Average-marginally-trapped surfaces in flat space

By an *average-marginally-trapped surface* I mean a (closed, topologically spherical) space-like 2-surface S with the property that the expansion of the out-going null normals to S is zero *on the average*. Equivalently, the average rate of change of the area along the out-going null normals is zero. This is a natural definition - if the rate of change of the area element is zero at each point, the surface is marginally trapped - and not original to me.

Now ask: can average MTSs exist in flat space? True MTSs cannot, because of Roger's original singularity theorem (Phys.Rev.Lett.**14**(1965)57-59). It is not hard to see that on the contrary average MTSs can exist in flat space - the idea is to have regions on S which shrink into the future faster than other regions expand. Here I want to describe a concrete example.

Working in a hyperplane of constant time, you pile up six spheres (say radius a) like an octahedron - four are shown solid and the other two are one above the other and shown dotted. Then you put a big sphere (say radius r, determined by a) around them and nearly touching them; then you tunnel through from the big sphere to each little sphere to make something topologically spherical - the tunnels are small, four are shown in solid and the other two dotted.

Since this surface lies in a constant-time hyperplane, the rate of change of area along the outgoing null normals equals the rate of change along the outgoing space-like normal in t=constant. If this surface moves along these outwards normals, the small spheres get smaller and the big one gets bigger, but the small ones 'win'.

To be qualitative, recall that the rate of change of the area of a surface along the out-going normals is the integral of the mean-curvature over the surface. The big sphere has mean-curvature $1/r$ and area $4\pi r^2$ so the mean-curvature integral for the big sphere is $4\pi r$ (less terms of order ϵ or δ for the tunnels); for the small ones it is therefore 6 times $-4\pi a$ (again less terms of order ϵ or δ; the sign is changed because the normal points into each small sphere); for the 'tunnels' the mean-curvature is $O(1/\epsilon + 1/\delta)$ and the area is $O(\epsilon\delta)$ so the mean-curvature integral is small, of order ϵ or δ. Finally, by a bit of trigonometry, $r=a(1+\sqrt{2}) + O(\epsilon)$.

For the whole surface therefore

$$\text{mean-curvature integral} \sim 4\pi\,(r-6a) + O(\epsilon,\delta) \sim -45a + O(\epsilon,\delta)$$

which is negative.

Since this is negative, we can make the small spheres smaller until it vanishes (imagine deflating them like balloons with the tunnels held fixed). When the mean-curvature integral vanishes, so will the average of ρ ('rho') and the surface will be 'average-trapped'.

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