Twistor Newsletter

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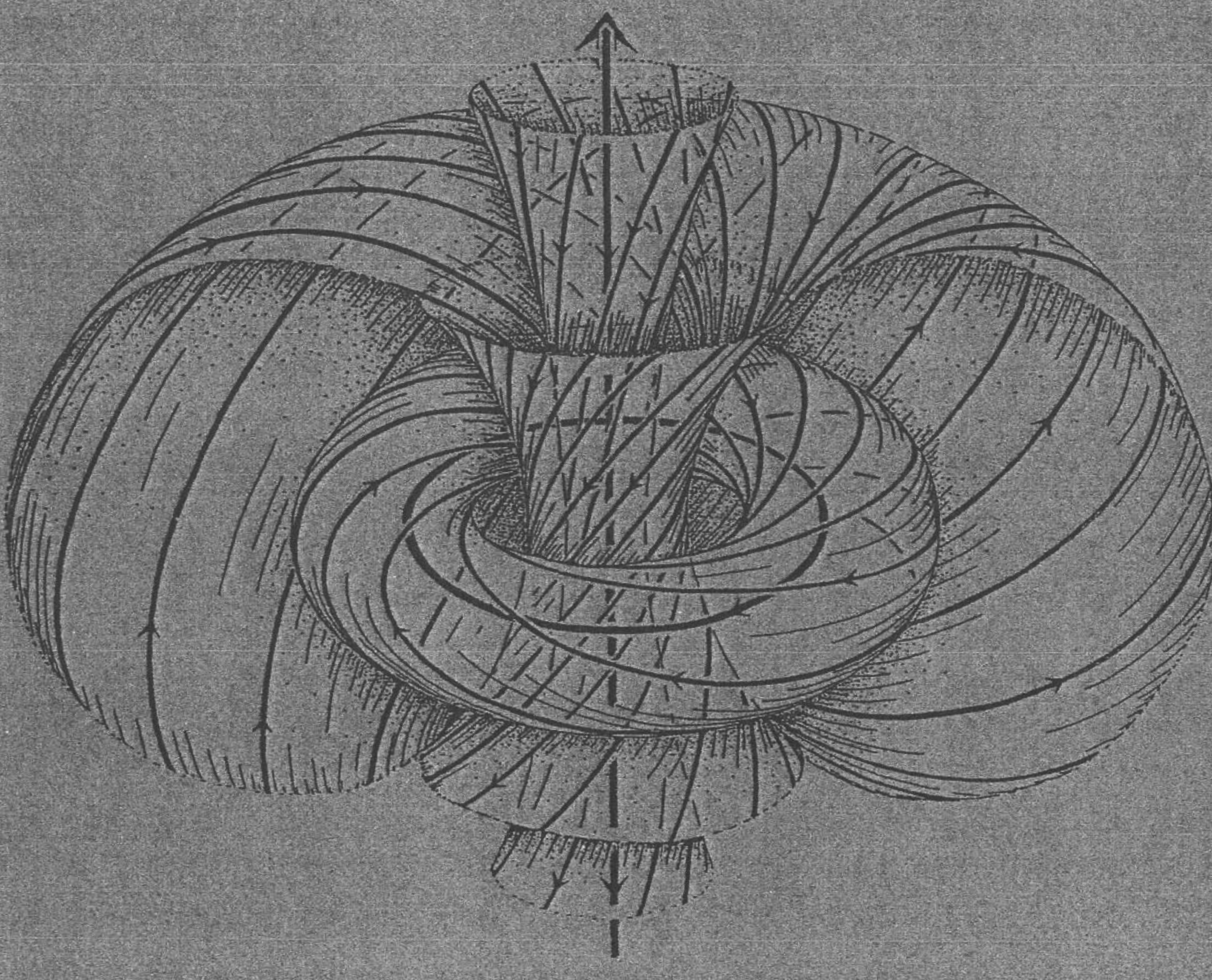


Mathematical Institute

Oxford, England

Googly Maps as Forms

by Roger Penrose

This is a brief interim report on some new developments that seem to be leading to the correct way of resolving the googly problem. This, we recall, is the problem of representing the self-dual part of the gravitational field (Weyl tensor) in terms of deformations of twistor space, in order to complement the standard "leg-break" non-linear graviton construction for anti-self-dual gravitation. These developments are closely related to some of those described in my article in TN40, and also to some earlier TN articles, but the new developments provide a vital new perspective on these issues.

The starting point is the same as that described in TN8, namely the canonical Poincaré-invariant exact sequence

$$0 \rightarrow \mathbb{S}^A \rightarrow \mathbb{T}^\alpha \rightarrow \mathbb{S}_{A'} \rightarrow 0$$

given, respectively, by the injection

$$\omega^A \mapsto (\omega^A, 0)$$

and the projection

$$(\omega^A, \pi_{A'}) \mapsto \pi_{A'} .$$

A space-time point (here an element $x \in \mathbb{M}$) is associated with a "splitting" of this sequence

$$0 \leftarrow \mathbb{S}^A \leftarrow \mathbb{T}^\alpha \leftarrow \mathbb{S}_{A'} \leftarrow 0$$

given by the projection

$$\omega^A - ix^{AA'}\pi_{A'} \leftarrow\!\shortmid (\omega^A, \pi_{A'})$$

and the injection

$$(\omega^A - ix^{AA'}\pi_{A'}, \pi_{A'}) \leftarrow\!\shortmid \pi_{A'} .$$

The (standard) leg-break construction may be regarded as a deformation of the tail end of this sequence, there still being a canonical projection

$$\mathcal{T} \rightarrow \mathbb{S}_{A'} - \{0\}$$

of the curved twistor space $\mathcal{T}$, the "space-time points" for the corresponding Ricci-flat complex space-time $\mathcal{M}$ being cross-sections of this projection, i.e. maps

$$\mathbb{S}_{A'} - \{0\} \rightarrow \mathcal{T}.$$

For the "dual" googly construction we should have (in some sense) a canonical injection

$$\mathbb{S}^A - \{0\} \rightarrow \mathcal{T}$$

instead, where the "space-time points" would correspond to projections

$$\mathcal{T} \rightarrow \mathbb{S}^A - \{0\},$$

these being what would be called the ~~googly~~ maps.

The new proposal is that these googly maps (and also other maps) should be represented in terms of <u>3-forms</u>. Since the (non-projective) twistor space $\mathcal{T}$ is 4-dimensional, any (non-zero) 3-form determines (locally) a family of curves that provide a foliation of $\mathcal{T}$. (The tangent vectors to these curves annihilate the 3-form.) Following these curves down to where $\mathbb{S}^A - \{0\}$ sits in $\mathcal{T}$, according to the canonical injection, we get the required projections corresponding to the googly points.

To understand better how this works, consider again the flat case. We can represent the point x, with position vector x^a, by the skew twistor

$$X^{\alpha\beta} = \begin{pmatrix} -\frac{1}{2}\varepsilon^{AB} x_c x^c & i x^A{}_{B'} \\ -i x_{A'}{}^B & \varepsilon_{A'B'} \end{pmatrix}$$

or, equivalently, by its dual

$$X_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta} X^{\gamma\delta} = \begin{pmatrix} \varepsilon_{AB} & i x_A{}^{B'} \\ -i x^{A'}{}_B & -\frac{1}{2}\varepsilon^{A'B'} x_c x^c \end{pmatrix},$$

where

$$X_{\alpha\beta} I^{\alpha\beta} = 2 = X^{\alpha\beta} I_{\alpha\beta}.$$

Equivalently, we can use the projection operators

$$L^\alpha{}_\beta = X^{\alpha\gamma} I_{\beta\gamma} = \begin{pmatrix} 0 & i x^{AB'} \\ 0 & \varepsilon_{A'}{}^{B'} \end{pmatrix}$$

and

$$G^\alpha{}_\beta = I^{\alpha\gamma} X_{\beta\gamma} = \begin{pmatrix} -\varepsilon^A{}_B & -i x^{AB'} \\ 0 & 0 \end{pmatrix}$$

(the "leg-break" and "googly" projections, respectively). These satisfy

$$L^\alpha{}_\beta L^\beta{}_\gamma = L^\alpha{}_\gamma, \quad G^\alpha{}_\beta G^\beta{}_\gamma = G^\alpha{}_\gamma, \quad L^\alpha{}_\beta G^\beta{}_\gamma = G^\alpha{}_\beta L^\beta{}_\gamma = 0, \quad L^\alpha{}_\beta + G^\alpha{}_\beta = \delta^\alpha_\beta.$$

The leg-break map corresponding to x (the composition of $\mathbb{T}^\alpha \to \mathbb{S}_{A'}$ with $\mathbb{S}_{A'} \xrightarrow{x} \mathbb{T}^\alpha$) is given by $Z^\alpha \mapsto L^\alpha{}_\beta Z^\beta$ and the googly map by $Z^\alpha \mapsto G^\alpha{}_\beta Z^\beta$. Note that there is also a more symmetrical quantity

$$J^\alpha{}_\beta = iG^\alpha{}_\beta - iL^\alpha{}_\beta = \begin{pmatrix} -i\varepsilon^A{}_B & 2x^{AB'} \\ 0 & -i\varepsilon_{A'}{}^{B'} \end{pmatrix}$$

(which is twistor-Hermitian, when x^a is real) satisfying

$$J^\alpha{}_\beta \, J^\beta{}_\gamma = -\delta^\alpha_\gamma$$

and which also represents the space-time point, but in a way that is biassed neither to self-duality nor to anti-self-duality.

The quantities **L** and **G** are closely associated with the respective 3-forms

$$\lambda = \tfrac{1}{2}\overset{x}{\omega}_A \, d\overset{x}{\omega}{}^A \wedge d\pi_{A'} \wedge d\pi^{A'} \quad \text{and} \quad \gamma = \tfrac{1}{2} d\overset{x}{\omega}_A \wedge d\overset{x}{\omega}{}^A \wedge \pi_{A'} d\pi^{A'},$$

where $\overset{x}{\omega}{}^A = \omega^A - ix^{AA'}\pi_{A'}$. We can also define the "ambidextrous" quantity

$$\xi = i\gamma - i\lambda$$

corresponding to **J**. Let us call the families of curves, whose tangent vectors annihilate these respective forms, the "integral curves" of these 3-forms. Following the integral curves of λ and γ down to where they vanish (at $\overset{x}{\omega}{}^A = 0$ and $\pi_{A'} = 0$, respectively), we obtain the leg-break and googly maps.

