'Special' Einstein–Weyl spaces from the heavenly equation

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In a recent $\mathbb{TN}$ article [2] one of us showed how real three-dimensional Einstein–Weyl spaces which come from four-dimensional hyper-Kähler metrics with triholomorphic conformal Killing vectors can be determined by solutions of the pair of equations

$$V = \frac{1+S\overline{S}}{S_{\overline{z}}+\overline{S}_z}, \qquad S_t + iS = 2iV_z. \tag{1}$$

Here S and V are, respectively, complex and real valued functions of three real coordinates (x, y, t) and $z = x + iy$. In this note we want to point out how equation (1) arises as a reduction of Plebański's first heavenly equation by a dilatation Killing vector.

Let $(w, z, \overline{w}, \overline{z})$ be a null coordinate system on $\mathbb{R}^4$. Each hyper-Kähler metric can be written as

$$g = \Omega_{w\overline{w}}dwd\overline{w} + \Omega_{w\overline{z}}dwd\overline{z} + \Omega_{z\overline{w}}dzd\overline{w} + \Omega_{z\overline{z}}dzd\overline{z} \tag{2}$$

where $\Omega = \Omega(w, z, \overline{w}, \overline{z})$ satisfies the first heavenly equation [1]

$$\Omega_{w\overline{w}}\Omega_{z\overline{z}} - \Omega_{w\overline{z}}\Omega_{z\overline{w}} = 1. \tag{3}$$

Let $K = w\partial_w + \overline{w}\partial_{\overline{w}}$ be the homothetic Killing vector (it turns out that there is no loss of generality in this choice). The Killing equation together with an appropriate choice of gauge yields $\mathcal{L}_K\Omega = \Omega$. Define real valued functions t and $\hat{t}$ by $\ln w = \hat{t} + it$. The general solution of (3) subject to the Killing equation is of the form $\Omega = e^{\hat{t}}F(z, \overline{z}, t)$. The heavenly equation (3) gives

$$F_{z\overline{z}}(F + F_{tt}) - (F_z + iF_{tz})(F_{\overline{z}} - iF_{t\overline{z}}) = 4. \tag{4}$$

Define $S = (F_z + iF_{tz})/2$ and $V = (F + F_{tt})/4$. It follows from a straightforward calculation that functions S and V satisfy equation (1). Conversely, the second equation in (1) gives the integrability condition for the existence of F, and then the first equation yields (4). With the definition $\zeta = \hat{t} + it$ the hyper-Kähler metric (2) becomes

$$g = e^{\hat{t}}[V^{-1}(1 + S\overline{S})dzd\overline{z} + Sdzd\overline{\zeta} + \overline{S}d\overline{z}d\zeta + Vd\zeta d\overline{\zeta}]. \tag{5}$$

From the last formula one finds the Einstein–Weyl metric and the associated one form to be

$$h = dzd\overline{z} + (Vdt - i(Sdz - \overline{S}d\overline{z})/2)^2, \; \alpha = (2V)^{-1}(Sdz + \overline{S}d\overline{z}).$$

References

[1] Plebański, J. F. (1975) Some solutions of complex Einstein Equations, J. Math. Phys. **16** 2395-2402.

[2] Tod, K.P. (1997) 'Special' Einstein-Weyl spaces, Twistor Newsletter **42**, 13-15.