A Recursion Operator for ASD Vacuums and Z.R.M Fields on ASD Backgrounds

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1 Introduction

In this note, we show that the recursion operator given in [1] for the ASD vacuum equations has a simple form on twistor space analogous to the Yang-Mills case.

Let $\mathcal{M}$ be an oriented complex four manifold with ASD vacuum metric g and let $\mu : \mathcal{PT} \longrightarrow \mathbb{CP}^1$ be a corresponding projective twistor space fibred over a Riemman sphere. Let $Z^\alpha = (\omega^A, \pi_{A'}) \in \mathcal{PT}$. Choose a constant spinor $o_{A'} = (0,1)$ on the base space and parametrise a section of μ by four complex coordinates

$$x^{AA'} = \frac{\partial \omega^A}{\partial \pi_{A'}}\Big|_{\pi_{A'}=o_{A'}}, \quad x^{A1'} = w^A, \; x^{A0'} = x^A. \tag{1}$$

In [1] (following [4])we showed that the existence of a two form on the fibres of μ with values in the pullback of $\mathcal{O}(2)$ guarantees the existence of a complex valued function satisfying the second Plebański equation

$$\frac{\partial^2\Theta}{\partial w^A \partial x_A} + \frac{1}{2}\frac{\partial^2\Theta}{\partial x^B \partial x^A}\frac{\partial^2\Theta}{\partial x_B \partial x_A} = 0. \tag{2}$$

With the parametrisation (1) the right flat metric on $\mathcal{M}$ is given by

$$g = dx_A dw^A + \frac{\partial^2\Theta}{\partial x^A \partial x^B} dw^A dw^B.$$

The converse problem of reconstructing a deformed twistor space given a Θ function on $\mathcal{M}$ was reformulated as a system of time dependent Hamilton's equations.

Linearised solutions to equation (2) satisfy the wave equation on ASD background given by Θ

$$\Box_g \delta\Theta = 0. \tag{3}$$

Let $\mathcal{W}_g$ be the space of solutions to equation (3). We constructed a (formal) recursion operator $R : \mathcal{W}_g \longrightarrow \mathcal{W}_g$ given by the relation

$$\nabla_{A1'}\phi = \nabla_{A0'} R\phi. \tag{4}$$

Here $\nabla_{AA'}$ is a null tetrad of volume preserving vector fields which in the adopted coordinate system takes form

$$\nabla_{A0'} = \frac{\partial}{\partial x^A}, \qquad \nabla_{A1'} = \frac{\partial}{\partial w^A} + \frac{\partial^2\Theta}{\partial x^A \partial x^B}\frac{\partial}{\partial x_B}. \tag{5}$$

The definition is formal because the the operator on the right hand side requires boundary conditions in order for it to be invertible and without these, there are ambiguities in the definition of $R\phi$.

The aim of this paper is to give a twistor description of a recursion procedure. We shall show that R is defined without ambiguities on twistor functions and the space-time relation (4) is replaced by a simple multiplication operator. We then extend the action of R to spaces of solutions to the left-handed zero-rest-mass equations on ASD vacuum background. We will get around the usual Buchdahl constraints by using the 'potential modulo gauge' description of negative helicity fields [5, 4]. In our treatment $\mathcal{W}_g$ will be identified with the space of linearised Hertz potentials.

2 Twistor description of recursion procedure

Let $\lambda = \pi_{0'}/\pi_{1'}$ be an affine coordinate on $\mathbb{CP}^1$. Cover $\mathcal{PT}$ by two sets, U and $\tilde{U}$ with $|\lambda| < 1+\epsilon$ on U and $|\lambda| > 1-\epsilon$ on $\tilde{U}$ with (ω^A, λ) coordinates on U and $(\tilde{\omega}^A, \lambda^{-1})$ on $\tilde{U}$. $\mathcal{PT}$ is then determined by the transition function $\tilde{\omega}^B = \tilde{\omega}^B(\omega^A, \pi_{A'})$ on $U \cap \tilde{U}$.