Note that (corresponding to $G^\alpha{}_\beta + L^\alpha{}_\beta = \delta^\alpha_\beta$ above)

$$\gamma + \lambda = \theta$$

where θ is the standard 3-form (projective volume form for $\mathbb{PT}^\alpha$) given by

$$\theta = \tfrac{1}{6}\epsilon_{\alpha\beta\gamma\delta} Z^\alpha dZ^\beta \wedge dZ^\gamma \wedge dZ^\delta$$

whose integral curves are those of the Euler vector field

$$\Upsilon = \theta \div \phi \qquad \text{where} \qquad d\theta = 4\phi,$$

these curves being the ones that we factor out by in the passage from $\mathbb{T}^\alpha$ to $\mathbb{PT}^\alpha$. Note that ϕ is the standard volume 4-form for $\mathbb{T}^\alpha$. Also there is a standard 1-form $\iota = \pi_{A'} d\pi^{A'}$ and 2-form $\tau = \tfrac{1}{2} d\pi_{A'} \wedge d\pi^{A'} = d\pi_{0'} \wedge d\pi_{1'}$ where

$$2\tau = d\iota, \qquad \tau \wedge \iota = 0, \qquad \text{and} \qquad \theta \wedge \iota = 0.$$

In addition, these forms have homogeneity degrees defined by

$$\pounds_{\Upsilon} \iota = 2\iota, \quad \pounds_{\Upsilon} \tau = 2\tau, \quad \pounds_{\Upsilon} \theta = 4\theta, \quad \pounds_{\Upsilon} \phi = 4\phi.$$

The last two of these are automatic consequences of $\Upsilon = \theta \div \phi$, with $\phi = \frac{1}{4}d\theta$, but the other two require the further condition

$$\theta \circ \tau = -\phi \otimes \iota$$

where the bilinear operation $\circ$, acting between an n-form α and a 2-form β is defined by

$$\alpha \circ (dp \wedge dq) = (\alpha \wedge dp) \otimes dq - (\alpha \wedge dp) \otimes dq.$$

In a leg-break curved twistor space $\mathcal{T}$, all the forms $\iota, \tau, \theta, \phi$ are retained, as is Υ, and they satisfy precisely the same local relations as listed above for the flat case. However, according to the discussion given in TN40, we anticipate that only the quantity

$$\Sigma = \phi \otimes \iota,$$

together with the integral curves of θ, should be retained from among these. There is a freedom in the choice of ϕ and ι arising from $\phi \mapsto \zeta\phi$, $\iota \mapsto \zeta^{-1}\iota$ $(\zeta \neq 0)$ and also, additionally, in the choice of θ for which $d\theta = 4\phi$. To cope with the first of these, we introduce the differential operator D, acting on quantities $\alpha \otimes \beta$, where α is an n-form and β a 1-form, subject to $\alpha \wedge \beta = 0$, given by

$$D(\alpha \otimes \beta) = (d\alpha) \otimes \beta - \alpha \circ (d\beta).$$

We find $D((\zeta\alpha) \otimes \beta) = D(\alpha \otimes (\zeta\beta))$, as is required for notational consistency. We envisage $\mathcal{T}$ as being covered by open sets, in each of which a choice of θ and ι are made, subject to (as is consistent with $D(\theta \otimes \iota) = 4\phi \otimes \iota - \theta \circ 2\tau = 6\phi \otimes \iota$)

$$D(\theta \otimes \iota) = 6\Sigma.$$

The kernel of D (keeping the integral curves of θ fixed) is given essentially by -6 deg. functions.

For a googly map, we keep the integral curves of γ consistent, but the actual googly 3-form γ varies from open set to open set of the covering. In each open set, γ satisfies

$$\lambda \circ \tau = -\Sigma$$

which simply states that the difference $\theta - \lambda$ has the form $\eta \wedge \tau$, for some 1-form η. The idea is that global conditions should be sufficient to restrict the googly maps to a 4-parameter family. There should also be a canonical 2-parameter family of 3-forms (corresponding to $\kappa_A d\omega^A \wedge \tau$, in the flat case, for each $\kappa_A \in \mathbb{S}_A$) such that two googly points are <u>null separated</u> iff the difference between their 3-forms is everywhere proportional to **one** 3-form of the family.

MORE LATER.

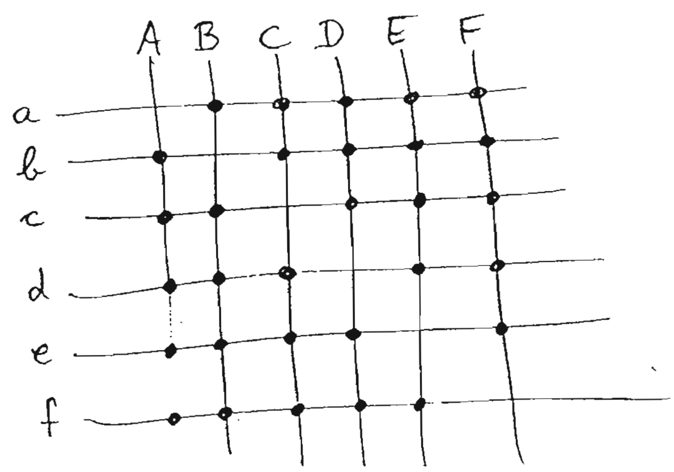
— Roger Penrose

A Letter from R. Penrose

On browsing Hilbert's & Cohn-Vossen's book "Anschauliche Geometrie", I came across the section on Geometric Configurations, and thereby across the "Schläfli Double-Six" Configurations of lines in 3-space. Shown there is a picture of a 3-dimensional symmetric model, and it came to my mind that I had indeed seen such a model in the old collection of mathematical models that our institute still possesses. (Most people nowadays don't know what the significance of these models should be.) It also came to mind that it might be possible to have such a configuration in $CP_3 = PT$ but now contained in PN instead of some real projective subspace (RP_3), so that it would have a space-time interpretation. While this is true, the actual interpretation in terms of observers and light signals sounds far-fetched and not very elegant. The question was whether there is at least a very symmetric special case which has a more elegant interpretation. After a while I asked Roger about it, and his reply was as follows.

"Thank you for your letter. As you may have guessed from my original "Twistor Algebra" paper, I did spend some time thinking about the implications of twistor projective geometry for the ordinary geometry of Minkowski space. I'm sure I thought a little about the double-six configuration and what it meant in Minkowski terms, but I don't think I had anything at all elegant, or I'd surely have put it in the paper!

The best I can give you, having now spent a little time thinking further about the matter is the following: Consider lines $A, B, \ldots, F, a, \ldots, f$ in PN intersecting thus



and label the corresponding points of M the same. Take f to be the point i^0, so $A, \ldots, E$ are all on $\mathscr{I}$. Then $A, \ldots, E$ may be thought of as null hyperplanes, i.e. (more visualizably) as planes in Euclidean 3-space moving with unit speed. Arrange things so that D and E are parallel but moving in opposite directions, and suppose that each of A, B, C is perpendicular to D and E (and thus remains so as each moves). There is a time – say time zero – when the planes D and E coincide. Then their common plane meets A, B, C in a triangle Δ. The vertices (at that time) are a, b, c. A little later (or earlier – but let's say later) the planes A, B, C come together and have a common line λ. When this happens D and E will have separated, and meet λ in e and d, respectively. A little thought convinces us that λ is the perpendicular to the plane of Δ through the incentre of Δ (where Δ is thought of as persisting with time). The final point F is the circumcentre of Δ, at time equal to the circumradius R of Δ. As far as I can make out, the double-six theorem tells us that the distance between the circumcentre and the incentre of Δ is given by the geometric mean of

$$R \text{ and } R - 2r$$

where r is the inradius of Δ. I suppose that this is a classical theorem, known to the ancient Greeks! Have I got it right? The points a, b, c, d, e all have to lie on the (past) light cone of F ...."

I could not easily find out whether this theorem on triangles was indeed known to the Greeks. It was known to Euler, and the generalization to n-gons was found by Jacobi. It is also related to Poncelet's Porism, but its relation to the double-six theorem is new. Considering the arguments given in Hilbert & Cohn-Vossen as a proof of the double-six theorem and continuing as Roger did, we have a proof of the triangle theorem "without calculation".

References:
D. Hilbert & S. Cohn-Vossen, Geometry and Imagination. Chelsea, N.Y. 1952.
Henderson, The 27 lines upon the cubic surface. Cambridge Tracts, Vol. 13 (1911).

H. Urbantke
Institute for Theoretical Physics
Boltzmanngasse 5
A-1090 Vienna, Austria
e-mail: urbantke@galileo.thp.univie.ac.at

Four-Twistor Particles

This note is written to clarify the algebraic structure of the constraint equations on internal operators and the interpretation in unitary space of a 4twistor particle.

A.) The Internal Operator Algebra

The internal transformations of a 4-twistor particle $Z_i \to U_i^k (Z_k + \Lambda_{k\ell} \Sigma^\ell)$ have the infinitesimal operators

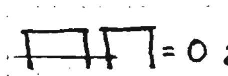
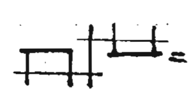
$$\underbrace{B^i_k = Z_k \Sigma^i}_{U(4)}, \quad \underbrace{d_{ik} = Z_i Z_k, \quad \bar{d}^{ik} = \Sigma^i \Sigma^k}_{\text{translations}} \quad (\Sigma^i = -\partial/\partial Z_i) \quad \cdots (1$$

where $i,k,.. = 1,2,3,4$. These satisfy the Lie-algebra commutators

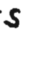
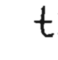
$$[B^i_j, B^k_\ell] = \delta^k_j B^i_\ell - \delta^i_\ell B^k_j \quad , \quad [d_{ij}, d_{k\ell}] = 0$$
$$[d_{ij}, B^k_\ell] = \delta^k_j d_{i\ell} - \delta^k_i d_{j\ell} \quad , \quad [d_{ij}, \bar{d}^{k\ell}] = 0, \quad \cdots\cdots (2)$$

and are subject to the algebraic constraints

$$d_{[ij} d_{k]\ell} = 0 \quad (a) \qquad d_{[ij} B^{[\ell}_{k]} \bar{d}^{mn]} = 0 \quad (b) \quad \cdots\cdots (3)$$

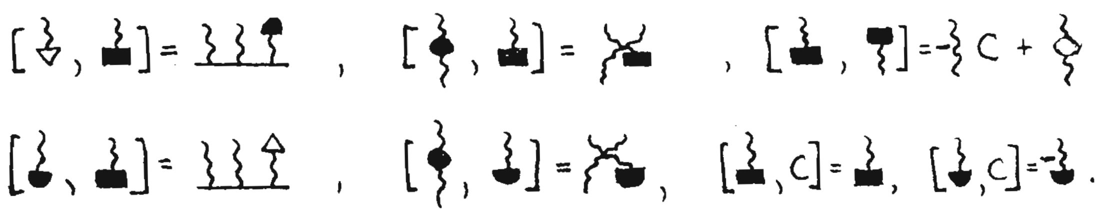
which follow from the identities  $=0$ and  $=0$, respectively.

