

Singularities of wavefronts and lightcones in the context of GR via null foliations

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(September 4, 1997)

We describe an approach to the issue of the singularities of null hypersurfaces, due to the focusing of null geodesics, in the context of the recently introduced formulation of GR via null foliations.

The null-surface approach to general relativity essentially reformulates general relativity in terms of two real functions on the bundle of null directions over the spacetime manifold (locally $M^4 \times S^2$ with a metric $g_{ab}(x^a)$ on M^4). These two functions are $Z(x^a, \zeta, \bar{\zeta})$, representing a sphere's worth of null foliations of the spacetime (i.e., such that $\tilde{g}^{ab} Z_{,a} Z_{,b} = 0$ for all values of ζ and for members \tilde{g}_{ab} of the conformal class of g_{ab}), and $\Omega(x^a, \zeta, \bar{\zeta})$, representing a sphere's worth of conformal factors with the role of picking the members of the the conformal class that satisfy the Einstein equations. More detail on this formulation can be found in [1].

Within the context of the null-surface approach to general relativity a family of null coordinate systems ($\theta^i, i = 0, 1, +, -$) is heavily used to derive dynamical equations for Z and Ω . The coordinates θ^i are defined by derivation from Z :

$$\theta^i = (Z, \partial\bar{\partial}Z, \bar{\partial}Z, \bar{\bar{\partial}}Z) \equiv (u, R, \omega, \bar{\omega}), \quad (1)$$

which, for fixed ζ , represents a transformation between an arbitrary coordinate system x^a and θ^i . The coordinates θ^i are adapted to the null foliations, so that the leaves $Z(x^a, \zeta, \bar{\zeta}) = \text{const.}$ of the foliation constitute surfaces of constant coordinate u .

There is some gauge freedom in the theory, due to the fact that there are different foliations which are all null with respect to the same metric, such as, for Minkowski spacetime, a foliation based on outgoing lightcones off a world line as opposed to a foliation by null planes. In the case of asymptotically flat spacetimes, we customarily fix the gauge by requiring that our null foliation consists of surfaces that asymptotically become null planes.

For every fixed value of ζ , our special coordinates have a well defined interpretation. The coordinate u labels leaves of the null foliation. The coordinates $(\omega, \bar{\omega})$ label null geodesics on a fixed leaf. The remaining coordinate R acts as a parameter along every null geodesic in the leaf. Because of our gauge choice of null surfaces becoming planes at null infinity, the null geodesics labeled by $(\omega, \bar{\omega})$ constitute bundles of asymptotically parallel null geodesics.

It is natural to raise the objection that, generically, any vacuum spacetime other than Minkowski focuses non-diverging bundles of null geodesics [2], and therefore our coordinate systems break down at the point of focusing, by assigning different labels to the same spacetime point. In this respect, although coordinate singularities are irrelevant to the physical content of the dynamical null-surface equations, we feel that the break down of these particular coordinates has a certain appeal since it entails the existence and location of caustics. Caustics and their singularities, as well as the singularities of wavefronts, have been classified by Arnol'd within a sophisticated mathematical context [3]. On the other hand, caustics are increasingly being considered in the field of astrophysical observations [4-7].

The null-surface approach provides a dual interpretation to the function Z . The condition $Z(x^a, \zeta, \bar{\zeta}) = u$ for fixed x^a picks up the points $(u, \zeta, \bar{\zeta})$ at scri which are connected to x^a by null geodesics. These points lie on a two-surface at scri, referred to as the lightcone cut of the point x^a . Generically, due to focusing in the interior, the lightcone cuts have self-intersections and typical wavefront singularities such as cusps and swallowtails. This appears to pose a technical difficulty regarding the perturbative approach to solving the null-surface equations, since the occurrence of this type of singularity generically entails divergences in the derivatives of the lightcone cuts.

In the following, we examine the occurrence of singularities of wavefronts and lightcones and its relevance to the null-surface approach.

