Googly Maps as Forms

by Roger Penrose

This is a brief interim report on some new developments that seem to be leading to the correct way of resolving the googly problem. This, we recall, is the problem of representing the self-dual part of the gravitational field (Weyl tensor) in terms of deformations of twistor space, in order to complement the standard "leg-break" non-linear graviton construction for anti-self-dual gravitation. These developments are closely related to some of those described in my article in TN40, and also to some earlier TN articles, but the new developments provide a vital new perspective on these issues.

The starting point is the same as that described in TN8, namely the canonical Poincaré-invariant exact sequence

$$0 \to \mathbb{S}^A \to \mathbb{T}^\alpha \to \mathbb{S}_{A'} \to 0$$

given, respectively, by the injection

$$\omega^A \mapsto (\omega^A, 0)$$

and the projection

$$(\omega^A, \pi_{A'}) \mapsto \pi_{A'}.$$

A space-time point (here an element $x \in \mathbb{M}$) is associated with a "splitting" of this sequence

$$0 \leftarrow \mathbb{S}^A \leftarrow \mathbb{T}^\alpha \leftarrow \mathbb{S}_{A'} \leftarrow 0$$

given by the projection

$$\omega^A - ix^{AA'}\pi_{A'} \leftarrow\!\shortmid (\omega^A, \pi_{A'})$$

and the injection

$$(\omega^A - ix^{AA'}\pi_{A'}, \pi_{A'}) \leftarrow\!\shortmid \pi_{A'}.$$

The (standard) leg-break construction may be regarded as a deformation of the tail end of this sequence, there still being a canonical projection

$$\mathcal{T} \to \mathbb{S}_{A'} - \{0\}$$

of the curved twistor space $\mathcal{T}$, the "space-time points" for the corresponding Ricci-flat complex space-time $\mathcal{M}$ being cross-sections of this projection, i.e. maps

$$\mathbb{S}_{A'} - \{0\} \to \mathcal{T}.$$

For the "dual" googly construction we should have (in some sense) a canonical injection

$$\mathbb{S}^A - \{0\} \to \mathcal{T}$$

instead, where the "space-time points" would correspond to projections

$$\mathcal{T} \to \mathbb{S}^A - \{0\},$$

these being what would be called the ~~googly~~ maps.

The new proposal is that these googly maps (and also other maps) should be represented in terms of 3-forms. Since the (non-projective) twistor space $\mathcal{T}$ is 4-dimensional, any (non-zero) 3-form determines (locally) a family of curves that provide a foliation of $\mathcal{T}$. (The tangent vectors to these curves annihilate the 3-form.) Following these curves down to where $\mathbb{S}^A - \{0\}$ sits in $\mathcal{T}$, according to the canonical injection, we get the required projections corresponding to the googly points.

To understand better how this works, consider again the flat case. We can represent the point x, with position vector x^a, by the skew twistor

$$X^{\alpha\beta} = \begin{pmatrix} -\frac{1}{2}\varepsilon^{AB}x_c x^c & ix^A{}_{B'} \\ -ix_{A'}{}^B & \varepsilon_{A'B'} \end{pmatrix}$$

or, equivalently, by its dual

$$X_{\alpha\beta} = \tfrac{1}{2}\epsilon_{\alpha\beta\gamma\delta}X^{\gamma\delta} = \begin{pmatrix} \varepsilon_{AB} & ix_A{}^{B'} \\ -ix^{A'}{}_B & -\frac{1}{2}\varepsilon^{A'B'}x_c x^c \end{pmatrix},$$

where

$$X_{\alpha\beta}I^{\alpha\beta} = 2 = X^{\alpha\beta}I_{\alpha\beta}.$$

Equivalently, we can use the projection operators

$$L^\alpha{}_\beta = X^{\alpha\gamma}I_{\beta\gamma} = \begin{pmatrix} 0 & ix^{AB'} \\ 0 & \varepsilon_{A'}{}^{B'} \end{pmatrix}$$

and

$$G^\alpha{}_\beta = I^{\alpha\gamma}X_{\beta\gamma} = \begin{pmatrix} -\varepsilon^A{}_B & -ix^{AB'} \\ 0 & 0 \end{pmatrix}$$

(the "leg-break" and "googly" projections, respectively). These satisfy

$$L^\alpha{}_\beta L^\beta{}_\gamma = L^\alpha{}_\gamma,\ G^\alpha{}_\beta G^\beta{}_\gamma = G^\alpha{}_\gamma,\ L^\alpha{}_\beta G^\beta{}_\gamma = G^\alpha{}_\beta L^\beta{}_\gamma = 0,\ L^\alpha{}_\beta + G^\alpha{}_\beta = \delta^\alpha_\beta.$$

The leg-break map corresponding to x (the composition of $\mathbb{T}^\alpha \to \mathbb{S}_{A'}$ with $\mathbb{S}_{A'} \xrightarrow{x} \mathbb{T}^\alpha$ is given by $Z^\alpha \mapsto L^\alpha{}_\beta Z^\beta$ and the googly map by $Z^\alpha \mapsto G^\alpha_\beta Z^\beta$. Note that there is also a more symmetrical quantity

$$J^\alpha{}_\beta = iG^\alpha{}_\beta - iL^\alpha{}_\beta = \begin{pmatrix} -i\varepsilon^A{}_B & 2x^{AB'} \\ 0 & -i\varepsilon_{A'}{}^{B'} \end{pmatrix}$$

(which is twistor-Hermitian, when x^a is real) satisfying

$$J^\alpha{}_\beta J^\beta{}_\gamma = -\delta^\alpha_\gamma$$

and which also represents the space-time point, but in a way that is biassed neither to self-duality nor to anti-self-duality.

