Four-Twistor Particles

This note is written to clarify the algebraic structure of the constraint equations on internal operators and the interpretation in unitary space of a 4-twistor particle.

A.) The Internal Operator Algebra

The internal transformations of a 4-twistor particle $Z_i \to U_i^k (Z_k + \Lambda_{kl} \Sigma^l)$ have the infinitesimal operators

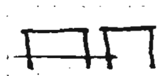
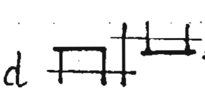
$$\underbrace{B^i_k = Z_k \Sigma^i}_{U(4)}, \quad \underbrace{d_{ik} = Z_i Z_k, \quad \bar{d}^{ik} = \Sigma^i \Sigma^k}_{\text{translations}} \qquad (\Sigma^i = -\partial/\partial Z_i) \quad \cdots(1)$$

where $i,k = 1,2,3,4$. These satisfy the Lie-algebra commutators

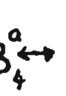
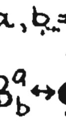
$$[B^i_j, B^k_l] = \delta^k_j B^i_l - \delta^i_l B^k_j, \qquad [d_{ij}, d_{kl}] = 0$$
$$[d_{ij}, B^k_l] = \delta^k_j d_{il} - \delta^k_i d_{jl}, \qquad [d_{ij}, \bar{d}^{kl}] = 0, \qquad \cdots(2)$$

and are subject to the algebraic constraints

$$d_{[ij}\, d_{k]l} = 0 \quad (a) \qquad d_{[ij}\, B^{[l}_{k]}\, \bar{d}^{mn]} = 0 \quad (b) \qquad \cdots(3)$$

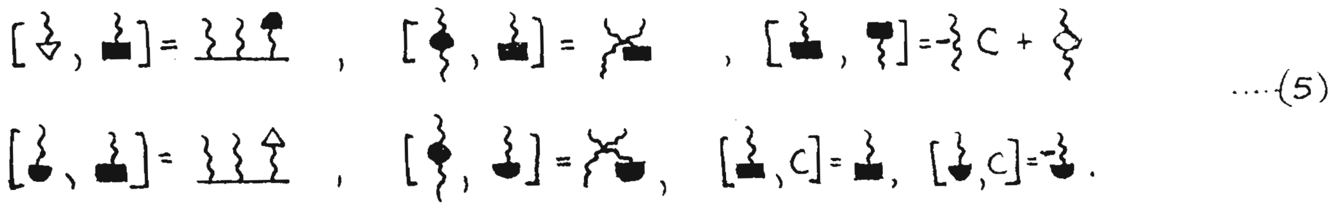
which follow from the identities  $=0$ and  $=0$, respectively.

How many independent conditions are contained in (3)? For a 3-twistor particle we have just a single one, of the form (3b), and this can even be eliminated by dropping the trace of B from among the generators (yielding ISU(3)). In order to answer the question, we now work with a 3+1 decomposition: $(a,b = 1,2,3)$

$d_{a4} \leftrightarrow$ , $\frac{1}{2}\epsilon^{abc} d_{bc} \leftrightarrow$ , $B^a_4 \leftrightarrow$ , $B^4_4 = C$, $B^a_b \leftrightarrow$ . $\cdots(4)$

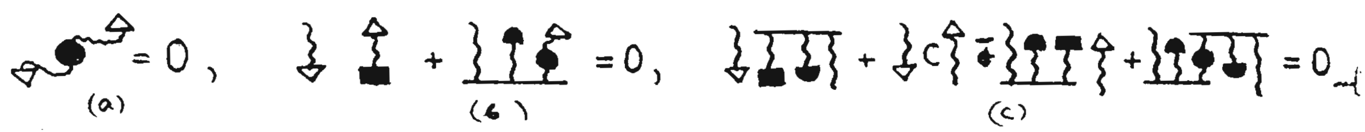
Commutators (2) are thus rewritten as the ISU(3) ones for

, and (Cf. Eq.(1.1.4) of preceding letter) plus



....(5)

In other combinations, the operators commute. The content of (3a) is =0 and (3b) yields the relations



Now (7b) and (7c) follow from (7a) and from (6). To show this, we compute the commutator of with , i.e, the ISU(3) basis vector (1.1.7). Consider first . Join here the vector from the right and use in the first term of . The result is (7b). Use of

yields (7c) in a similar way.

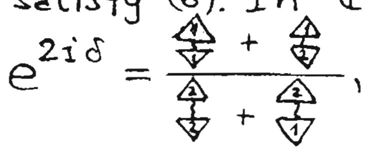
Thus we have established that the independent algebraic constraints on the 4-twistor operators are (6) and (7a).

B.) Interpretation in Unitary Space

The algebraic equation (7a) can be satisfied by taking the trace-free part . We then have the internal operators and C. These represent 27 Hermitian quantities with two Hermitian relations =0 among them. Now a point particle has only 14 parameters whereas two points have too many parameters. We seek for two point particles (in $\mathbb{C}^3$) with special properties.

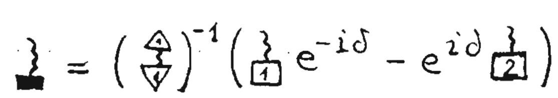
We denote the momentum and the angular momentum of the particles by , and , , respectively. For their center of mass we put

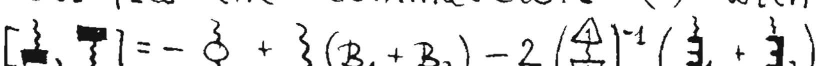
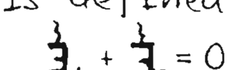
 & Herm. conj. .....(8)

The angular momentum and the momentum have the correct ISU(3) commutation properties, therefore we do the identification with the 4-twistor operators suggested by our notation. A similar identification with the relative momentum and angular momentum is not so straightforward. If unitary space was real, the vector would, for satisfy (6). In $\mathbb{C}^3$ we must introduce a phase factor  in the definition of the relative momentum  .....(9)

Now satisfies (6) and has the commutation properties required in (5). In order to retain the usual notion of a relative momentum in the real limit, the quantity δ must be Hermitian. Hence , i.e., the masses of the two unitary particles are equal.

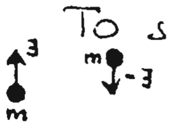
The orbital momentum of each of the particles is given by and , respectively, constructed according to the recipe (1.1.7) from the particle's operators. The relative orbital momentum

 ....(10)

satisfies the commutators (5) with the exception of . We obtain the correct result if we identify $-C = B_1 + B_2 - B - 2$ where the B of the center of mass is defined in (1.1.5), and if we impose on the particles  (their spin be oppositely equal). How many conditions are these?

Out of the 8 Hermitian components of the angular momentum, the orbital vector carries 4. The spin tensor contains only 3 independent Hermitian data, and the Hermitian scalar B contains one more spin which has 3 essential components!

A pair of unitary point particles has, generally, 28 parameters. Now the masses are equal (1 less parameter) and the unitary spins are oppositely equal (3 less parameters). Thus the system has 25 free parameters. This is what we want since 27 internal operators – 2 constraints(=0 make 25, and C can be expressed algebraically from (7c).

To sum up, a 4-twistor particle is pictured by point masses 

in the unitary space. If we believe in an interaction between these points then we end up with something like a Kählerian hydrogen atom.

[1] Picturing intrinsic properties of particles, TN8 (1979)

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