^{AA'_1 \dots A'_n} V^{BB'_1 \dots B'_n}. \tag{12}$$

(iii) If one considers $\mathcal{N} = \mathcal{M} \times \mathcal{X}$ as being foliated by four dimensional slices $t^{Ai} = const$ then structures (1)-(3) on $\mathcal{PT}$ induce an anti-self-dual vacuum metrics on the leaves of the foliation. Consider $\Theta(x^{AA'}, \mathbf{t})$ where $\mathbf{t} = \{t^{Ai}, i = 2...n\}$. For each fixed $\mathbf{t}$ the function Θ satisfies the 2nd heavenly equation. The ASD metric on a corresponding four-dimensional slice $\mathcal{N}_{\mathbf{t}=\mathbf{t}_0}$ is given by

$$ds^2 = \varepsilon_{AB} dx^{A1'} dx^{B0'} + \frac{\partial^2 \Theta}{\partial x^{A0'} \partial x^{B0'}} dx^{A1'} dx^{B1'}.$$

One would like to determine this metric from the structure of the $\mathcal{O}^A(n)$ twistor space.

If we fix $2n - 2$ parameters in the expansion (5) then the normal vector $V = V^A \partial / \partial \omega^A$ is given by

$$V^A = \delta \omega^A = \lambda^{n-1} V^{A1'} + \lambda^n V^{A0'} + \lambda^{n+1} \frac{\partial \delta \Theta}{\partial x^{0'}_A} + ...$$

where $\delta \Theta = V^{AA'} \partial \Theta / \partial x^{AA'}$. The metric is

$$g(U, V) = \frac{\alpha^{C'} \beta_{C'}}{(\pi_{0'})^{2n-2} \alpha^{A'} \pi_{A'} \beta^{B'} \pi_{B'}} \Sigma(U(\pi^{D'}), V(\pi^{D'})). \qquad (13)$$

The last formula follows also from (12) if one puts

$$V^{AA'_1 ... A'_n} = V^{A(A'_1} o^{A'_2} .. o^{A'_n)}$$

for V tangent to $t^{Ai} = const.$ Note that it is sufficient to consider the slice $\mathbf{t} = 0$. This is because an appropriate (canonical) coordinate transformation of $\mathcal{PT}$ induces the transformation of parameters

$$\omega^A \to \hat{\omega}^A(\omega^B, \lambda) \quad \text{yields} \quad \{\mathbf{t} = \mathbf{t}_0\} \to \{\hat{\mathbf{t}} = 0\}.$$

(iv) From Merkulov's work [2] it follows that $\mathcal{N}$ is equipped with an affine connection. It would be interesting to relate his approach to ours and in particular to express the connection on $\mathcal{N}$ as a function of Θ.

Thanks to Paul Tod.

References

[1] Dunajski, M. & Mason, L.J. (1996) Heavenly Hierarchies and Curved Twistor Spaces, Twistor Newsletter 41.

[2] Merkulov, S (1995) Relative deformation theory and differential geometry, in *Twistor theory*, ed. S. Huggett, Lecture notes in pure and applied mathematics, **169**, Marcel Dekker.

Asymptotic Young Tableaux for Discrete Series

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We give an example, how to use Young tableaux for the decomposition of products of *positive discrete series* representations.
Consider $SU(2,1)$ for example. We label its positive discrete series representations $\gamma_{a,b}$ by two integers $a \geq 2$, $b \geq 0$ which give the weight of their *lowest* weight vector. The representations $\gamma_{a,0}$ are on a wall of the dominant Weyl chamber and do not, strictly speaking, fall under the discrete series. They are *ladder representations* which we can include without problems. We have to exclude the other type $\gamma_{1,b}$ of positive ladder representations though, because they do not satisfy the following *asymptotic* properties (for natural reasons):

Multiplicities of $\gamma_{a,b}$: The multiplicities (dimensions) of the weight spaces $V_{c,d}$ of the representation $\gamma_{a,b}$ are the *same* as those of $V_{n-c-d,d}$ in the representation $\Gamma_{n-a-b,b}$ for $n \to \infty$.

