Triviality of the Grassmann bundles on hypersurfaces in $\mathbb{R}^{m+1}$

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The triviality of the bundles of spinors on spheres has been recognized in connection with work on Killing spinors [1] and used to obtain an explicit expression for the eigenfunctions of the Dirac operator on these spaces [2]. Every hypersurface M in $\mathbb{R}^{m+1}$ has a pin$^-$ structure and the associated complex bundle $\Sigma \to M$ of spinors is trivial [3]. If the dimension m of the hypersurface M is even, then the trivial bundle $\Sigma \otimes \Sigma$ is isomorphic to $\mathbb{C} \otimes \bigwedge TM$ even though the tangent bundle $TM \to M$ is not trivial, in general. In this Letter, I present a few simple results on the triviality of the exterior algebra (Grassmann) bundles of hypersurfaces in $\mathbb{R}^{m+1}$.

Let the vector space $\mathbb{R}^{m+1}$ be given the standard, positive-definite quadratic form h and an orientation; these data define the Hodge map $\star : \bigwedge\mathbb{R}^{m+1} \to \bigwedge\mathbb{R}^{m+1}$ such that $\star\star = (-1)^{\frac{1}{2}m(m+1)}\mathrm{id}_{\bigwedge\mathbb{R}^{m+1}}$. Consider a hypersurface M in $\mathbb{R}^{m+1}$, i.e. a connected smooth manifold M, of dimension m, together with an immersion $i : M \to \mathbb{R}^{m+1}$. The tangent space T_xM to M at $x \in M$ is identified with its image by T_xi, this image being considered as an m-dimensional vector subspace of $\mathbb{R}^{m+1}$. This identification extends, in a natural manner, to a linear injection $\bigwedge T_xM \to \bigwedge\mathbb{R}^{m+1}$. The same letter is used to denote an element of $\bigwedge T_xM$ and its image in $\bigwedge\mathbb{R}^{m+1}$. Let $\bigwedge_0\mathbb{R}^{m+1}$ denote the even subalgebra of $\bigwedge\mathbb{R}^{m+1}$ and let $\bigwedge_0 TM$ be the bundle of even multivectors on M.

Proposition 1. *If the hypersurface M is orientable, then the vector bundle $\bigwedge TM \to M$ is trivial.*

Proof. Since M is orientable, there is a vector field $n : M \to \mathbb{R}^{m+1}$ of unit normals to M. A trivialization $f : \bigwedge TM \to M \times \bigwedge_0\mathbb{R}^{m+1}$ is defined as follows. Let $a \in \bigwedge T_xM$ be either even or odd; if a is even, then $f(a) = (x, a)$; if a is odd, then $f(a) = (x, n_x \wedge a)$.

Proposition 2. *If the hypersurface M is even-dimensional, then the vector bundle $\bigwedge TM \to M$ is trivial.*

Proof. The trivializing map $f : \bigwedge TM \to M \times \bigwedge_0 \mathbb{R}^{m+1}$ is now defined as follows: $f(a) = (x, a)$ for a even and $f(a) = (x, \star a)$ for a odd, $a \in \bigwedge T_x M$.

Proposition 3. *If the hypersurface M is of dimension $m \equiv 3 \bmod 4$, then the vector bundle $\bigwedge_0 TM \to M$ is trivial.*

Proof. If $m \equiv 3 \bmod 4$, then $\star\star = \mathrm{id}_{\bigwedge \mathbb{R}^{m+1}}$. Let $\bigwedge_0^+ \mathbb{R}^{m+1}$ be the vector space of self-dual, even multivectors over $\mathbb{R}^{m+1}$. A trivializing map $f : \bigwedge_0 TM \to M \times \bigwedge_0^+ \mathbb{R}^{m+1}$ is defined by $f(a) = (x, a + \star a)$ for $a \in \bigwedge_0 T_x M$. To prove that the map f is an isomorphism of vector bundles, one constructs the inverse map $f^{-1} : M \times \bigwedge_0^+ \mathbb{R}^{m+1} \to \bigwedge_0 TM$ as follows. Given $x \in M$, let l be a unit vector orthogonal to $T_x M$. Denoting by λ the 1-form associated with l by h, one has $\lambda \lrcorner l = 1$ and $\bigwedge T_x M = \{c \in \bigwedge \mathbb{R}^{m+1} : \lambda \lrcorner c = 0\}$. By virtue of the identity $\lambda \lrcorner \star c = \star(l \wedge c)$ one has $f^{-1}(x, b) = \lambda \lrcorner \star (\lambda \lrcorner b)$ for every $b \in \bigwedge_0^+ \mathbb{R}^{m+1}$.

If $m \equiv 1 \bmod 4$, then $\star\star = -\mathrm{id}_{\bigwedge \mathbb{R}^{m+1}}$. Upon complexification, one can define a trivializing map $f : \mathbb{C} \otimes \bigwedge_0 TM \to M \times \bigwedge_0^+ \mathbb{C}^{m+1}$ by putting $f(a) = (x, a - \mathrm{i} \star a)$, where now $\bigwedge_0^+ \mathbb{C}^{m+1} = \{b \in \bigwedge_0 \mathbb{C}^{m+1} : \star b = \mathrm{i} b\}$. This proves

Proposition 4. *If the hypersurface M is odd-dimensional. then the complex vector bundle $\mathbb{C} \otimes \bigwedge_0 TM \to M$ is trivial.*

Questions. Does there exist a non-orientable, odd-dimensional hypersurface M in $\mathbb{R}^{m+1}$ such that the vector bundle $\bigwedge TM \to M$ is not trivial? Are there hypersurfaces of dimension $m \not\equiv 3 \bmod 4$ such that $\bigwedge_0 TM \to M$ is not trivial?

References

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