How many independent conditions are contained in (3)? For a 3-twistor particle we have just a single one, of the form (3b), and this can even be eliminated by dropping the trace of B from among the generators (yielding ISU(3)). In order to answer the question, we now work with a 3+1 decomposition: $(a,b,.. = 1,2,3)$

$$d_{a4} \leftrightarrow$$  $, \quad \tfrac{1}{2}\epsilon^{abc} d_{bc} \leftrightarrow$  $, \quad B^a_4 \leftrightarrow$  $, \quad B^4_4 = C, \quad B^a_b \leftrightarrow$  $. \quad \cdots (4)$

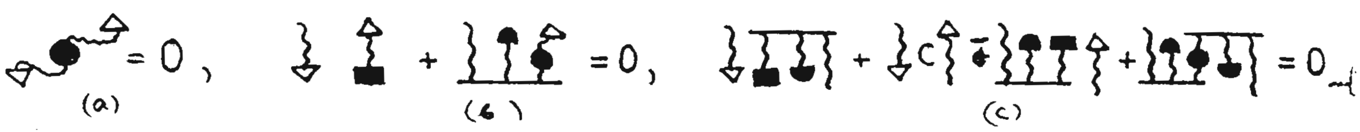
Commutators (2) are thus rewritten as the ISU(3) ones for

, and (Cf. Eq. (1.1.4) of preceding letter) plus



....(5)

In other combinations, the operators commute. The content of (3a) is =0 and (3b) yields the relations



Now (7b) and (7c) follow from (7a) and from (6). To show this, we compute the commutator of with = − , i.e., the ISU(3) basis vector (1.1.7). Consider first − = . Join here the vector from the right and use [,] = in the first term of − = . The result is (7b). Use of

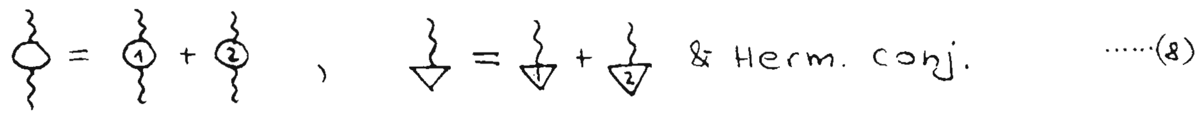
 yields (7c) in a similar way.

Thus we have established that the independent algebraic constraints on the 4-twistor operators are (6) and (7a).

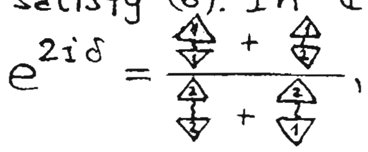
B.) Interpretation in Unitary Space

The algebraic equation (7a) can be satisfied by taking the trace-free part = − + $\frac{1}{3}$. We then have the internal operators , , , , , , and C. These represent 27 Hermitian quantities with two Hermitian relations =0 among them. Now a point particle has only 14 parameters whereas two points have too many parameters. We seek for two point particles (in $\mathbb{C}^3$) with special properties.

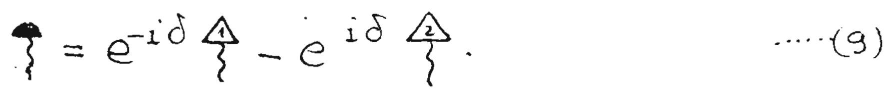
We denote the momentum and the angular momentum of the particles by , and , , respectively. For their center of mass we put



The angular momentum and the momentum have the correct ISU(3) commutation properties, therefore we do the identification with the 4-twistor operators suggested by our notation. A similar identification with the relative momentum and angular momentum is not so straightforward. If unitary space was real, the vector would, for satisfy (6). In $\mathbb{C}^3$ we must introduce a phase factor

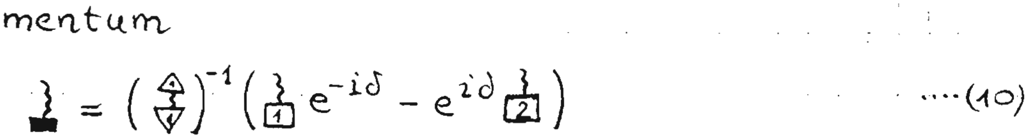


in the definition of the relative momentum

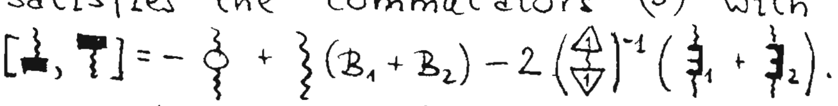


Now satisfies (6) and has the commutation properties required in (5). In order to retain the usual notion of a relative momentum in the real limit, the quantity δ must be Hermitian. Hence , i.e., the masses of the two unitary particles are equal.

The orbital momentum of each of the particles is given by and , respectively, constructed according to the recipe (1.1.7) from the particle's operators. The relative orbital momentum



satisfies the commutators (5) with the exception of

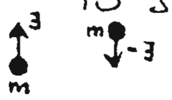


We obtain the correct result if we identify $-C = B_1 + B_2 - B - 2$ where the B of the center of mass is defined in (1.1.5), and if we impose on the particles (their spin be oppositely equal). How many conditions are these?

Out of the 8 Hermitian components of the angular momentum , the orbital vector carries 4. The spin tensor contains only 3 independent Hermitian data, and the Hermitian scalar B contains one more spin which has 3 essential components!

A pair of unitary point particles has, generally, 28 parameters. Now the masses are equal (1 less parameter) and the unitary spins are oppositely equal (3 less parameters). Thus the system has 25 free parameters. This is what we want since 27 internal operators − 2 constraints ( =0) make 25, and C can be expressed algebraically from (7c).

To sum up, a 4-twistor particle is pictured by point masses



in the unitary space. If we believe in an interaction between these points then we end up with something like a Kählerian hydrogen atom.

[1] Picturing intrinsic properties of particles, TN 8 (1979)

Zoltán Perjés

Average-marginally-trapped surfaces in flat space

By an *average-marginally-trapped surface* I mean a (closed, topologically spherical) space-like 2-surface S with the property that the expansion of the out-going null normals to S is zero *on the average*. Equivalently, the average rate of change of the area along the out-going null normals is zero. This is a natural definition - if the rate of change of the area element is zero at each point, the surface is marginally trapped - and not original to me.

Now ask: can average MTSs exist in flat space? True MTSs cannot, because of Roger's original singularity theorem (Phys.Rev.Lett.**14**(1965)57-59). It is not hard to see that on the contrary average MTSs can exist in flat space - the idea is to have regions on S which shrink into the future faster than other regions expand. Here I want to describe a concrete example.

Working in a hyperplane of constant time, you pile up six spheres (say radius a) like an octahedron - four are shown solid and the other two are one above the other and shown dotted. Then you put a big sphere (say radius r, determined by a) around them and nearly touching them; then you tunnel through from the big sphere to each little sphere to make something topologically spherical - the tunnels are small, four are shown in solid and the other two dotted.

Since this surface lies in a constant-time hyperplane, the rate of change of area along the outgoing null normals equals the rate of change along the outgoing space-like normal in t=constant. If this surface moves along these outwards normals, the small spheres get smaller and the big one gets bigger, but the small ones 'win'.

To be qualitative, recall that the rate of change of the area of a surface along the out-going normals is the integral of the mean-curvature over the surface. The big sphere has mean-curvature $1/r$ and area $4\pi r^2$ so the mean-curvature integral for the big sphere is $4\pi r$ (less terms of order ε or δ for the tunnels); for the small ones it is therefore 6 times $-4\pi a$ (again less terms of order ε or δ; the sign is changed because the normal points into each small sphere); for the 'tunnels' the mean-curvature is $O(1/\varepsilon + 1/\delta)$ and the area is $O(\varepsilon\delta)$ so the mean-curvature integral is small, of order ε or δ. Finally, by a bit of trigonometry, $r=a(1+\sqrt{2}) + O(\varepsilon)$.

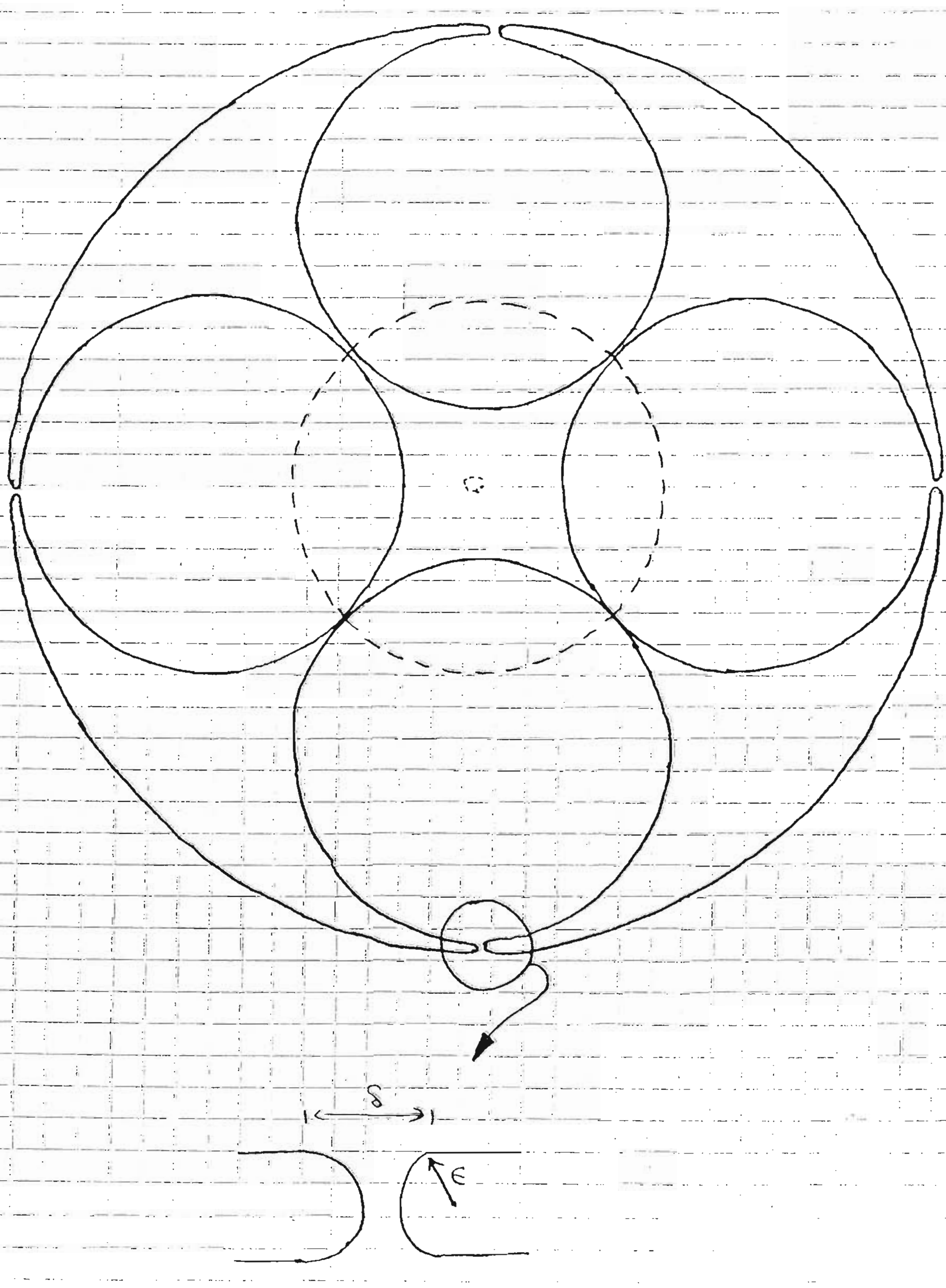
For the whole surface therefore

$$\text{mean-curvature integral} \sim 4\pi(r-6a) + O(\varepsilon, \delta) \sim -45a + O(\varepsilon, \delta)$$

which is negative.