Here $\Gamma_{a,b}$ denotes the representation of the compact form $SU(3)$ of $SU(2,1)$ with *highest* weight (a,b). With such a representation we associate a Young tableau whose first row has $a+b$ boxes and whose second row has b boxes. Thus, if we want to decompose $\gamma_{a,b} \otimes \gamma_{c,d}$ say, we choose a large n, decompose $\Gamma_{n-a-b,b} \otimes \Gamma_{n-c-d,d}$ according to Littlewood-Richardson [1] and translate back, ignoring that n is finite. The representation with *highest* highest weight in this product will be $\Gamma_{2n-a-b-c-d,b+d}$ which has to match $\gamma_{a+b,c+d}$, since the *lowest* lowest weight of $\gamma_{a,b} \otimes \gamma_{c,d}$ is of course $(a+b, c+d)$. Thus, on the way back from representations of the compact group to discrete series, we have to choose the obvious matching parameter ($2n$ in this exaple). In practice one would determine the kind of representations occurring in $\Gamma_{n-a-b,b} \otimes \Gamma_{n-c-d,d}$ for $n \to \infty$ starting from a typical large enough n and translate back, ignoring the finiteness of n.

The above simple observation can be used to prove several conjectural formulas in [2]. Its proof and generalisations will be given elsewhere.

References:

[1] e.g.: Fulton and Harris: A first course in Representation Theory. GTM. Springer.

[2] F.Müller: Twistor Diagrams as Projection Operators. DPhil. Thesis. Oxford 1993.

Coherent Quantum Measurements

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This article continues the discussion of coherent states I give in TN 41 and I shall use the same notation and conventions as I use there in what follows. I apply the basic underlying geometry to motivate a result which concerns the role of coherent states in quantum mechanical measurement. In particular I argue that there exists a critical amplitude associated with a pair of coherent states, above which reduction from their closest quantum mechanical mean to the states themselves is favoured over reduction to their corresponding classical average.

Suppose we take $|\psi_1\rangle$ and $|\psi_2\rangle$ to be coherent state vectors based on ξ^a, $-\xi^a \in \mathcal{H}^1$ respectively according to the exponential map, as detailed in TN 41. The complex projective line joining $P|\psi_1\rangle$ and $P|\psi_2\rangle$ is a topological sphere on which these states are represented as points, and since any pair of coherent states has non-zero overlap, these points are never antipodal. They still define a unique circle C of states equidistant from $P|\psi_{1,2}\rangle$ whose state vectors are given by $\{|\psi_1\rangle + \lambda|\psi_2\rangle\}$ for $|\lambda| = 1$. We the define the abstract Hermitian field operator

$$F^a \equiv F^{\alpha,\alpha'} = A^\alpha \oplus C^{\alpha'}$$

which is the sum of the abstract annihilation and creation operators. The expectation of this operator in the general state $P[|\psi_1\rangle + \lambda|\psi_2\rangle]$ is given by

$$\langle F^a\rangle = \xi^\alpha[1 - |\lambda|^2 + e^{-\Lambda}(\bar{\lambda} - \lambda)] \oplus \bar{\xi}^{\alpha'}[1 - |\lambda|^2 + e^{-\Lambda}(\lambda - \bar{\lambda})]$$

where $\Lambda = \xi^\alpha \bar{\xi}_\alpha$. Thus on the circle C there are precisely two points at which $\langle F\rangle$ vanishes, namely at $\lambda = \pm 1$, and the choice $\lambda = +1$ gives the state closest to $P|\psi_{1,2}\rangle$. We shall focus attention on this case taking $|S\rangle$ to be $\frac{1}{2}(|\psi_1\rangle + |\psi_2\rangle)$ and defining also $P|C\rangle$ to be the coherent vacuum state. From Lemma 1 of TN 41 the state $P|S\rangle$ lies off the coherent state submanifold, and the situation is depicted below.