The quantities **L** and **G** are closely associated with the respective 3-forms

$$\lambda = \tfrac{1}{2}\overset{x}{\omega}_A d\overset{x}{\omega}^A \wedge d\pi_{A'} \wedge d\pi^{A'} \quad \text{and} \quad \gamma = \tfrac{1}{2} d\overset{x}{\omega}_A \wedge d\overset{x}{\omega}^A \wedge \pi_{A'} d\pi^{A'},$$

where $\overset{x}{\omega}^A = \omega^A - ix^{AA'}\pi_{A'}$. We can also define the "ambidextrous" quantity

$$\xi = i\gamma - i\lambda$$

corresponding to **J**. Let us call the families of curves, whose tangent vectors annihilate these respective forms, the "integral curves" of these 3-forms. Following the integral curves of λ and γ down to where they vanish (at $\overset{x}{\omega}^A = 0$ and $\pi_{A'} = 0$, respectively), we obtain the leg-break and googly maps.

Note that (corresponding to $G^\alpha_\beta + L^\alpha_\beta = \delta^\alpha_\beta$ above)

$$\gamma + \lambda = \theta$$

where θ is the standard 3-form (projective volume form for $\mathbb{P}\mathbb{T}^\alpha$) given by

$$\theta = \tfrac{1}{6}\epsilon_{\alpha\beta\gamma\delta} Z^\alpha dZ^\beta \wedge dZ^\gamma \wedge dZ^\delta$$

whose integral curves are those of the Euler vector field

$$\Upsilon = \theta \div \phi \qquad \text{where} \qquad d\theta = 4\phi,$$

these curves being the ones that we factor out by in the passage from $\mathbb{T}^\alpha$ to $\mathbb{P}\mathbb{T}^\alpha$. Note that ϕ is the standard volume 4-form for $\mathbb{T}^\alpha$. Also there is a standard 1-form $\iota = \pi_{A'} d\pi^{A'}$ and 2-form $\tau = \frac{1}{2} d\pi_{A'} \wedge d\pi^{A'} = d\pi_{0'} \wedge d\pi_{1'}$ where

$$2\tau = d\iota, \qquad \tau \wedge \iota = 0, \quad \text{and} \quad \theta \wedge \iota = 0.$$

In addition, these forms have homogeneity degrees defined by

$$\mathcal{L}_{\Upsilon}\iota = 2\iota, \quad \mathcal{L}_{\Upsilon}\tau = 2\tau, \quad \mathcal{L}_{\Upsilon}\theta = 4\theta, \quad \mathcal{L}_{\Upsilon}\phi = 4\phi.$$

The last two of these are automatic consequences of $\Upsilon = \theta \div \phi$, with $\phi = \frac{1}{4}d\theta$, but the other two require the further condition

$$\theta \odot \tau = -\phi \otimes \iota$$

where the bilinear operation $\odot$, acting between an n-form α and a 2-form β is defined by

$$\alpha \odot (dp \wedge dq) = (\alpha \wedge dp) \otimes dq - (\alpha \wedge dp) \otimes dq.$$

In a leg-break curved twistor space $\mathcal{T}$, all the forms $\iota, \tau, \theta, \phi$ are retained, as is Υ, and they satisfy precisely the same local relations as listed above for the flat case. However, according to the discussion given in TN40, we anticipate that only the quantity

$$\Sigma = \phi \otimes \iota,$$

together with the integral curves of θ, should be retained from among these. There is a freedom in the choice of ϕ and ι arising from $\phi \mapsto \zeta\phi$, $\iota \mapsto \zeta^{-1}\iota$ $(\zeta \neq 0)$ and also, additionally, in the choice of θ for which $d\theta = 4\phi$. To cope with the first of these, we introduce the differential operator D, acting on quantities $\alpha \otimes \beta$, where α is an n-form and β a 1-form, subject to $\alpha \wedge \beta = 0$, given by

$$D(\alpha \otimes \beta) = (d\alpha) \otimes \beta - \alpha \odot (d\beta).$$

We find $D((\zeta\alpha) \otimes \beta) = D(\alpha \otimes (\zeta\beta))$, as is required for notational consistency. We envisage $\mathcal{T}$ as being covered by open sets, in each of which a choice of θ and ι are made, subject to (as is consistent with $D(\theta\otimes\iota) = 4\phi\otimes\iota - \theta\odot 2\tau = 6\phi\otimes\iota$)

$$D(\theta \otimes \iota) = 6\Sigma.$$

The kernel of D (keeping the integral curves of θ fixed) is given essentially by -6 deg. functions.

For a googly map, we keep the integral curves of γ consistent, but the actual googly 3-form γ varies from open set to open set of the covering. In each open set, γ satisfies

$$\lambda \odot \tau = -\Sigma$$

which simply states that the difference $\theta - \lambda$ has the form $\eta \wedge \tau$, for some 1-form η. The idea is that global conditions should be sufficient to restrict the googly maps to a 4-parameter family. There should also be a canonical 2-parameter family of 3-forms (corresponding to $\kappa_A d\omega^A \wedge \tau$, in the flat case, for each $\kappa_A \in \mathbb{S}_A$) such that two googly points are null separated iff the difference between their 3-forms is everywhere proportional to one 3-form of the family.

MORE LATER.

— Roger Penrose