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A. Singularities of the null coordinates

Consider a foliation of spacetime by past null cones from points $(u, \zeta, \bar{\zeta})$ at scri along a fixed null generator ζ . (This is the compactified version of a foliation by null surfaces that are asymptotically null planes). Every past lightcone in this foliation has singularities, in the sense that the lightcone "folds" and self-intersects, due to focusing in the interior spacetime, as shown in Fig. 1. The points where the past lightcone is singular are points where neighboring null geodesics of the congruence intersect, and are thus conjugate to the point at scri. Translating this into the physical non-compactified spacetime, these points in the interior are "focal points", such that light rays emitted from them are asymptotically parallel. How can we locate these "focal points" in terms of null-surface variables?

A preliminary answer to this question can be approached quite directly from a consideration of the geodesic deviation vector of the congruence that becomes asymptotically parallel in a direction $(\zeta, \bar{\zeta})$ at scri. Every null geodesic in this congruence is characterized by fixed values of $(u, \zeta, \bar{\zeta}, \omega, \bar{\omega})$. The geodesic deviation vector has been derived earlier in [8]. Along a fixed null geodesic, the cross sectional area of the congruence has the expression:

$$A_p(R; u, \zeta, \bar{\zeta}, \omega, \bar{\omega}) = \frac{1}{\Omega^2 \sqrt{1 - \Lambda_{,1} \bar{\Lambda}_{,1}}}, \quad (2)$$

where $_{,1}$ represents $\partial/\partial R$. This is an expression of the area A_p in terms of the two null-surface variables

$$\Omega = \Omega(x^a, \zeta, \bar{\zeta}) \quad \Lambda \equiv \partial^2 Z(x^a, \zeta, \bar{\zeta}). \quad (3)$$

In (2), the quantities $\Omega(x^a, \zeta, \bar{\zeta})$ and $\Lambda(x^a, \zeta, \bar{\zeta})$ are evaluated at fixed $(\zeta, \bar{\zeta})$ and at values of x^a along the null geodesic given by $(u, \zeta, \bar{\zeta}, \omega, \bar{\omega})$.

It is relevant to point out that the variable Ω actually is the product of two factors, one of which is an arbitrary conformal factor for the conformal class (and as such, it does not depend on ζ), whereas the other factor carries conformal information via its ζ -dependence. Therefore it is not surprising to find that it plays a role in a completely conformally invariant matter such as the determination of conjugate points of null geodesic congruences.

Focusing takes place at the value of R such that $A_p = 0$. However, the only way for the area to vanish is that the denominator become infinite. The square root in the denominator can not diverge before becoming pure imaginary. This leaves us with the previously unsuspected result that along a fixed null geodesic in the past lightcone from a point $(u, \zeta, \bar{\zeta})$ at scri, Ω must blow up for focusing to take place.

The vanishing of A_p is also related to the vanishing of the determinant of the metric, since in these coordinates we have

$$g \equiv \det(g_{ij}) = (\det(g^{ij}))^{-1} = \frac{1}{\Omega^8 (1 - \Lambda_{,1} \bar{\Lambda}_{,1})} = \Omega^{-4} A_p^2 \quad (4)$$

Thus both the vanishing of A_p and the fact that Ω diverges result in the vanishing of the 4-dimensional volume element.

This divergence of Ω has another significant consequence. Along the null geodesics labeled by $(\omega, \bar{\omega})$ there is a choice of an affine parameter s , which is related to R via

$$\frac{ds}{dR} = \Omega^{-2} \quad \text{or alternatively} \quad \frac{dR}{ds} = \Omega^2. \quad (5)$$

Since the affine parameter is regular, it follows that dR/ds blows up as well at the point where Ω does. Thus R is a bad coordinate (as we might have suspected) in the neighborhood of a focal point. Both R and Ω are, however, determined by the function Z through the same second derivative (See [1] for details):

$$R \equiv \partial \bar{\partial} Z(x^a, \zeta, \bar{\zeta}), \quad \Omega^2 \equiv g^{ab} Z_{,a} \partial \bar{\partial} Z_{,b}, \quad (6)$$

thus it is consistent to attribute both complications to $\partial \bar{\partial} Z(x^a, \zeta, \bar{\zeta})$ becoming singular, since g^{ab} is smooth in a good choice of coordinates x^a .