Since this is negative, we can make the small spheres smaller until it vanishes (imagine deflating them like balloons with the tunnels held fixed). When the mean-curvature integral vanishes, so will the average of ρ ('rho') and the surface will be 'average-trapped'.

Paul Tod



'Special' Einstein–Weyl spaces from the heavenly equation

Paul Tod Maciej Dunajski

In a recent $\mathbb{TN}$ article [2] one of us showed how real three-dimensional Einstein–Weyl spaces which come from four-dimensional hyper-Kähler metrics with triholomorphic conformal Killing vectors can be determined by solutions of the pair of equations

$$V = \frac{1 + S\overline{S}}{S_{\overline{z}} + \overline{S}_z}, \qquad S_t + iS = 2iV_z. \tag{1}$$

Here S and V are, respectively, complex and real valued functions of three real coordinates (x, y, t) and $z = x + iy$. In this note we want to point out how equation (1) arises as a reduction of Plebański's first heavenly equation by a dilatation Killing vector.

Let $(w, z, \overline{w}, \overline{z})$ be a null coordinate system on $\mathbb{R}^4$. Each hyper-Kähler metric can be written as

$$g = \Omega_{w\overline{w}} dw d\overline{w} + \Omega_{w\overline{z}} dw d\overline{z} + \Omega_{z\overline{w}} dz d\overline{w} + \Omega_{z\overline{z}} dz d\overline{z} \tag{2}$$

where $\Omega = \Omega(w, z, \overline{w}, \overline{z})$ satisfies the first heavenly equation [1]

$$\Omega_{w\overline{w}}\Omega_{z\overline{z}} - \Omega_{w\overline{z}}\Omega_{z\overline{w}} = 1. \tag{3}$$

Let $K = w\partial_w + \overline{w}\partial_{\overline{w}}$ be the homothetic Killing vector (it turns out that there is no loss of generality in this choice). The Killing equation together with an appropriate choice of gauge yields $\mathcal{L}_K\Omega = \Omega$. Define real valued functions t and $\hat{t}$ by $\ln w = \hat{t} + it$. The general solution of (3) subject to the Killing equation is of the form $\Omega = e^{\hat{t}}F(z, \overline{z}, t)$. The heavenly equation (3) gives

$$F_{z\overline{z}}(F + F_{tt}) - (F_z + iF_{tz})(F_{\overline{z}} - iF_{t\overline{z}}) = 4. \tag{4}$$

Define $S = (F_z + iF_{tz})/2$ and $V = (F + F_{tt})/4$. It follows from a straightforward calculation that functions S and V satisfy equation (1). Conversely, the second equation in (1) gives the integrability condition for the existence of F, and then the first equation yields (4). With the definition $\zeta = \hat{t} + it$ the hyper-Kähler metric (2) becomes

$$g = e^{\hat{t}}[V^{-1}(1 + S\overline{S})dz d\overline{z} + S dz d\overline{\zeta} + \overline{S} d\overline{z} d\zeta + V d\zeta d\overline{\zeta}]. \tag{5}$$

From the last formula one finds the Einstein–Weyl metric and the associated one form to be

$$h = dz d\overline{z} + (V dt - i(S dz - \overline{S} d\overline{z})/2)^2, \ \alpha = (2V)^{-1}(S dz + \overline{S} d\overline{z}).$$

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Singularities of wavefronts and lightcones in the context of GR via null foliations

Simonetta Frittelli*[a,b], Ezra T. Newman[a]
[a] *Department of Physics and Astronomy, University of Pittsburgh, Pittsburgh, PA 15260, USA.*
[b] *Department of Physics, Duquesne University, Pittsburgh, PA 15282*
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We describe an approach to the issue of the singularities of null hypersurfaces, due to the focusing of null geodesics, in the context of the recently introduced formulation of GR via null foliations.

The null-surface approach to general relativity essentially reformulates general relativity in terms of two real functions on the bundle of null directions over the spacetime manifold (locally $M^4 \times S^2$ with a metric $g_{ab}(x^a)$ on M^4). These two functions are $Z(x^a, \zeta, \bar{\zeta})$, representing a sphere's worth of null foliations of the spacetime (i.e., such that $\tilde{g}^{ab} Z_{,a} Z_{,b} = 0$ for all values of ζ and for members $\tilde{g}_{ab}$ of the conformal class of g_{ab}), and $\Omega(x^a, \zeta, \bar{\zeta})$, representing a sphere's worth of conformal factors with the role of picking the members of the the conformal class that satisfy the Einstein equations. More detail on this formulation can be found in [1].

Within the context of the null-surface approach to general relativity a family of null coordinate systems $(\theta^i, i = 0, 1, +, -)$ is heavily used to derive dynamical equations for Z and Ω. The coordinates θ^i are defined by derivation from Z:

$$\theta^i = (Z, \eth\bar{\eth}Z, \eth Z, \bar{\eth}Z) \equiv (u, R, \omega, \bar{\omega}), \tag{1}$$

which, for fixed ζ, represents a transformation between an arbitrary coordinate system x^a and θ^i. The coordinates θ^i are adapted to the null foliations, so that the leaves $Z(x^a, \zeta, \bar{\zeta}) = const.$ of the foliation constitute surfaces of constant coordinate u.

There is some gauge freedom in the theory, due to the fact that there are different foliations which are all null with respect to the same metric, such as, for Minkowski spacetime, a foliation based on outgoing lightcones off a world line as opposed to a foliation by null planes. In the case of asymptotically flat spacetimes, we customarily fix the gauge by requiring that our null foliation consists of surfaces that asymptotically become null planes.

For every fixed value of ζ, our special coordinates have a well defined interpretation. The coordinate u labels leaves of the null foliation. The coordinates $(\omega, \bar{\omega})$ label null geodesics on a fixed leaf. The remaining coordinate R acts as a parameter along every null geodesic in the leaf. Because of our gauge choice of null surfaces becoming planes at null infinity, the null geodesics labeled by $(\omega, \bar{\omega})$ constitute bundles of asymptotically parallel null geodesics.

It is natural to raise the objection that, generically, any vacuum spacetime other than Minkowski focuses non-diverging bundles of null geodesics [2], and therefore our coordinate systems break down at the point of focusing, by assigning different labels to the same spacetime point. In this respect, although coordinate singularities are irrelevant to the physical content of the dynamical null-surface equations, we feel that the break down of these particular coordinates has a certain appeal since it entails the existence and location of caustics. Caustics and their singularities, as well as the singularities of wavefronts, have been classified by Arnol'd within a sophisticated mathematical context [3]. On the other hand, caustics are increasingly being considered in the field of astrophysical observations [4–7].

The null-surface approach provides a dual interpretation to the function Z. The condition $Z(x^a, \zeta, \bar{\zeta}) = u$ for fixed x^a picks up the points $(u, \zeta, \bar{\zeta})$ at scri which are connected to x^a by null geodesics. These points lie on a two-surface at scri, referred to as the lightcone cut of the point x^a. Generically, due to focusing in the interior, the lightcone cuts have self-intersections and typical wavefront singularities such as cusps and swallowtails. This appears to pose a technical difficulty regarding the perturbative approach to solving the null-surface equations, since the occurrence of this type of singularity generically entails divergences in the derivatives of the lightcone cuts.

In the following, we examine the occurrence of singularities of wavefronts and lightcones and its relevance to the null-surface approach.

*e-mail: simo@artemis.phyast.pitt.edu

A. Singularities of the null coordinates

Consider a foliation of spacetime by past null cones from points $(u, \zeta, \bar{\zeta})$ at scri along a fixed null generator ζ. (This is the compactified version of a foliation by null surfaces that are asymptotically null planes). Every past lightcone in this foliation has singularities, in the sense that the lightcone "folds" and self-intersects, due to focusing in the interior spacetime, as shown in Fig. 1. The points where the past lightcone is singular are points where neighboring null geodesics of the congruence intersect, and are thus conjugate to the point at scri. Translating this into the physical non-compactified spacetime, these points in the interior are "focal points", such that light rays emitted from them are asymptotically parallel. How can we locate these "focal points" in terms of null-surface variables?

A preliminary answer to this question can be approached quite directly from a consideration of the geodesic deviation vector of the congruence that becomes asymptotically parallel in a direction $(\zeta, \bar{\zeta})$ at scri. Every null geodesic in this congruence is characterized by fixed values of $(u, \zeta, \bar{\zeta}, \omega, \bar{\omega})$. The geodesic deviation vector has been derived earlier in [8]. Along a fixed null geodesic, the cross sectional area of the congruence has the expression:

$$A_p(R; u, \zeta, \bar{\zeta}, \omega, \bar{\omega}) = \frac{1}{\Omega^2 \sqrt{1 - \Lambda_{,1} \bar{\Lambda}_{,1}}}, \tag{2}$$

where $_{,1}$ represents $\partial/\partial R$. This is an expression of the area A_p in terms of the two null-surface variables

$$\Omega = \Omega(x^a, \zeta, \bar{\zeta}) \qquad\qquad \Lambda \equiv \eth^2 Z(x^a, \zeta, \bar{\zeta}). \tag{3}$$

In (2), the quantities $\Omega(x^a, \zeta, \bar{\zeta})$ and $\Lambda(x^a, \zeta, \bar{\zeta})$ are evaluated at fixed $(\zeta, \bar{\zeta})$ and at values of x^a along the null geodesic given by $(u, \zeta, \bar{\zeta}, \omega, \bar{\omega})$.

It is relevant to point out that the variable Ω actually is the product of two factors, one of which is an arbitrary confromal factor for the conformal class (and as such, it does not depend on ζ), whereas the other factor carries conformal information via its ζ-dependence. Therefore it is not surprising to find that it plays a role in a completely conformally invariant matter such as the determination of conjugate points of null geodesic congruences.