It is well known that infinitesimal deformations are given by elements of $H^1(\mathcal{PT}, \Theta)$, where Θ denotes a sheaf of germs of holomorphic vector fields. Let

$$Y = f^A(\omega^B, \pi_{B'})\frac{\partial}{\partial \omega^A} \in H^1(\mathcal{PT}, \Theta)$$

be defined on the overlap $U \cap \tilde{U}$. Infinitesimal deformation is given by

$$\tilde{\omega}^A = (1 + tY)(\omega^A). \tag{6}$$

From the globality of $\Sigma(\lambda) = d\omega^A \wedge d\omega_A$ it follows that Y can be taken to be a hamiltonian vector field with a hamiltonian $f \in H^1(\mathcal{PT}, \mathcal{O}(2))$ with respect to the symplectic structure Σ. The finite version of (6) is given by integrating

$$\frac{d\tilde{\omega}^A}{dt} = \varepsilon^{BA}\frac{\partial f}{\partial \tilde{\omega}^A}.$$

from $t = 0$ to 1 with $\tilde{\omega}^A(0) = \omega^A$ to obtain $\tilde{\omega}^A = \tilde{\omega}^A(1)$. We are interested in the linearised version of the last formula

$$\delta\tilde{\omega}^A = \epsilon\frac{\partial \delta f}{\partial \tilde{\omega}_A}. \tag{7}$$

This should be understood as follows: $\tilde{\omega}^A$ is the patching function obtained by exponentiating the Hamiltonian vector field of f and corresponds to the ASD metric determined by Θ and $\delta f^A = \varepsilon^{BA}\partial\delta f/\partial\omega^B$ (or more simply δf) is a linearised deformation corresponding to $\delta\Theta \in \mathcal{W}_g$.

The construction of a hierarchy of curved twistor spaces from a linearised deformation is given by

Proposition 1 *Let R be the recursion operator defined by (4). Its twistor counterpart is the multiplication operator*

$$R\,\delta f = \frac{\pi_{1'}}{\pi_{0'}}\delta f = \lambda^{-1}\delta f. \tag{8}$$

We see that the twistor description of the recursion operator is simpler than the space-time one. It is also better defined since R acts on δf without ambiguity (alternatively, the ambiguity in boundary condition for the definition of R on space-time is absorbed into the choice of explicit representative for the cohomology class determined by δf).

Proof. We work on the primed spin bundle. Restrict δf to the section of μ and represent it as a coboundary

$$\delta f(\pi_{A'}, x^a) = h(\pi_{A'}, x^a) - \tilde{h}(\pi_{A'}, x^a) \tag{9}$$

where h and $\tilde{h}$ are holomorphic on U and $\tilde{U}$ respectively (here we abuse notation and denote by U and $\tilde{U}$ the open sets on the spin bundle that are the preimage of U and $\tilde{U}$ on twistor space). Splitting

(9) is given by

$$h = \frac{1}{2\pi i}\oint_\Gamma \frac{(\pi^{A'}o_{A'})^3}{(\rho^{C'}\pi_{C'})(\rho^{B'}o_{B'})^3}\delta f(\rho_{E'})\rho_{D'}d\rho^{D'}, \tag{10}$$

$$\tilde{h} = \frac{1}{2\pi i}\oint_{\tilde\Gamma} \frac{(\pi^{A'}\iota_{A'})^3}{(\rho^{C'}\pi_{C'})(\rho^{B'}\iota_{B'})^3}\delta f(\rho_{E'})\rho_{D'}d\rho^{D'}.$$