B. Singularities of the lightcone cuts

In asymptotically flat spacetimes the function Z can also be viewed as describing the intersection of the lightcone of a point x^a with scri, via $u = Z(x^a, \zeta, \bar{\zeta})$ where $(u, \zeta, \bar{\zeta})$ are Bondi coordinates on scri. This intersection is a two-surface

in a three-dimensional space and can be thought as one member of a series of “wavefronts” obtained by slicing the lightcone of a point x^a with a one-parameter family of past lightcones, the last one of them being scri itself. In this respect, the two-surface at scri given by $u = Z(x^a, \zeta, \bar{\zeta})$ can be thought of as a two-dimensional wavefront in a three-dimensional space, for which there is a standard treatment in singularity theory.

Wavefronts are considered as projections of smooth two-dimensional Legendrian manifolds in a five-dimensional space down to a three-dimensional space. The singularities of the wavefront are the places where the projection is singular. The Legendrian manifold itself is obtained from a generating function. More specifically, consider the 1-jet bundle over the two dimensional configuration space (x^1, x^2) , given in coordinates (z, x^1, x^2, p_1, p_2) , and a function $S(p_1, x^2)$. A Legendrian manifold is a two dimensional subspace of points (z, x^1, x^2, p_1, p_2) that can be specified by [3]

$$x^1 = \frac{\partial S}{\partial p_1}, \quad p_2 = -\frac{\partial S}{\partial x^2}, \quad z = S(p_1, x^2) - x^1 p_1, \quad \text{parametrized by } x^2 \text{ and } p_1. \quad (7)$$

The projection of this Legendrian manifold down to (z, x^1, x^2) is a two-surface in three dimensions, representing a wavefront. This wavefront is singular where the projection breaks down, namely at points such that the 3×2 Jacobian matrix

$$\begin{pmatrix} \frac{\partial x^1}{\partial x^2} & \frac{\partial x^1}{\partial p_1} \\ \frac{\partial x^2}{\partial x^2} & \frac{\partial x^2}{\partial p_1} \\ \frac{\partial z}{\partial x^2} & \frac{\partial z}{\partial p_1} \end{pmatrix} \quad (8)$$

has rank less than 2.

In our case, we take (x^1, x^2) as coordinates on the sphere by defining $\zeta = x^1 + ix^2$ and interpret the projection of z as the lightcone cut function $Z(x^a, \zeta, \bar{\zeta})$ (the dependence on the spacetime points x^a is considered parametric in the instance of lightcone cuts, being regarded as fixed here). The lightcone cut is regular except at some values of ζ at which the projection breaks down. Carrying through the calculation of the determinant of the Jacobian matrix in these terms, we obtain the result that the projection breaks down at points ζ such that

$$\frac{1}{\partial^2 Z(x^a, \zeta, \bar{\zeta})} = 0 \quad \text{and} \quad \frac{1}{\partial \bar{\partial} Z(x^a, \zeta, \bar{\zeta})} = 0. \quad (9)$$

This calculation is parametric in x^a , where x^a represents the apex of the lightcone intersecting scri at the lightcone cut surface. This is equivalent to the statement that both Λ and R must blow up at particular values of ζ for the lightcone cut of a given spacetime point x^a (See Fig.2). For every point on the cut, the quantities Λ and R have finite values, except at the singular points shown in Fig.2. Given the lightcone cut function $Z(x^a, \zeta, \bar{\zeta})$ for fixed x^a , the values of ζ for which Λ and R diverge determine the null geodesics for which x^a is a focal point. This is consistent with the result found in the previous section.

C. Singularities of the lightcones

A related issue which we are also concerned with is the location of the singularities of the lightcone of a point in the interior spacetime. Consider the lightcone of a point, namely, the congruence of all the null geodesics through that point, and follow one null geodesic in the congruence out to the future. Generically, due to curvature, there is at least one future point along this null geodesic where neighboring geodesics intersect, referred to as a point conjugate to the apex. The cross sectional area of the congruence vanishes at points which are conjugate to the apex along any null geodesic of the congruence, the locust of all such points being sometimes referred to as the caustic surface (although it would be more accurate to call it the singularity of the lightcone). The lightcone folds and self-intersects beyond the occurrence of the earliest of the points conjugate to the apex, as shown in Fig. 2. How do we formulate the condition for the location of such singular points in terms of null-surface variables?