Focusing takes place at the value of R such that $A_p = 0$. However, the only way for the area to vanish is that the denominator become infinite. The square root in the denominator can not diverge before becoming pure imaginary. This leaves us with the previously unsuspected result that along a fixed null geodesic in the past lightcone from a point $(u, \zeta, \bar{\zeta})$ at scri, Ω must blow up for focusing to take place.

The vanishing of A_p is also related to the vanishing of the determinant of the metric, since in these coordinates we have

$$g \equiv \det(g_{ij}) = \left(\det(g^{ij})\right)^{-1} = \frac{1}{\Omega^8 (1 - \Lambda_{,1} \bar{\Lambda}_{,1})} = \Omega^{-4} A_p^2 \tag{4}$$

Thus both the vanishing of A_p and the fact that Ω diverges result in the vanishing of the 4-dimensional volume element.

This divergence of Ω has another significant consequence. Along the null geodesics labeled by $(\omega, \bar{\omega})$ there is a choice of an affine parameter s, which is related to R via

$$\frac{ds}{dR} = \Omega^{-2} \qquad \text{or alternatively} \qquad \frac{dR}{ds} = \Omega^2. \tag{5}$$

Since the affine parameter is regular, it follows that dR/ds blows up as well at the point where Ω does. Thus R is a bad coordinate (as we might have suspected) in the neighborhood of a focal point. Both R and Ω are, however, determined by the function Z through the same second derivative (See [1] for details):

$$R \equiv \eth\bar{\eth} Z(x^a, \zeta, \bar{\zeta}), \qquad\qquad \Omega^2 \equiv g^{ab} Z_{,a} \eth\bar{\eth} Z_{,b}\,, \tag{6}$$

thus it is consistent to attribute both complications to $\eth\bar{\eth} Z(x^a, \zeta, \bar{\zeta})$ becoming singular, since g^{ab} is smooth in a good choice of coordinates x^a.

B. Singularities of the lightcone cuts

In asymptotically flat spacetimes the function Z can also be viewed as describing the intersection of the lightcone of a point x^a with scri, via $u = Z(x^a, \zeta, \bar{\zeta})$ where $(u, \zeta, \bar{\zeta})$ are Bondi coordinates on scri. This intersection is a two-surface in a three-dimensional space and can be thought as one member of a series of "wavefronts" obtained by slicing the lightcone of a point x^a with a one-parameter family of past lightcones, the last one of them being scri itself. In this respect, the two-surface at scri given by $u = Z(x^a, \zeta, \bar{\zeta})$ can be thought of as a two-dimensional wavefront in a three-dimensional space, for which there is a standard treatment in singularity theory.

Wavefronts are considered as projections of smooth two-dimensional Legendrian manifolds in a five-dimensional space down to a three-dimensional space. The singularities of the wavefront are the places where the projection is singular. The Legendrian manifold itself is obtained from a generating function. More specifically, consider the 1-jet bundle over the two dimensional configuration space (x^1, x^2), given in coordinates (z, x^1, x^2, p_1, p_2), and a function $S(p_1, x^2)$. A Legendrian manifold is a two dimensional subspace of points (z, x^1, x^2, p_1, p_2) that can be specified by [3]

$$x^1 = \frac{\partial S}{\partial p_1}, \qquad p_2 = -\frac{\partial S}{\partial x^2}, \qquad z = S(p_1, x^2) - x^1 p_1, \qquad \text{parametrized by } x^2 \text{ and } p_1. \tag{7}$$

The projection of this Legendrian manifold down to (z, x^1, x^2) is a two-surface in three dimensions, representing a wavefront. This wavefront is singular where the projection breaks down, namely at points such that the 3×2 Jacobian matrix

$$\begin{pmatrix} \frac{\partial x^1}{\partial x^2} & \frac{\partial x^1}{\partial p_1} \\ \frac{\partial x^2}{\partial x^2} & \frac{\partial x^2}{\partial p_1} \\ \frac{\partial z}{\partial x^2} & \frac{\partial z}{\partial p_1} \end{pmatrix} \tag{8}$$

has rank less than 2.

In our case, we take (x^1, x^2) as coordinates on the sphere by defining $\zeta = x^1 + ix^2$ and interpret the projection of z as the lightcone cut function $Z(x^a, \zeta, \bar{\zeta})$ (the dependence on the spacetime points x^a is considered parametric in the instance of lightcone cuts, being regarded as fixed here). The lightcone cut is regular except at some values of ζ at which the projection breaks down. Carrying through the calculation of the determinant of the Jacobian matrix in these terms, we obtain the result that the projection breaks down at points ζ such that

$$\frac{1}{\eth^2 Z(x^a, \zeta, \bar{\zeta})} = 0 \qquad \text{and} \qquad \frac{1}{\eth\bar{\eth} Z(x^a, \zeta, \bar{\zeta})} = 0. \tag{9}$$

This calculation is parametric in x^a, where x^a represents the apex of the lightcone intersecting scri at the lightcone cut surface. This is equivalent to the statement that both Λ and R must blow up at particular values of ζ for the lightcone cut of a given spacetime point x^a (See Fig.2). For every point on the cut, the quantities Λ and R have finite values, except at the singular points shown in Fig.2. Given the lightcone cut function $Z(x^a, \zeta, \bar{\zeta})$ for fixed x^a, the values of ζ for which Λ and R diverge determine the null geodesics for which x^a is a focal point. This is consistent with the result found in the previous section.

C. Singularities of the lightcones

A related issue which we are also concerned with is the location of the singularities of the lightcone of a point in the interior spacetime. Consider the lightcone of a point, namely, the congruence of all the null geodesics through that point, and follow one null geodesic in the congruence out to the future. Generically, due to curvature, there is at least one future point along this null geodesic where neighboring geodesics intersect, referred to as a point conjugate to the apex. The cross sectional area of the congruence vanishes at points which are conjugate to the apex along any null geodesic of the congruence, the locust of all such points being sometimes referred to as the caustic surface (although it would be more accurate to call it the singularity of the lightcone). The lightcone folds and self-intersects beyond the occurrence of the earliest of the points conjugate to the apex, as shown in Fig. 2. How do we formulate the condition for the location of such singular points in terms of null-surface variables?

We can approach this issue from an analysis of the geodesic deviation vector along a fixed null ray ζ of the lightcone of an interior point x_0^a. The geodesic deviation vector in this case can be derived solely from knowledge of $Z(x^a, \zeta, \bar{\zeta})$ through a procedure that is standard to the null-surface approach. The coordinate transformation (1) can, in principle, be piece-wise inverted, yielding, perhaps with different branches,

$$x^a = f^a(u,\omega,\bar{\omega},R,\zeta,\bar{\zeta}) \tag{10}$$

for fixed ζ. The lightcone of a point x_0^a can be obtained now by substituting

$$u = Z(x_0^a,\zeta,\bar{\zeta}) \qquad \omega = \eth Z(x_0^a,\zeta,\bar{\zeta}) \qquad \bar{\omega} = \bar{\eth} Z(x_0^a,\zeta,\bar{\zeta}) \tag{11}$$

and

$$R = \eth\bar{\eth} Z(x_0^a,\zeta,\bar{\zeta}) + r \tag{12}$$

into (10), in this manner obtaining, perhaps with several branches,

$$x^a = F^a(r;x_0^a,\zeta,\bar{\zeta}) \tag{13}$$

as the lightcone of the point x_0^a parametrized by the directions $(\zeta,\bar{\zeta})$ labeling the null geodesics, and the parameter r along each null geodesic. The geodesic deviation vector is

$$M^a = \eth F^a = \frac{\partial f^a}{\partial \theta^i}\eth\theta^i + \eth' f^a \tag{14}$$

where $\eth'$ is taken keeping θ^i fixed. This expression can be worked out straightforwardly to yield M^a in terms of Z and its derivatives, and the cross sectional area of the congruence along the null ray ζ can subsequently be obtained:

$$A_{lc}(r;x_0^a,\zeta,\bar{\zeta}) = \frac{(\Lambda-\Lambda_0)(\bar{\Lambda}-\bar{\Lambda}_0)-(R-R_0)^2}{\Omega^2\sqrt{1-\Lambda_{,1}\bar{\Lambda}_{,1}}} \tag{15}$$

The sublabel 0 indicates evaluation at $r=0$ (the apex). The quantities Λ_0 and R_0 appearing in (15) can be thought of as referring to the lightcone cut of the apex, whereas Λ and R correspond to the lightcone cut of the point at r along the null geodesic labeled by ζ (See Fig. 3). By comparing with (2) we can see that

$$A_{lc} = A_p\, H(r;x_0^a,\zeta,\bar{\zeta}) \qquad \text{with} \qquad H(r;x_0^a,\zeta,\bar{\zeta}) \equiv (\Lambda-\Lambda_0)(\bar{\Lambda}-\bar{\Lambda}_0)-(R-R_0)^2. \tag{16}$$

This equation relates the cross sectional areas of two different congruences containing the same null ray ζ, namely the cone of lightrays through x_0^a and the congruence of asymptotically parallel rays parallel to the null ray ζ. This implies a relationship between the focusing of either congruence and the behavior of the quantities Λ and R. We distinguish three alternatives.

1. $A_{lc}=0$ and $A_p=0$ at some value r. Then H must not diverge at a rate faster than A_p^{-1} at that point.
2. $A_{lc}=0$ with $A_p\neq 0$ at some value r. Then H must vanish at that point.
3. $A_p=0$ with $A_{lc}\neq 0$ at some value r. Then H must blow up at a rate faster than A_p^{-1} at that point.

As an example of the first instance, if $A_p=0$ at the apex then the apex is a focal point, because $H=0$, and is therefore finite, at $r=0$ (See Fig 4). So should be any point conjugate to it along the null ray ζ; however, at such points Λ_0, R_0, Λ and R all blow up, therefore H becomes quite intractable and its behavior has yet to be verified (See Fig. 5).

In the second case, the apex is not a focal point, and its conjugate point is located at r such that

$$H(r;x_0^a,\zeta,\bar{\zeta}) = 0. \tag{17}$$

This is the generic situation illustrated in Fig.2. See also Fig. 6.

In the third case, the point r at which A_p vanishes is a focal point along the null ray ζ, and therefore Λ and R both blow up, thus again H becomes intractable but is not unlikely that it blows up, since Λ_0 and R_0 are finite in this case. See Fig. 6.

This analysis applies to every null ray ζ in the lightcone. For some null ray ζ_m the conjugate point occurs closest to the apex, for some other direction ζ_f the apex has a conjugate point at infinity, and finally there are values of ζ for which no focusing takes place at any value of r, as can be seen from Fig.2.

D. Conclusion

Although the work reported on here is very much in progress, we believe we are finding significant clues as to what consequences the occurrence of focusing of null geodesics has for the null-surface formulation of general relativity. Our ultimate goal is to find a way to integrate the singularity issue with the null-surface dynamical equations, which played no role in this discussion.

