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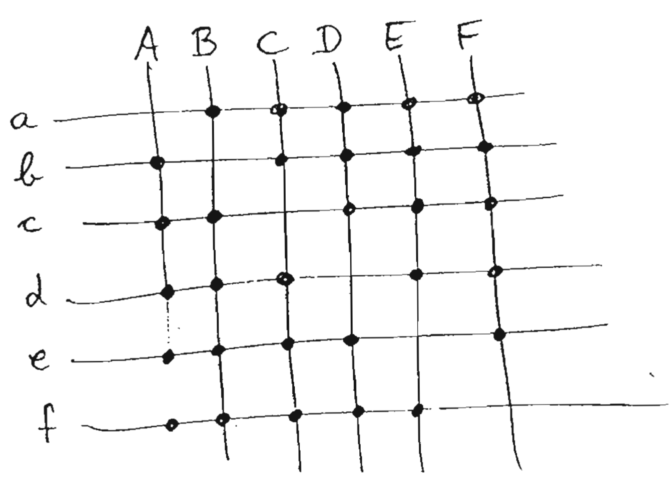
A Letter from R. Penrose

On browsing Hilbert's & Cohn-Vossen's book "Anschauliche Geometrie", I came across the section on Geometric Configurations, and thereby across the "Schläfli Double-Six" Configurations of lines in 3-space. Shown there is a picture of a 3-dimensional symmetric model, and it came to my mind that I had indeed seen such a model in the old collection of mathematical models that our institute still possesses. (Most people nowadays don't know what the significance of these models should be.) It also came to mind that it might be possible to have such a configuration in $CP_3 = PT$ but now contained in PN instead of some real projective subspace (RP_3), so that it would have a space-time interpretation. While this is true, the actual interpretation in terms of observers and light signals sounds far-fetched and not very elegant. The question was whether there is at least a very symmetric special case which has a more elegant interpretation. After a while I asked Roger about it, and his reply was as follows.

"Thank you for your letter. As you may have guessed from my original "Twistor Algebra" paper, I did spend some time thinking about the implications of twistor projective geometry for the ordinary geometry of Minkowski space. I'm sure I thought a little about the double-six configuration and what it meant in Minkowski terms, but I don't think I had anything at all elegant, or I'd surely have put it in the paper!

The best I can give you, having now spent a little time thinking further about the matter is the following: Consider lines $A, B, \ldots, F, a, \ldots, f$ in PN intersecting thus



and label the corresponding points of M the same. Take f to be the point i^0, so $A, \ldots, E$ are all on $\mathscr{I}$. Then $A, \ldots, E$ may be thought of as null hyperplanes, i.e. (more visualizably) as planes in Euclidean 3-space moving with unit speed. Arrange things so that D and E are parallel but moving in opposite directions, and suppose that each of A, B, C is perpendicular to D and E (and thus remains so as each moves). There is a time – say time zero – when the planes D and E coincide. Then their common plane meets A, B, C in a triangle Δ. The vertices (at that time) are a, b, c. A little later (or earlier – but let's say later) the planes A, B, C come together and have a common line λ. When this happens D and E will have separated, and meet λ in e and d, respectively. A little thought convinces us that λ is the perpendicular to the plane of Δ through the incentre of Δ (where Δ is thought of as persisting with time). The final point F is the circumcentre of Δ, at time equal to the circumradius R of Δ. As far as I can make out, the double-six theorem tells us that the distance between the circumcentre and the incentre of Δ is given by the geometric mean of

$$R \text{ and } R - 2r$$

where r is the inradius of Δ. I suppose that this is a classical theorem, known to the ancient Greeks! Have I got it right? The points a, b, c, d, e all have to lie on the (past) light cone of F...."

I could not easily find out whether this theorem on triangles was indeed known to the Greeks. It was known to Euler, and the generalization to n-gons was found by Jacobi. It is also related to Poncelet's Porism, but its relation to the double-six theorem is new. Considering the arguments given in Hilbert & Cohn-Vossen as a proof of the double-six theorem and continuing as Roger did, we have a proof of the triangle theorem "without calculation".

References:

D. Hilbert & S. Cohn-Vossen, Geometry and Imagination. Chelsea, N.Y. 1952.
Henderson, The 27 lines upon the cubic surface. Cambridge Tracts, Vol. 13 (1911).

H. Urbantke
Institute for Theoretical Physics
Boltzmanngasse 5
A-1090 Vienna, Austria
e-mail: urbantke@galileo.thp.univie.ac.at