We can approach this issue from an analysis of the geodesic deviation vector along a fixed null ray ζ of the lightcone of an interior point x_0^a . The geodesic deviation vector in this case can be derived solely from knowledge of $Z(x^a, \zeta, \bar{\zeta})$ through a procedure that is standard to the null-surface approach. The coordinate transformation (1) can, in principle, be piece-wise inverted, yielding, perhaps with different branches,

$$x^a = f^a(u, \omega, \bar{\omega}, R, \zeta, \bar{\zeta}) \quad (10)$$

for fixed ζ . The lightcone of a point x_0^a can be obtained now by substituting

$$u = Z(x_0^a, \zeta, \bar{\zeta}) \quad \omega = \bar{\partial}Z(x_0^a, \zeta, \bar{\zeta}) \quad \bar{\omega} = \bar{\partial}\bar{Z}(x_0^a, \zeta, \bar{\zeta}) \quad (11)$$

and

$$R = \bar{\partial}\bar{\partial}Z(x_0^a, \zeta, \bar{\zeta}) + r \quad (12)$$

into (10), in this manner obtaining, perhaps with several branches,

$$x^a = F^a(r; x_0^a, \zeta, \bar{\zeta}) \quad (13)$$

as the lightcone of the point x_0^a parametrized by the directions $(\zeta, \bar{\zeta})$ labeling the null geodesics, and the parameter r along each null geodesic. The geodesic deviation vector is

$$M^a = \bar{\partial}F^a = \frac{\partial f^a}{\partial \theta^i} \bar{\partial}\theta^i + \bar{\partial}' f^a \quad (14)$$

where $\bar{\partial}'$ is taken keeping θ^i fixed. This expression can be worked out straightforwardly to yield M^a in terms of Z and its derivatives, and the cross sectional area of the congruence along the null ray ζ can subsequently be obtained:

$$A_{lc}(r; x_0^a, \zeta, \bar{\zeta}) = \frac{(\Lambda - \Lambda_0)(\bar{\Lambda} - \bar{\Lambda}_0) - (R - R_0)^2}{\Omega^2 \sqrt{1 - \Lambda_{,1} \bar{\Lambda}_{,1}}} \quad (15)$$

The sublabel 0 indicates evaluation at $r = 0$ (the apex). The quantities Λ_0 and R_0 appearing in (15) can be thought of as referring to the lightcone cut of the apex, whereas Λ and R correspond to the lightcone cut of the point at r along the null geodesic labeled by ζ (See Fig. 3). By comparing with (2) we can see that

$$A_{lc} = A_p H(r; x_0^a, \zeta, \bar{\zeta}) \quad \text{with} \quad H(r; x_0^a, \zeta, \bar{\zeta}) \equiv (\Lambda - \Lambda_0)(\bar{\Lambda} - \bar{\Lambda}_0) - (R - R_0)^2. \quad (16)$$

This equation relates the cross sectional areas of two different congruences containing the same null ray ζ , namely the cone of lightrays through x_0^a and the congruence of asymptotically parallel rays parallel to the null ray ζ . This implies a relationship between the focusing of either congruence and the behavior of the quantities Λ and R . We distinguish three alternatives.

1. $A_{lc} = 0$ and $A_p = 0$ at some value r . Then H must not diverge at a rate faster than A_p^{-1} at that point.
2. $A_{lc} = 0$ with $A_p \neq 0$ at some value r . Then H must vanish at that point.
3. $A_p = 0$ with $A_{lc} \neq 0$ at some value r . Then H must blow up at a rate faster than A_p^{-1} at that point.

As an example of the first instance, if $A_p = 0$ at the apex then the apex is a focal point, because $H = 0$, and is therefore finite, at $r = 0$ (See Fig 4). So should be any point conjugate to it along the null ray ζ ; however, at such points Λ_0, R_0, Λ and R all blow up, therefore H becomes quite intractable and its behavior has yet to be verified (See Fig. 5).

In the second case, the apex is not a focal point, and its conjugate point is located at r such that

$$H(r; x_0^a, \zeta, \bar{\zeta}) = 0. \quad (17)$$

This is the generic situation illustrated in Fig.2. See also Fig. 6.