Here $\iota_{A'}$ is a constant spinor satisfying $o_{A'}\iota^{A'}=1$ and $\rho_{A'}$ are homogeneous coordinates of $\mathbb{CP}^1$ pulled back to the spin bundle. The contours Γ and $\tilde\Gamma$ are homologous to the equator of $\mathbb{CP}^1$ in $U\cap\tilde U$ and are such that $\Gamma-\tilde\Gamma$ surrounds the point $\rho_{A'}=\pi_{A'}$. Functions h and $\tilde h$ do not descend to $\mathcal{PT}$. They are global and homogeneous of degree 2 in $\pi_{A'}$ therefore

$$\pi^{A'}\nabla_{AA'}h = \pi^{A'}\nabla_{AA'}\tilde h = \pi^{A'}\pi^{B'}\pi^{C'}\Sigma_{AA'B'C'} \tag{11}$$

where $\Sigma_{AA'B'C'}$ is one of the four potentials for a linearised ASD Weyl spinor. $\Sigma_{AA'B'C'}$ is defined modulo terms of the form $\nabla_{A(A'}\gamma_{B'C')}$ but a part of this gauge freedom is fixed by choosing the Plebański's coordinate system[1] in which $\Sigma_{AA'B'C'} = o_{A'}o_{B'}o_{C'}\nabla_{A0'}\delta\Theta$. The condition $\nabla_{A(A'}\Sigma^A{}_{B'C')}=0$ follows from equation (11) which with the Plebański gauge choice implies $\delta\Theta\in\mathcal{W}_g$. Define δf_A by $\nabla_{AA'}\delta f = \rho_{A'}\delta f_A$. Equation (11) becomes

$$\oint_\Gamma \frac{\delta f_A(\rho_{E'})}{(\rho^{B'}o_{B'})^3}\rho_{D'}d\rho^{D'} = 2\pi i\nabla_{A0'}\delta\Theta. \tag{13}$$

The twistor function δf is not constrained by the RHS of (13) being a gradient. To see this define δf_{AB} by $\nabla_{AA'}(\delta f_B\rho_{B'}) = \delta f_{AB}\rho_{A'}\rho_{B'}$ and note that in ASD vacuum δf_{AB} is symmetric[2] which implies $\nabla^A{}_{A'}\delta f_A = 0$. Therefore the RHS of (13) is also a solution of a neutrino equation so (in ASD vacuum) it must be given by $\alpha^{A'}\nabla_{AA'}\phi$ where $\alpha^{A'}$ is a constant spinor and $\phi\in\mathcal{W}_g$. Equation (13) gives the formula for a linearisation of the second heavenly equation

$$\delta\Theta = \frac{1}{2\pi i}\oint_\Gamma \frac{\delta f}{(\rho^{B'}o_{B'})^4}\rho_{D'}d\rho^{D'}. \tag{14}$$

[1] There is also a freedom in $\delta\Theta$ which we shall now describe. Let M be volume preserving vector field on $\mathcal{M}$. Define $\delta^0_M\nabla_{AA'} = [M,\nabla_{AA'}]$. This is a pure gauge transformation. Once a Plebański coordinate system has been selected, the field equation will not be invariant under all the $sdiff(\mathcal{M})$ transformations. We restrict ourselves to transformations which preserve the SD two-forms $dw_A\wedge dw^A$ and $dx_A\wedge dw^A$. Such transformations are generated by

$$M = \frac{\partial h}{\partial w_A}\frac{\partial}{\partial w^A} + \left(\frac{\partial g}{\partial w_A} - x^B\frac{\partial^2 h}{\partial w_A\partial w^B}\right)\frac{\partial}{\partial x^A}$$

where $h=h(w^A)$ and $g=g(w^A)$. Space-time is now viewed as a cotangent bundle $\mathcal{M}=T^*\mathcal{N}^2$ with w^A being coordinates on a two-dimensional complex manifold $\mathcal{N}^2$. The full $sdiff(\mathcal{M})$ symmetry breaks down to $sdiff(\mathcal{N}^2)$ which acts on $\mathcal{M}$ by a Lie lift. The 'pure gauge' elements of $\mathcal{W}_g$ are given by

$$\begin{aligned}\delta^0_M\Theta &= F + x_AG^A + x_Ax_B\frac{\partial^2 g}{\partial w_A\partial w_B} + x_Ax_Bx_C\frac{\partial^3 h}{\partial w_A\partial w_B\partial w_C}\\ &+\frac{\partial g}{\partial w_A}\frac{\partial\Theta}{\partial x^A} + \frac{\partial h}{\partial w_A}\frac{\partial\Theta}{\partial w^A} - x^B\frac{\partial^2 h}{\partial w_A\partial w^B}\frac{\partial\Theta}{\partial x^A}\end{aligned} \tag{12}$$

where F, G^A, g, h are functions of w^B only.