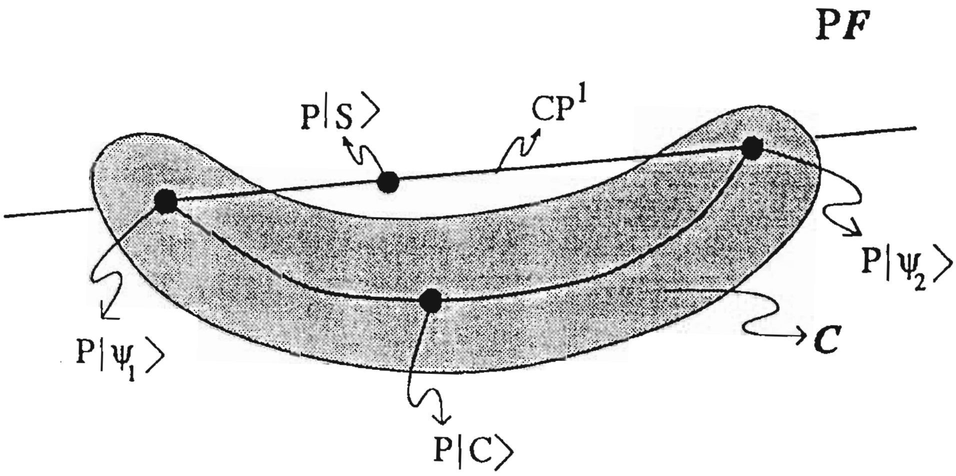

Fig. 1-1. Coherent state submanifold, embedded inside projective Fock space.

We analyze the comparitive distance d in the ambient Fubini-Study geometry, of $P|S\rangle$ from the vacuum and from $P|\psi_{1,2}\rangle$. An intriguing phenomenon arises in this regard. We have, with $\Lambda = \xi^\alpha \bar{\xi}_\alpha$, transition probabilities given by

$$\mathbf{P}[P|S\rangle \mapsto P|0\rangle] = \cos^2(\frac{1}{2}d[P|S\rangle, P|0\rangle]) = \frac{1}{\cosh \Lambda},$$

$$\mathbf{P}[P|S\rangle \mapsto P|\psi_{1,2}\rangle] = \cos^2(\frac{1}{2}d[P|S\rangle, P|\psi_{1,2}\rangle]) = e^{-\Lambda}\cosh \Lambda.$$

Thus for small values of $\Lambda > 0$ the probability of a transition from $P|S\rangle$ to the vacuum is greater than that for the coherent states $P|\psi_{1,2}\rangle$, and correspondingly $d[P|S\rangle, P|0\rangle] < d[P|S\rangle, P|\psi_{1,2}\rangle]$. Moreover for small values of Λ the vacuum state is the closest coherent state to $P|S\rangle$, as one can easily verify. Now if we allow Λ to increase, there exists a critical value Λ_0 for which the above distances are in fact equal, and for $\Lambda > \Lambda_0$ there is a cross-over with $d[P|S\rangle, P|0\rangle] > d[P|S\rangle, P|\psi_{1,2}\rangle]$. The critical value Λ_0 is given by the solution of $e^{\Lambda_0} = \cosh^2 \Lambda_0$ which, setting $\Omega = e^{\Lambda_0}$, becomes

$$\Omega^4 - 4\Omega^3 + 2\Omega^2 + 1 \equiv (\Omega - 1)(\Omega^3 - 3\Omega^2 - \Omega - 1) = 0.$$

The solution $\Omega = 1$ clearly corresponds to the trivial case where all three states under consideration coincide at the vacuum. The cubic term above has a minimum at $\Omega = 1 + 2/\sqrt{3}$ and a root $\Omega_0 \in (3, 4)$. Thus the critical value of the amplitude $\Lambda_0 = \log \Omega_0$ lies in the range $1 < \Lambda_0 < \frac{3}{2}$. In physical terms this is telling us that 'collapse of the wavefunction' to $P|\psi_{1,2}\rangle$ is more likely to occur when the expectation of the total number operator $\hat{N}$ exceeds unity. With a photon field (a beam of coherent light) for example, the asymmetric collapse would occur when on average more than a single photon were present in the field. The result is notably independent of the underlying classical field theory or of any dimensional considerations.