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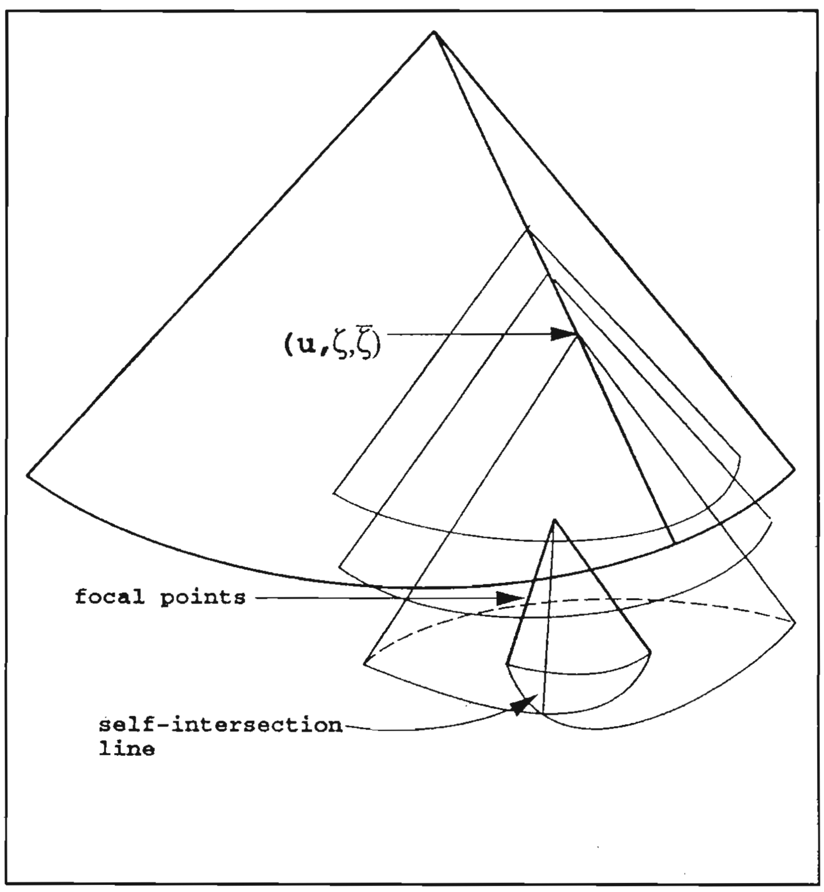

FIG. 1. A null foliation by past lightcones of points at scri.

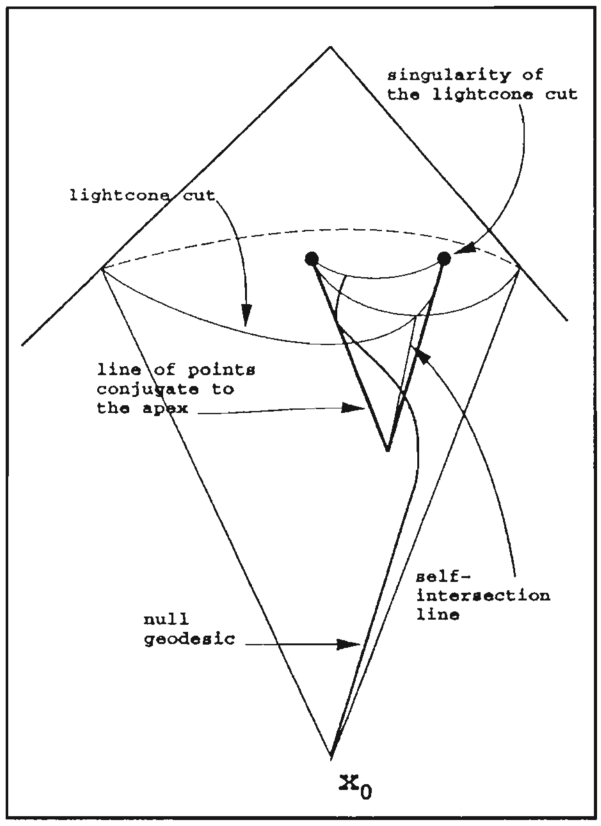

FIG. 2. The lightcone and lightcone cut of a point x_0^a. For the null geodesic shown, both $\Lambda(x_0^a, \zeta, \bar{\zeta})$ and $R(x_0^a, \zeta, \bar{\zeta})$ are finite.

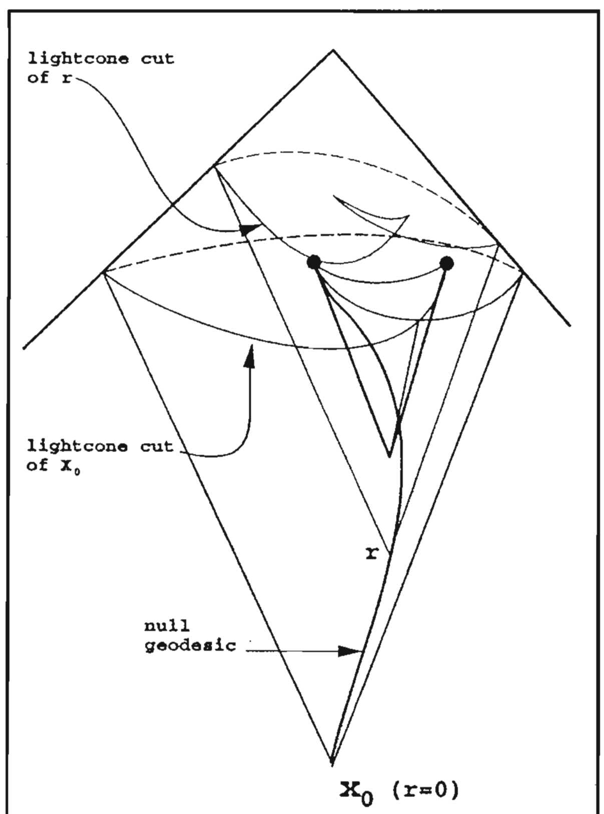

FIG. 3. Evaluating Λ and R at points s along the null geodesic. In this case, Λ_0 and R_0 blow up, where Λ and R are finite at the point r further up, since the null geodesic does not hit a singularity of the lightcone cut of the point r.

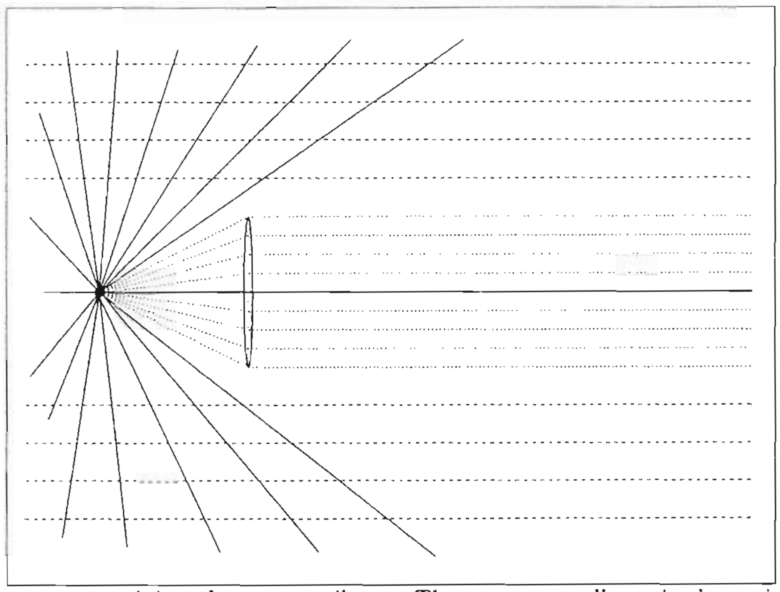

FIG. 4. Two null congruences containing the same null ray. The common null ray is shown in boldface. The solid lin represent the lightcone congruence, whereas the dashed lines represent the congruence asymptotically parallel. The dotted ra are also common to both congruences. In this case, the apex is a focal point.

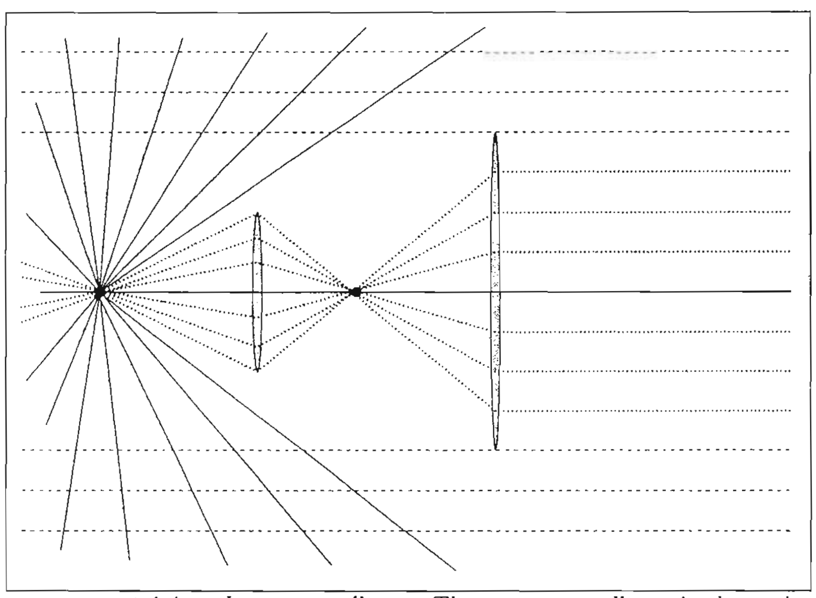

FIG. 5. Two null congruences containing the same null ray. The common null ray is shown in boldface. The solid lin represent the lightcone congruence, whereas the dashed lines represent the congruence asymptotically parallel. The dotted ra are also common to both congruences. In this case, the apex is a focal point and so is the focusing point between the tw lenses.

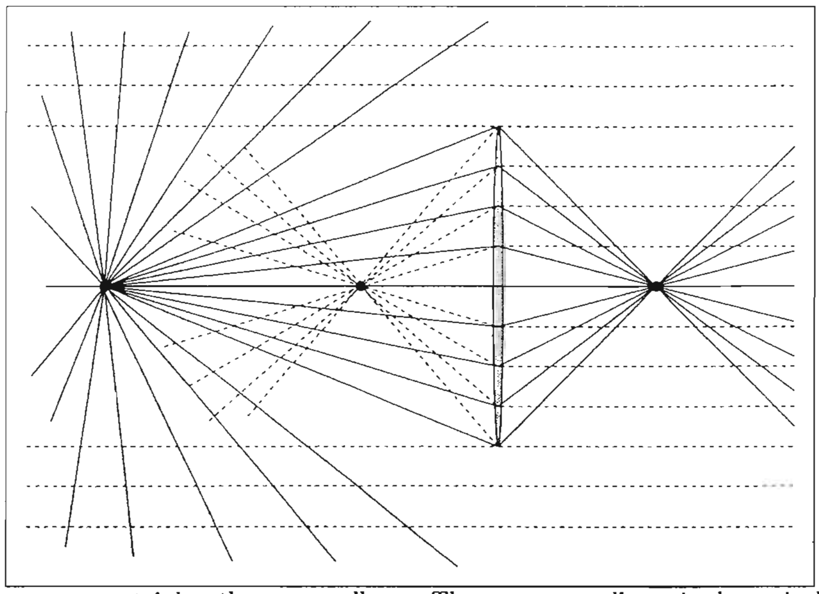

FIG. 6. Two null congruences containing the same null ray. The common null ray is shown in boldface. The solid lines represent the lightcone congruence, whereas the dashed lines represent the congruence asymptotically parallel. In this case, the apex is not a focal point, nor is its conjugate point beyond the lens. The focal point is between the lens and the apex.