The twistorial origins of the $Sdiff(\mathcal{N}^2)$ symmetry come from the existence of a symplectic form $\Sigma = d\omega^A\wedge d\omega_A$ on the fibers of $\mu:\mathcal{PT}\longrightarrow CP^1$. Consider a canonical transformation of each fiber of μ leaving Σ invariant. Let $H = H(x^a,\lambda) = \sum_{i=0}^\infty h_i\lambda^i$ be the hamiltonian for this transformation pulled back to the projective spin bundle. Functions h_i depend on space time coordinates only. In particular h_0 and h_1 are identified with h and g from the previous construction (12). This can be seen by calculating how Θ transforms if $\omega^A = w^A + \lambda x^A + \lambda^2\partial\Theta/\partial x_A + ...\longrightarrow\hat\omega^A$. Now Θ is treated as an object on the first jet bundle of a fixed fibre of $\mathcal{PT}$.

[2] In flat space $\delta f_{AB} = \partial^2\delta f/\partial\omega^A\partial\omega^B$.

Now recall formula (4) defining R. Let $R\delta f$ be the twistor function corresponding to $R\delta\Theta$ by (14). The recursion relations yield

$$\oint_\Gamma \frac{R\delta f_A}{(\rho^{B'}o_{B'})^3}\rho_{D'}d\rho^{D'} = \oint_\Gamma \frac{\delta f_A}{(\rho^{B'}o_{B'})^2(\rho^{B'}\iota_{B'})}\rho_{D'}d\rho^{D'}$$

so $R\delta f = \lambda^{-1}\delta f$.

□

Let $\delta\Omega$ be the linearisation of the first Plebański's potential. In [1] it was shown that $\delta\Omega = R^2\delta\Theta$. As a consequence

$$\delta\Omega = \frac{1}{2\pi i}\oint_\Gamma \frac{\delta f}{(\rho_{A'}o^{A'})^2(\rho_{B'}\iota^{B'})^2}\rho_{C'}d\rho^{C'}.$$

3 Z.R.M. fields on a heavenly background

Now consider a more general situation. If δf is homogeneous of degree n then contour integrals that give a splitting on the spin bundle can be chosen to be

$$h = \frac{1}{2\pi i}\oint_\Gamma \frac{(\pi^{A'}o_{A'})^{n+1}}{(\rho^{C'}\pi_{C'})(\rho^{B'}o_{B'})^{n+1}}\delta f(\rho_{E'})\rho_{D'}d\rho^{D'}$$

and similarly for $\tilde{h}$. The equality $\pi^{A'}\nabla_{AA'}h = \pi^{A'}\nabla_{AA'}\tilde{h}$ defines a global, homogeneity $n+1$ function

$$\pi^{A'}\nabla_{AA'}h = \pi^{A'_1}\pi^{A'_2}...\pi^{A'_{n+1}}\Sigma_{AA'_1A'_2...A'_{n+1}}.$$

With the chosen splitting formulae, $\Sigma_{AA'_1A'_2...A'_{n+1}} = o_{A'_1}o_{A'_2}...o_{A'_{n+1}}\nabla_{A0'}\delta\Theta$ which can be thought of as a potential for the spin $(n+2)/2$ field (the field itself is well defined only in flat space)

$$\psi_{A_1A_2...A_{n+2}} = \nabla_{A_10'}\nabla_{A_20'}...\nabla_{A_{n+2}0'}\delta\Theta$$

where

$$\delta\Theta = \frac{1}{2\pi i}\oint_\Gamma \frac{\delta f}{(\rho^{B'}o_{B'})^{n+2}}\rho_{D'}d\rho^{D'}.$$