I am grateful to Lane Hughston, David Robinson, Ray Streater and Gerard Watts for a valuable discussion following a seminar that Lane Hughston gave at King's College, London, 27th November 1996.

T.R. Field

Solutions to Puzzles in TN41

① Adding -12 to each term in

$$\ldots, 28, 0, 21, 4, 18, 0, \text{(?)}, 24, 18, 20, 21, 24, 28, \ldots$$

and then multiplying the n^{th} term T_n by n ((?) being the 0^{th} term), we get $12 F_n$, where F_n is the n^{th} Fibonacci number. Accordingly, the n^{th} term must be

$$T_n = 12\left(\frac{F_n}{n} + 1\right)$$

$$= \frac{12}{n}\left(\frac{\tau^n - \tilde{\tau}^n}{\tau - \tilde{\tau}}\right)$$

where $\tau = \frac{1+\sqrt{5}}{2}$ and $\tilde{\tau} = \frac{1-\sqrt{5}}{2} = -\frac{1}{\tau}$.

To get T_0 we use "l'Horpital's rule", and find

$$\text{(?)} = T_0 = \frac{12}{\sqrt{5}}\left\{\log\tau - \log\tilde{\tau}\right\} + 12$$

$$= \frac{24}{\sqrt{5}}\left(\log\tau + \frac{i\pi}{2}\right) + 12 \doteqdot \underline{17.16490729 + i\,16.85955535}$$

Of course, an alternative answer is the complex conjugate of this, but just taking the real part would be wrong. (But you could add any integer multiple of $i\frac{24\pi}{\sqrt{5}}$.)