In the third case, the point r at which A_p vanishes is a focal point along the null ray ζ , and therefore Λ and R both blow up, thus again H becomes intractable but is not unlikely that it blows up, since Λ_0 and R_0 are finite in this case. See Fig. 6.

This analysis applies to every null ray ζ in the lightcone. For some null ray ζ_m the conjugate point occurs closest to the apex, for some other direction ζ_f the apex has a conjugate point at infinity, and finally there are values of ζ for which no focusing takes place at any value of r , as can be seen from Fig.2.

D. Conclusion

Although the work reported on here is very much in progress, we believe we are finding significant clues as to what consequences the occurrence of focusing of null geodesics has for the null-surface formulation of general relativity. Our ultimate goal is to find a way to integrate the singularity issue with the null-surface dynamical equations, which played no role in this discussion.

This work has been supported by the NSF under grant No. PHY 92-05109.

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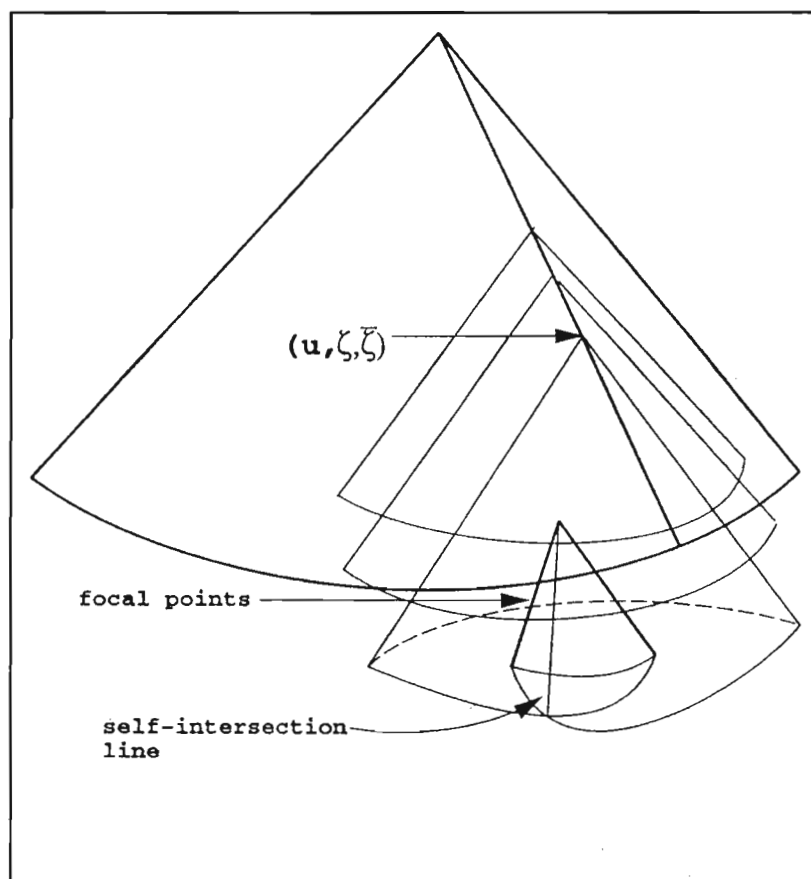


FIG. 1. A null foliation by past lightcones of points at scri.