Triviality of the Grassmann bundles on hypersurfaces in $\mathbb{R}^{m+1}$

Andrzej Trautman
Instytut Fizyki Teoretycznej, Uniwersytet Warszawski
Hoża 69, 00681 Warszawa, Poland

The triviality of the bundles of spinors on spheres has been recognized in connection with work on Killing spinors [1] and used to obtain an explicit expression for the eigenfunctions of the Dirac operator on these spaces [2]. Every hypersurface M in $\mathbb{R}^{m+1}$ has a pin$^-$ structure and the associated complex bundle $\Sigma \to M$ of spinors is trivial [3]. If the dimension m of the hypersurface M is even, then the trivial bundle $\Sigma \otimes \Sigma$ is isomorphic to $\mathbb{C} \otimes \bigwedge TM$ even though the tangent bundle $TM \to M$ is not trivial, in general. In this Letter, I present a few simple results on the triviality of the exterior algebra (Grassmann) bundles of hypersurfaces in $\mathbb{R}^{m+1}$.

Let the vector space $\mathbb{R}^{m+1}$ be given the standard, positive-definite quadratic form h and an orientation; these data define the Hodge map $\star : \bigwedge\mathbb{R}^{m+1} \to \bigwedge\mathbb{R}^{m+1}$ such that $\star\star = (-1)^{\frac{1}{2}m(m+1)}\mathrm{id}_{\bigwedge\mathbb{R}^{m+1}}$. Consider a hypersurface M in $\mathbb{R}^{m+1}$, i.e. a connected smooth manifold M, of dimension m, together with an immersion $i : M \to \mathbb{R}^{m+1}$. The tangent space T_xM to M at $x \in M$ is identified with its image by T_xi, this image being considered as an m-dimensional vector subspace of $\mathbb{R}^{m+1}$. This identification extends, in a natural manner, to a linear injection $\bigwedge T_xM \to \bigwedge\mathbb{R}^{m+1}$. The same letter is used to denote an element of $\bigwedge T_xM$ and its image in $\bigwedge\mathbb{R}^{m+1}$. Let $\bigwedge_0\mathbb{R}^{m+1}$ denote the even subalgebra of $\bigwedge\mathbb{R}^{m+1}$ and let $\bigwedge_0 TM$ be the bundle of even multivectors on M.

Proposition 1. *If the hypersurface M is orientable, then the vector bundle $\bigwedge TM \to M$ is trivial.*

Proof. Since M is orientable, there is a vector field $n : M \to \mathbb{R}^{m+1}$ of unit normals to M. A trivialization $f : \bigwedge TM \to M \times \bigwedge_0\mathbb{R}^{m+1}$ is defined as follows. Let $a \in \bigwedge T_xM$ be either even or odd; if a is even, then $f(a) = (x, a)$; if a is odd, then $f(a) = (x, n_x \wedge a)$.

Proposition 2. *If the hypersurface M is even-dimensional, then the vector bundle $\bigwedge TM \to M$ is trivial.*

Proof. The trivializing map $f : \bigwedge TM \to M \times \bigwedge_0 \mathbb{R}^{m+1}$ is now defined as follows: $f(a) = (x, a)$ for a even and $f(a) = (x, \star a)$ for a odd, $a \in \bigwedge T_x M$.

Proposition 3. *If the hypersurface M is of dimension $m \equiv 3 \bmod 4$, then the vector bundle $\bigwedge_0 TM \to M$ is trivial.*

Proof. If $m \equiv 3 \bmod 4$, then $\star\star = \mathrm{id}_{\bigwedge \mathbb{R}^{m+1}}$. Let $\bigwedge_0^+ \mathbb{R}^{m+1}$ be the vector space of self-dual, even multivectors over $\mathbb{R}^{m+1}$. A trivializing map $f : \bigwedge_0 TM \to M \times \bigwedge_0^+ \mathbb{R}^{m+1}$ is defined by $f(a) = (x, a + \star a)$ for $a \in \bigwedge_0 T_x M$. To prove that the map f is an isomorphism of vector bundles, one constructs the inverse map $f^{-1} : M \times \bigwedge_0^+ \mathbb{R}^{m+1} \to \bigwedge_0 TM$ as follows. Given $x \in M$, let l be a unit vector orthogonal to $T_x M$. Denoting by λ the 1-form associated with l by h, one has $\lambda \lrcorner l = 1$ and $\bigwedge T_x M = \{c \in \bigwedge \mathbb{R}^{m+1} : \lambda \lrcorner c = 0\}$. By virtue of the identity $\lambda \lrcorner \star c = \star(l \wedge c)$ one has $f^{-1}(x, b) = \lambda \lrcorner \star (\lambda \lrcorner b)$ for every $b \in \bigwedge_0^+ \mathbb{R}^{m+1}$.

If $m \equiv 1 \bmod 4$, then $\star\star = -\mathrm{id}_{\bigwedge \mathbb{R}^{m+1}}$. Upon complexification, one can define a trivializing map $f : \mathbb{C} \otimes \bigwedge_0 TM \to M \times \bigwedge_0^+ \mathbb{C}^{m+1}$ by putting $f(a) = (x, a - \mathrm{i} \star a)$, where now $\bigwedge_0^+ \mathbb{C}^{m+1} = \{b \in \bigwedge_0 \mathbb{C}^{m+1} : \star b = \mathrm{i} b\}$. This proves

Proposition 4. *If the hypersurface M is odd-dimensional. then the complex vector bundle $\mathbb{C} \otimes \bigwedge_0 TM \to M$ is trivial.*

Questions. Does there exist a non-orientable, odd-dimensional hypersurface M in $\mathbb{R}^{m+1}$ such that the vector bundle $\bigwedge TM \to M$ is not trivial? Are there hypersurfaces of dimension $m \not\equiv 3 \bmod 4$ such that $\bigwedge_0 TM \to M$ is not trivial?

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A Recursion Operator for ASD Vacuums and Z.R.M Fields on ASD Backgrounds

Maciej Dunajski Lionel J. Mason

1 Introduction

In this note, we show that the recursion operator given in [1] for the ASD vacuum equations has a simple form on twistor space analogous to the Yang-Mills case.

Let $\mathcal{M}$ be an oriented complex four manifold with ASD vacuum metric g and let $\mu : \mathcal{PT} \longrightarrow \mathbb{CP}^1$ be a corresponding projective twistor space fibred over a Riemman sphere. Let $Z^\alpha = (\omega^A, \pi_{A'}) \in \mathcal{PT}$. Choose a constant spinor $o_{A'} = (0,1)$ on the base space and parametrise a section of μ by four complex coordinates

$$x^{AA'} = \frac{\partial \omega^A}{\partial \pi_{A'}}\Big|_{\pi_{A'}=o_{A'}}, \quad x^{A1'} = w^A, \ x^{A0'} = x^A. \tag{1}$$

In [1] (following [4])we showed that the existence of a two form on the fibres of μ with values in the pullback of $\mathcal{O}(2)$ guarantees the existence of a complex valued function satisfying the second Plebański equation

$$\frac{\partial^2\Theta}{\partial w^A \partial x_A} + \frac{1}{2}\frac{\partial^2\Theta}{\partial x^B \partial x^A}\frac{\partial^2\Theta}{\partial x_B \partial x_A} = 0. \tag{2}$$

With the parametrisation (1) the right flat metric on $\mathcal{M}$ is given by

$$g = dx_A dw^A + \frac{\partial^2\Theta}{\partial x^A \partial x^B} dw^A dw^B.$$

The converse problem of reconstructing a deformed twistor space given a Θ function on $\mathcal{M}$ was reformulated as a system of time dependent Hamilton's equations.

Linearised solutions to equation (2) satisfy the wave equation on ASD background given by Θ

$$\Box_g \delta\Theta = 0. \tag{3}$$

Let $\mathcal{W}_g$ be the space of solutions to equation (3). We constructed a (formal) recursion operator $R : \mathcal{W}_g \longrightarrow \mathcal{W}_g$ given by the relation

$$\nabla_{A1'}\phi = \nabla_{A0'} R\phi. \tag{4}$$

Here $\nabla_{AA'}$ is a null tetrad of volume preserving vector fields which in the adopted coordinate system takes form

$$\nabla_{A0'} = \frac{\partial}{\partial x^A}, \qquad \nabla_{A1'} = \frac{\partial}{\partial w^A} + \frac{\partial^2\Theta}{\partial x^A \partial x^B}\frac{\partial}{\partial x_B}. \tag{5}$$

The definition is formal because the the operator on the right hand side requires boundary conditions in order for it to be invertible and without these, there are ambiguities in the definition of $R\phi$.

The aim of this paper is to give a twistor description of a recursion procedure. We shall show that R is defined without ambiguities on twistor functions and the space-time relation (4) is replaced by a simple multiplication operator. We then extend the action of R to spaces of solutions to the left-handed zero-rest-mass equations on ASD vacuum background. We will get around the usual Buchdahl constraints by using the 'potential modulo gauge' description of negative helicity fields [5, 4]. In our treatment $\mathcal{W}_g$ will be identified with the space of linearised Hertz potentials.

2 Twistor description of recursion procedure

Let $\lambda = \pi_{0'}/\pi_{1'}$ be an affine coordinate on $\mathbb{CP}^1$. Cover $\mathcal{PT}$ by two sets, U and $\tilde{U}$ with $|\lambda| < 1+\epsilon$ on U and $|\lambda| > 1-\epsilon$ on $\tilde{U}$ with (ω^A, λ) coordinates on U and $(\tilde{\omega}^A, \lambda^{-1})$ on $\tilde{U}$. $\mathcal{PT}$ is then determined by the transition function $\tilde{\omega}^B = \tilde{\omega}^B(\omega^A, \pi_{A'})$ on $U \cap \tilde{U}$.