Differentiating under the integral one shows that $\psi_{A_1A_2...A_{n+2}}$ satisfies

$$\nabla^{A_{n+2}A'_{n+2}}\psi_{A_1A_2...A_{n+2}} = C_{BCA_1(A_2}\nabla^{BA'_1}\nabla^{CA'_2}\nabla^{A_3}{}_{A'_3}...\nabla^{A_n}{}_{A'_n}\Sigma_{A_{n+1})A'_1A'_2...A'_n}{}^{A'_{n+2}}. \quad (15)$$

The last formula generalises the one given in [7] for a left-handed Rarita–Schwinger field. The Weyl spinor C_{ABCD} is present because one needs to use expressions like $\nabla_{CC'}\delta f_{AB}$. Note that the Buchdahl constraints do not appear. This can be seen by operating on (15) with $\nabla^{A_{n+1}}{}_{C'}$. The usual algebraic expression will cancel out with the RHS. (Note, however, that the definition of the field is not independent of the gauge choices as it would be in flat space.)

The notion of the recursion operator generalises to solutions of equations of type (15). We restrict ourselves to the case of ASD neutrino and Maxwell fields on an ASD background. For these two case the RHS of equation (15) vanishes and fields are gauge invariant. Define the recursion relations

$$R^*\psi_A := \nabla_{A0'}R\delta\Theta$$

for a neutrino field, and

$$R^*\psi_{AB} := \nabla_{A0'}\nabla_{B0'}R\delta\Theta$$

for a Maxwell field. It is easy to see that R^* maps solutions into solutions.

As an application consider the following example. Modify the tetrad (5) to

$$\nabla_{A0'} = \frac{\partial}{\partial x^A}, \qquad \nabla_{A1'} = \frac{\partial}{\partial w^A} + \frac{\partial \Theta_B}{\partial x^A}\frac{\partial}{\partial x_B} \tag{16}$$

where Θ_B is a pair of functions. It turns out that the second heavenly equation can be written as

$$\Box_g \Theta_A = 0, \qquad \nabla_{A0'}\Theta^A = 0. \tag{17}$$

The second condition in (17) guarantees the existence of a scalar function Θ such that $\Theta_A = \nabla_{A0'}\Theta$. The linearisations of (17) satisfy the neutrino equation

$$\nabla^A{}_{A'}\delta\Theta_A = 0. \tag{18}$$

Now one can apply R^* to generate new solutions of (18). There is a reason for rewriting (2) in the strange looking form (17). If one drops the condition $\nabla_{A0'}\Theta^A = 0$ then the tetrad (16) is not vacuum. It does however define (the most general) hyperhermitian metric and its symmetries can still be generated using R^*. A different form of equation (17) was first given by Finley and Plebański [3] in the context of 'weak heavens'. It can be given a twistorial interpretation by means of 'Twisted Photon' construction on ASD vacuum background [2].

Let us finish with the following remark. In all the consideration we assumed the vanishing of the cosmological constant. This underlies the original nonlinear graviton construction. The ASD Einstein metrics with $\Lambda \neq 0$ have a natural twistor construction [10] where the extra information about a scalar curvature is encoded into a contact structure on $\mathcal{PT}$. On the other hand Przanowski [9] reduced $\Lambda \neq 0$ case to the single second order PDE for one scalar function $u(w, \tilde{w}, z, \tilde{z})$

$$u_{w\tilde{w}}u_{z\tilde{z}} - u_{w\tilde{z}}u_{z\tilde{w}} - (2u_{w\tilde{w}} + u_w u_{\tilde{w}})e^{-u} = 0. \tag{19}$$

We remark that the linearisations of (19) satisfy

$$(\Box_g + 4\Lambda)\delta u = 0$$

and the recursion relations (4) are still valid. It should be possible to derive Przanowski's result from the structure of a curved twistor space.

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