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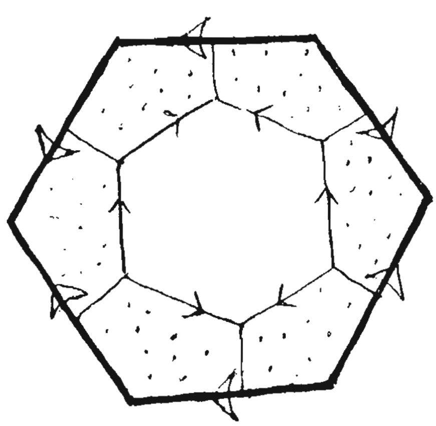
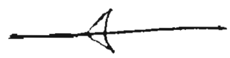
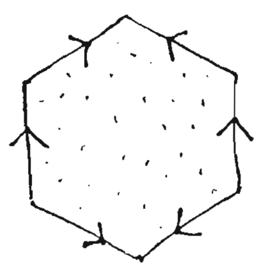
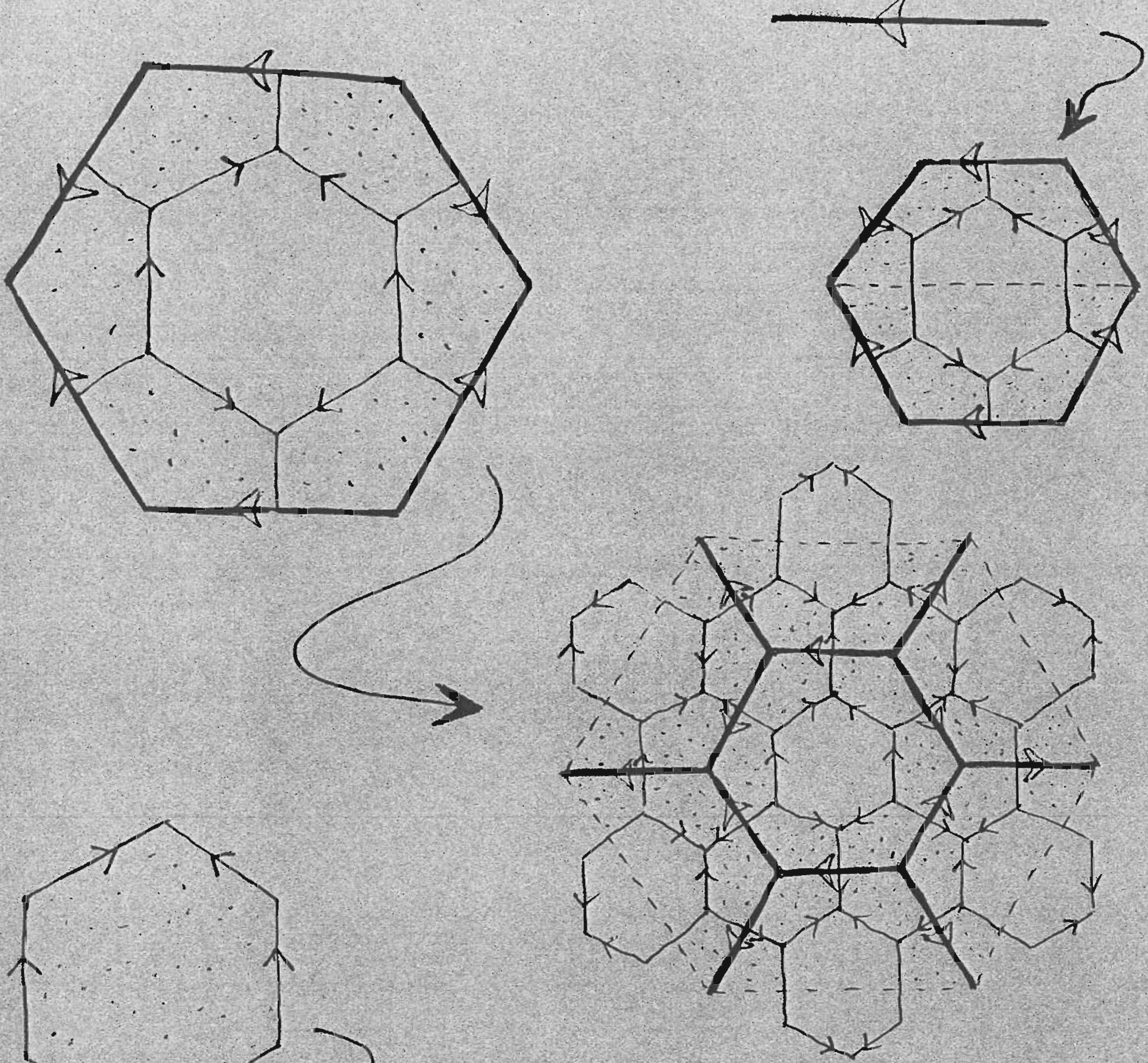
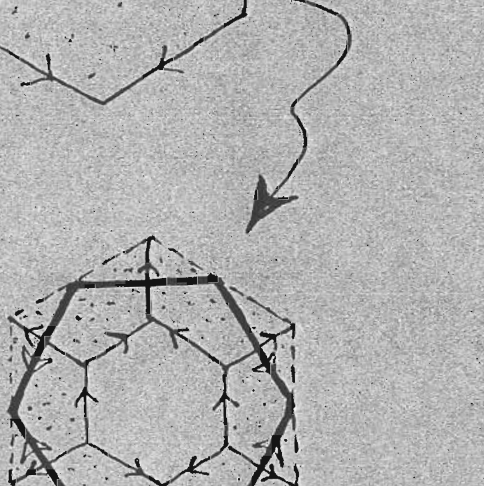
② Any tiling with , where the edges match 

and the corners match

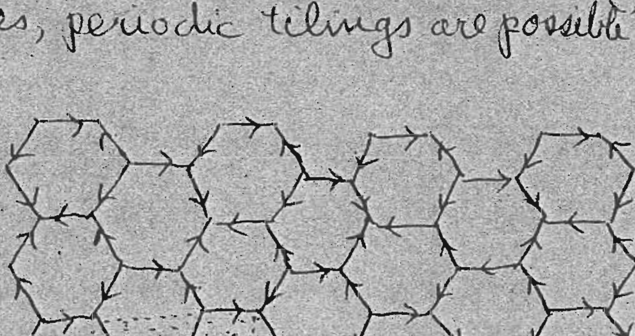


must be non-periodic, and is composed according to the hierarchical scheme





NB: without the double matching rules, periodic tilings are possible, eg.



Roger

TWISTOR NEWSLETTER No. 42

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