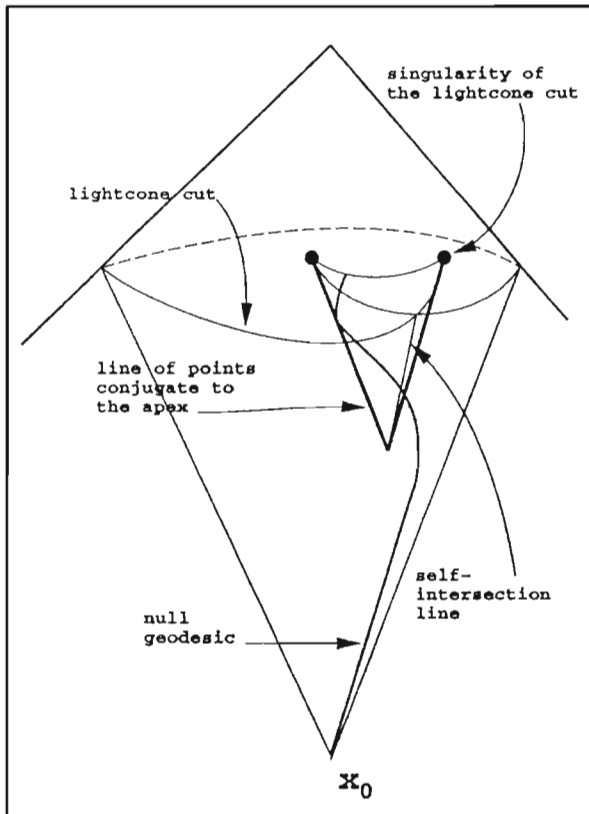


FIG. 2. The lightcone and lightcone cut of a point x_0^a . For the null geodesic shown, both $\Lambda(x_0^a, \zeta, \bar{\zeta})$ and $R(x_0^a, \zeta, \bar{\zeta})$ are finite.

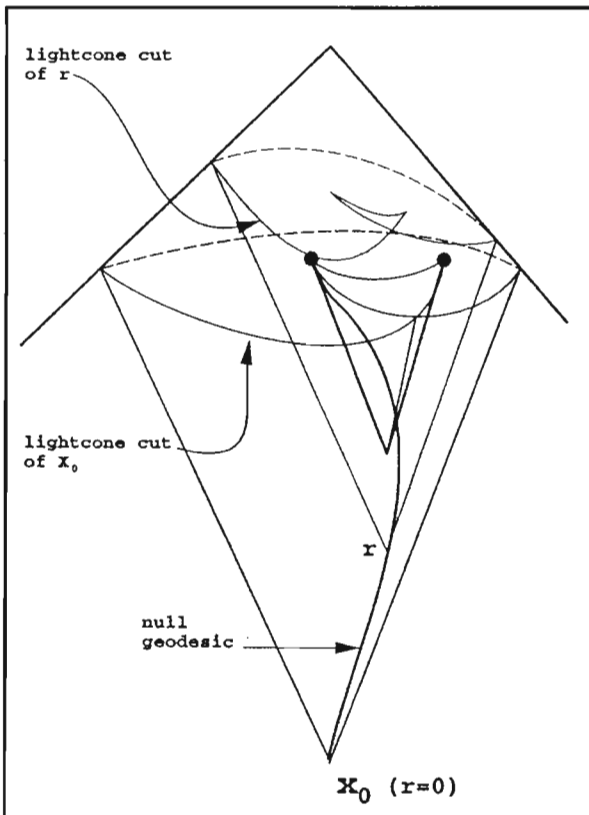


FIG. 3. Evaluating Λ and R at points s along the null geodesic. In this case, Λ_0 and R_0 blow up, where Λ and R are finite at the point r further up, since the null geodesic does not hit a singularity of the lightcone cut of the point r .

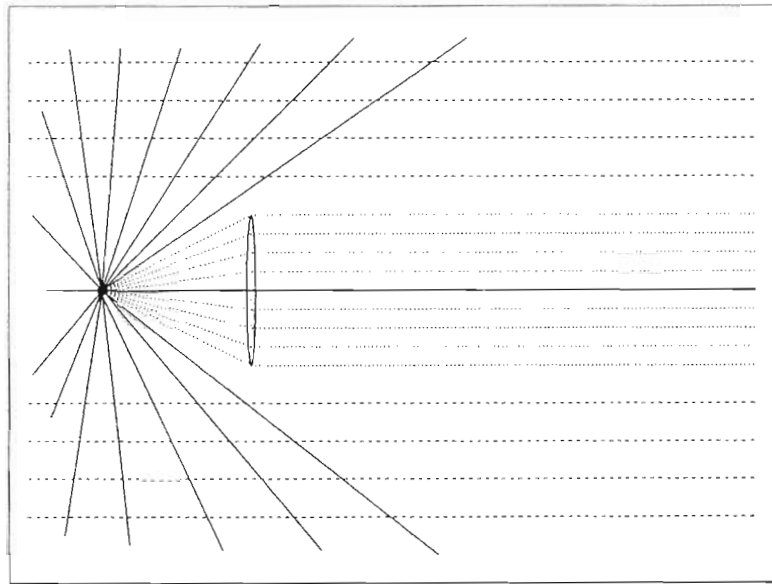


FIG. 4. Two null congruences containing the same null ray. The common null ray is shown in boldface. The solid lines represent the lightcone congruence, whereas the dashed lines represent the congruence asymptotically parallel. The dotted rays are also common to both congruences. In this case, the apex is a focal point.

