An integral formula in General Relativity

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In this note I want to present an integral formula which is valid in a general relativistic space-time M. In particular, given a hypersurface Σ embedded in M which can be foliated by two-dimensional space-like compact surfaces S_s, $s \in \mathbb{R}$, then one can relate the rate of change of the integral over S_s of the divergence of outgoing light rays to the geometric (convexity and curvature) properties of S_s and the energy-momentum content of M.

The derivation is based on the well-known Sparling-Witten-Nester identity: given two spinor fields λ_A and μ_A on M we can define a two-form L and a three-form S by the expressions

$$L = -i\bar{\mu}_{A'}\mathrm{d}\lambda_A\theta^{AA'}, \tag{1}$$

$$S = -i\mathrm{d}\bar{\mu}_{A'}\mathrm{d}\lambda_A\theta^{AA'}, \tag{2}$$

for which we have the identity

$$\mathrm{d}\,L = -i\mathrm{d}\bar{\mu}_{A'}\mathrm{d}\lambda_A\theta^{AA'} - i\bar{\mu}_{A'}\mathrm{d}^2\lambda_A\theta^{AA'} = S + E. \tag{3}$$

The important fact is that the three-form E which contains the second derivatives, can be expressed entirely in terms of the Einstein tensor which, in turn, can be replaced by the energy-momentum tensor T^{ab} of the matter content of the space-time by virtue of the Einstein equation $G_{ab} = -8\pi G T_{ab}$ (for notation and conventions cf. [1]). Thus, we obtain the identity

$$d\,L = S + 4\pi G\, V^a T_a{}^b \Sigma_b, \tag{4}$$

where $V^a = \lambda^A \bar{\mu}^{A'}$. This is the starting point for the various proofs of positivity of mass in General Relativity. These can be obtained from (4) by integrating over a space-like hypersurface Σ and choosing the spinor fields so that they satisfy an appropriate differential equation on Σ which makes the right hand side positive definite provided the energy-momentum tensor satisfies the dominant energy condition while the left hand side reduces to the mass expression.

Instead of imposing a differential condition on the spinor fields one can just as well try to fix them geometrically and this will be pursued here. Thus, I will assume that Σ is diffeomorphic to $S \times I$, S two-dimensional and compact, I an open interval so that S_s is the image of $S \times \{s\}$ under this diffeomorphism, which is supposed to be space-like. Then there exist two unique null directions at each point of S_s orthogonal to S_s. Let l^a and n^a be null vectors along those directions. We choose the spinor fields $\lambda_A = \mu_A$ and so that the flag pole points along the same null direction, say along l^a. This fixes the spinor field up to a scaling on S_s. This can be almost fixed by noting that the light rays coming out from S_s along the chosen null direction form a null hypersurface $\mathcal{N}$. We may assume without loss of generality that the normal to $\mathcal{N}$ is a gradient and we choose the spinor field on S_s so that its flag pole agrees with that gradient on S_s. This fixes the spinor field up to the multiplication with a real constant (on S_s) and an arbitrary phase function.

Having fixed things on one surface S_s we now have to relate several surfaces. Let Z^a be a vector field on Σ with $Z^a \nabla_a s = 1$ and consider the integral

$$I(s) = \oint_{S_s} L. \tag{5}$$

We are interested in the rate of change of that integral with the parameter s which is

$$\frac{d}{ds} \oint L = \oint L_Z L = \oint i_Z dL = \oint i_Z S + \oint i_Z E \tag{6}$$

where we have used the formula for the Lie derivative of any differential form $L_Z \omega = d i_Z \omega + i_Z d\omega$, Stokes' theorem and the identity (4). This formula shows that we may assume without loss of generality that Z^a is orthogonal to the surfaces S_s. A component tangent to the surfaces would correspond to applying an infinitesimal diffeomorphism to the integrand which does not change the value of the integral because there are no boundaries. Thus, we can write $Z^a = Z l^a + Z' n^a$ and we can exploit the fact that Z^a is hypersurface orthogonal in the evaluation of the integrals in (6). This is a somewhat lengthy calculation which is described in detail in [2]. It is best done using the formalism of two-dimensional Sen connections developed by L. Szabados [3] and makes use of all the assumptions above. The final result is