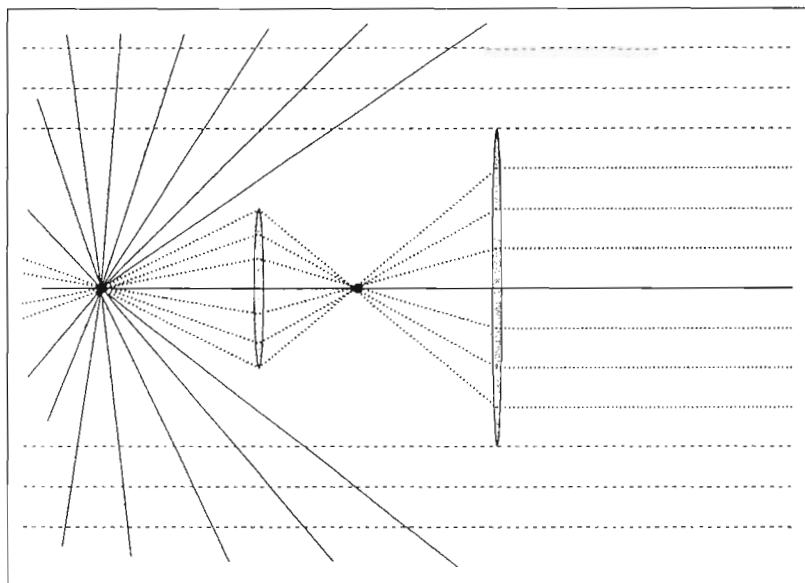


FIG. 5. Two null congruences containing the same null ray. The common null ray is shown in boldface. The solid lines represent the lightcone congruence, whereas the dashed lines represent the congruence asymptotically parallel. The dotted rays are also common to both congruences. In this case, the apex is a focal point and so is the focusing point between the two lenses.

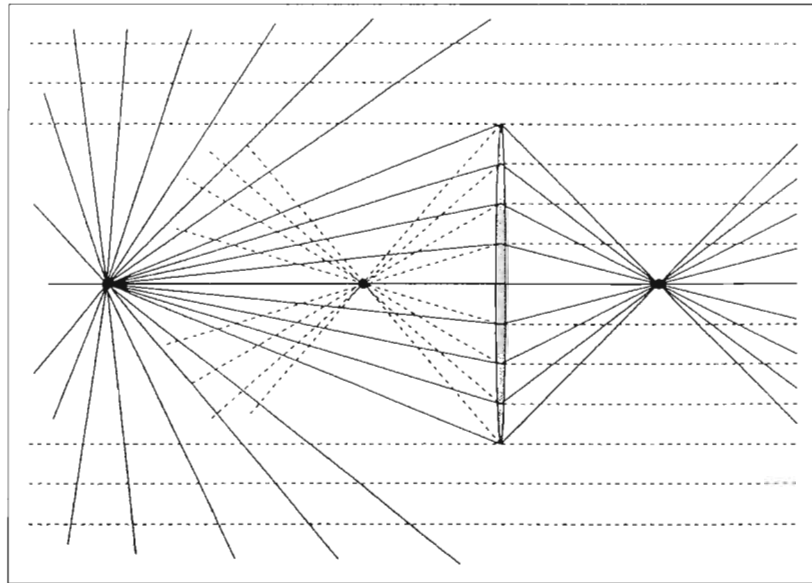


FIG. 6. Two null congruences containing the same null ray. The common null ray is shown in boldface. The solid lines represent the lightcone congruence, whereas the dashed lines represent the congruence asymptotically parallel. In this case, the apex is not a focal point, nor is its conjugate point beyond the lens. The focal point is between the lens and the apex.