$$\begin{aligned} \frac{d}{ds} \oint \phi\rho \, d^2 A = \oint \rho\dot{\phi} \, d^2 A + Z'\phi \oint \left(\frac{\mathcal{R}}{8} - \tau\bar{\tau} \right) d^2 A \\ + \oint Z\phi \left(\sigma\bar{\sigma} - \rho^2 \right) d^2 A + 4\pi G \oint \phi \left(l^a T_a{}^b p_{bc} Z^c \right) d^2 A, \end{aligned} \tag{7}$$

where we have used the standard spin coefficients with respect to a spin frame (o^A, ι^A) which is adapted to the null directions but not specified further. p_{ab} is the volume element in the two-dimensional orthogonal complement to S_s. The function ϕ is an arbitrary positive $(-1,-1)$ weighted function on Σ (in the sense of the GHP-formalism [4]) which satisfies the equation $\eth\phi = \tau\phi$ on each two-surface S_s and $\dot{\phi} = Z^e \nabla_e \phi$.

Several points are worth mentioning:

- The term on the left hand side is the integrated divergence of the null geodesic congruence emanating from S_s.
- On the right hand side we find terms which have to do with the geometric properties of the two-surfaces, namely the scalar curvature $\mathcal{R}$ and the expression $\rho^2 - \sigma\bar{\sigma}$ which is the determinant of the extrinsic curvature associated with the normal vector l^a and which describes (part of) the convexity properties of S_s.

- The τ-terms have no immediately apparent geometric meaning. It is, however, worthwhile to mention that in the case where Σ is space-like and when the dominant energy condition holds the τ-terms and the energy-momentum terms combine with the same sign so that one could view them as some kind of gravitational contribution to the total energy.
- We have not yet fixed the scaling of the spin frame. Under special circumstances this can be specialised to further simplify the formula.
- The term involving the scalar curvature $\mathcal{R}$ of S_s integrates to a constant by the Gauß-Bonnet theorem.
- The term involving the derivative of ϕ is present because it guarantees the invariance under reparametrisation of the foliation.
- Finally, there exists a "primed" version of this integral formula corresponding to choosing the spinor fields along the other outgoing null direction.

I now want to briefly discuss some special applications for the integral formula. The first of these concerns the case when Σ is itself a null hypersurface. Then one can choose the conormal, say l_a, to be a gradient. The vector l^a is tangent to the generators of Σ and we choose s to be an affine parameter. From a starting surface S_0 we obtain a foliation by the level sets of s. Thus, we can take $\phi = 1$ and $Z^a = l^a$ and we obtain

$$\frac{d}{ds}\oint \rho\, d^2A = \oint \left(\sigma\bar{\sigma} - \rho^2\right) d^2A + 4\pi G \oint T_{ab}l^a l^b d^2A. \qquad (8)$$

This equation is an integrated version of one of the optical equations which governs the focusing of light rays.

The next application is to the case when Σ approaches $\mathcal{J}$ in an asymptotically flat space-time. We use the asymptotic solution of the field equations from [5] and choose the foliation so that it agrees with the cuts of $\mathcal{J}$ defined by a Bondi retarded time coordinate u. Then we specify the spin-frame and the function ϕ appropriately and we obtain the Bondi mass loss formula. Again, this is discussed in more detail in [2]. This shows again that the focusing of light rays is closely related to the positivity of mass, a fact which has been exploited earlier [6].

The final special case is in space-times with spherical symmetry. This assumption entails a tremendous simplification in the integral formula and one finds that, for a space-like asymptotically flat hypersurface Σ, the two formulae corresponding to the two different null directions are completely equivalent to the constraint equations in that case. Furthermore, one can recover a formulation of the constraints due to O'Murchadha and Malec [7]. With this form of the constraints it is easy to prove the spherically symmetric version of the Penrose inequality.

The integral formula presented above seems to have a rather broad range of validity and some interesting special applications. In particular, in the spherically symmetric case one can obtain results on the occurrence of trapped surfaces and singularities in the domain of dependence of Σ from the special form of the constraints. It is hoped that one can derive similar results from the more general formula. There are several arbitrary pieces in the formula which need to be chosen appropriately. But exactly how, remains to be seen. Work is in progress.

References

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