It is well known that infinitesimal deformations are given by elements of $H^1(\mathcal{PT}, \Theta)$, where Θ denotes a sheaf of germs of holomorphic vector fields. Let

$$Y = f^A(\omega^B, \pi_{B'})\frac{\partial}{\partial \omega^A} \in H^1(\mathcal{PT}, \Theta)$$

be defined on the overlap $U \cap \tilde{U}$. Infinitesimal deformation is given by

$$\tilde{\omega}^A = (1 + tY)(\omega^A). \tag{6}$$

From the globality of $\Sigma(\lambda) = d\omega^A \wedge d\omega_A$ it follows that Y can be taken to be a hamiltonian vector field with a hamiltonian $f \in H^1(\mathcal{PT}, \mathcal{O}(2))$ with respect to the symplectic structure Σ. The finite version of (6) is given by integrating

$$\frac{d\tilde{\omega}^A}{dt} = \varepsilon^{BA}\frac{\partial f}{\partial \tilde{\omega}^A}.$$

from $t = 0$ to 1 with $\tilde{\omega}^A(0) = \omega^A$ to obtain $\tilde{\omega}^A = \tilde{\omega}^A(1)$. We are interested in the linearised version of the last formula

$$\delta\tilde{\omega}^A = \epsilon\frac{\partial \delta f}{\partial \tilde{\omega}_A}. \tag{7}$$

This should be understood as follows: $\tilde{\omega}^A$ is the patching function obtained by exponentiating the Hamiltonian vector field of f and corresponds to the ASD metric determined by Θ and $\delta f^A = \varepsilon^{BA}\partial\delta f/\partial\omega^B$ (or more simply δf) is a linearised deformation corresponding to $\delta\Theta \in \mathcal{W}_g$.

The construction of a hierarchy of curved twistor spaces from a linearised deformation is given by

Proposition 1 *Let R be the recursion operator defined by (4). Its twistor counterpart is the multiplication operator*

$$R\,\delta f = \frac{\pi_{1'}}{\pi_{0'}}\delta f = \lambda^{-1}\delta f. \tag{8}$$

We see that the twistor description of the recursion operator is simpler than the space-time one. It is also better defined since R acts on δf without ambiguity (alternatively, the ambiguity in boundary condition for the definition of R on space-time is absorbed into the choice of explicit representative for the cohomology class determined by δf).

Proof. We work on the primed spin bundle. Restrict δf to the section of μ and represent it as a coboundary

$$\delta f(\pi_{A'}, x^a) = h(\pi_{A'}, x^a) - \tilde{h}(\pi_{A'}, x^a) \tag{9}$$

where h and $\tilde{h}$ are holomorphic on U and $\tilde{U}$ respectively (here we abuse notation and denote by U and $\tilde{U}$ the open sets on the spin bundle that are the preimage of U and $\tilde{U}$ on twistor space). Splitting

(9) is given by

$$h = \frac{1}{2\pi i}\oint_\Gamma \frac{(\pi^{A'}o_{A'})^3}{(\rho^{C'}\pi_{C'})(\rho^{B'}o_{B'})^3}\delta f(\rho_{E'})\rho_{D'}d\rho^{D'}, \tag{10}$$

$$\tilde{h} = \frac{1}{2\pi i}\oint_{\tilde\Gamma} \frac{(\pi^{A'}\iota_{A'})^3}{(\rho^{C'}\pi_{C'})(\rho^{B'}\iota_{B'})^3}\delta f(\rho_{E'})\rho_{D'}d\rho^{D'}.$$

Here $\iota_{A'}$ is a constant spinor satisfying $o_{A'}\iota^{A'} = 1$ and $\rho_{A'}$ are homogeneous coordinates of $\mathbb{CP}^1$ pulled back to the spin bundle. The contours Γ and $\tilde\Gamma$ are homologous to the equator of $\mathbb{CP}^1$ in $U \cap \tilde U$ and are such that $\Gamma - \tilde\Gamma$ surrounds the point $\rho_{A'} = \pi_{A'}$. Functions h and $\tilde h$ do not descend to $\mathcal{PT}$. They are global and homogeneous of degree 2 in $\pi_{A'}$ therefore

$$\pi^{A'}\nabla_{AA'}h = \pi^{A'}\nabla_{AA'}\tilde h = \pi^{A'}\pi^{B'}\pi^{C'}\Sigma_{AA'B'C'} \tag{11}$$

where $\Sigma_{AA'B'C'}$ is one of the four potentials for a linearised ASD Weyl spinor. $\Sigma_{AA'B'C'}$ is defined modulo terms of the form $\nabla_{A(A'}\gamma_{B'C')}$ but a part of this gauge freedom is fixed by choosing the Plebański's coordinate system[1] in which $\Sigma_{AA'B'C'} = o_{A'}o_{B'}o_{C'}\nabla_{A0'}\delta\Theta$. The condition $\nabla_{A(A'}\Sigma^A{}_{B'C')} = 0$ follows from equation (11) which with the Plebański gauge choice implies $\delta\Theta \in \mathcal{W}_g$. Define δf_A by $\nabla_{AA'}\delta f = \rho_{A'}\delta f_A$. Equation (11) becomes

$$\oint_\Gamma \frac{\delta f_A(\rho_{E'})}{(\rho^{B'}o_{B'})^3}\rho_{D'}d\rho^{D'} = 2\pi i\nabla_{A0'}\delta\Theta. \tag{13}$$

The twistor function δf is not constrained by the RHS of (13) being a gradient. To see this define δf_{AB} by $\nabla_{AA'}(\delta f_B\rho_{B'}) = \delta f_{AB}\rho_{A'}\rho_{B'}$ and note that in ASD vacuum δf_{AB} is symmetric[2] which implies $\nabla^A{}_{A'}\delta f_A = 0$. Therefore the RHS of (13) is also a solution of a neutrino equation so (in ASD vacuum) it must be given by $\alpha^{A'}\nabla_{AA'}\phi$ where $\alpha^{A'}$ is a constant spinor and $\phi \in \mathcal{W}_g$. Equation (13) gives the formula for a linearisation of the second heavenly equation

$$\delta\Theta = \frac{1}{2\pi i}\oint_\Gamma \frac{\delta f}{(\rho^{B'}o_{B'})^4}\rho_{D'}d\rho^{D'}. \tag{14}$$

---

[1]There is also a freedom in $\delta\Theta$ which we shall now describe. Let M be volume preserving vector field on $\mathcal{M}$. Define $\delta^0_M\nabla_{AA'} = [M, \nabla_{AA'}]$. This is a pure gauge transformation. Once a Plebański coordinate system has been selected, the field equation will not be invariant under all the $sdiff(\mathcal{M})$ transformations. We restrict ourselves to transformations which preserve the SD two-forms $dw_A \wedge dw^A$ and $dx_A \wedge dw^A$. Such transformations are generated by

$$M = \frac{\partial h}{\partial w_A}\frac{\partial}{\partial w^A} + \left(\frac{\partial g}{\partial w_A} - x^B\frac{\partial^2 h}{\partial w_A\partial w^B}\right)\frac{\partial}{\partial x^A}$$

where $h = h(w^A)$ and $g = g(w^A)$. Space-time is now viewed as a cotangent bundle $\mathcal{M} = T^*\mathcal{N}^2$ with w^A being coordinates on a two-dimensional complex manifold $\mathcal{N}^2$. The full $sdiff(\mathcal{M})$ symmetry breaks down to $sdiff(\mathcal{N}^2)$ which acts on $\mathcal{M}$ by a Lie lift. The 'pure gauge' elements of $\mathcal{W}_g$ are given by

$$\begin{aligned}\delta^0_M\Theta &= F + x_AG^A + x_Ax_B\frac{\partial^2 g}{\partial w_A\partial w_B} + x_Ax_Bx_C\frac{\partial^3 h}{\partial w_A\partial w_B\partial w_C}\\ &+\frac{\partial g}{\partial w_A}\frac{\partial\Theta}{\partial x^A} + \frac{\partial h}{\partial w_A}\frac{\partial\Theta}{\partial w^A} - x^B\frac{\partial^2 h}{\partial w_A\partial w^B}\frac{\partial\Theta}{\partial x^A}\end{aligned} \tag{12}$$

where F, G^A, g, h are functions of w^B only.

The twistorial origins of the $Sdiff(\mathcal{N}^2)$ symmetry come from the existence of a symplectic form $\Sigma = d\omega^A \wedge d\omega_A$ on the fibers of $\mu : \mathcal{PT} \longrightarrow CP^1$. Consider a canonical transformation of each fiber of μ leaving Σ invariant. Let $H = H(x^a, \lambda) = \sum_{i=0}^{\infty} h_i\lambda^i$ be the hamiltonian for this transformation pulled back to the projective spin bundle. Functions h_i depend on space time coordinates only. In particular h_0 and h_1 are identified with h and g from the previous construction (12). This can be seen by calculating how Θ transforms if $\omega^A = w^A + \lambda x^A + \lambda^2\partial\Theta/\partial x_A + \ldots \longrightarrow \hat\omega^A$. Now Θ is treated as an object on the first jet bundle of a fixed fibre of $\mathcal{PT}$.

[2]In flat space $\delta f_{AB} = \partial^2\delta f/\partial\omega^A\partial\omega^B$.

Now recall formula (4) defining R. Let $R\delta f$ be the twistor function corresponding to $R\delta\Theta$ by (14). The recursion relations yield

$$\oint_\Gamma \frac{R\delta f_A}{(\rho^{B'}o_{B'})^3}\rho_{D'}d\rho^{D'} = \oint_\Gamma \frac{\delta f_A}{(\rho^{B'}o_{B'})^2(\rho^{B'}\iota_{B'})}\rho_{D'}d\rho^{D'}$$

so $R\delta f = \lambda^{-1}\delta f$.

□

Let $\delta\Omega$ be the linearisation of the first Plebański's potential. In [1] it was shown that $\delta\Omega = R^2\delta\Theta$. As a consequence

$$\delta\Omega = \frac{1}{2\pi i}\oint_\Gamma \frac{\delta f}{(\rho_{A'}o^{A'})^2(\rho_{B'}\iota^{B'})^2}\rho_{C'}d\rho^{C'}.$$

3 Z.R.M. fields on a heavenly background

Now consider a more general situation. If δf is homogeneous of degree n then contour integrals that give a splitting on the spin bundle can be chosen to be

$$h = \frac{1}{2\pi i}\oint_\Gamma \frac{(\pi^{A'}o_{A'})^{n+1}}{(\rho^{C'}\pi_{C'})(\rho^{B'}o_{B'})^{n+1}}\delta f(\rho_{E'})\rho_{D'}d\rho^{D'}$$

and similarly for $\tilde{h}$. The equality $\pi^{A'}\nabla_{AA'}h = \pi^{A'}\nabla_{AA'}\tilde{h}$ defines a global, homogeneity $n+1$ function

$$\pi^{A'}\nabla_{AA'}h = \pi^{A'_1}\pi^{A'_2}...\pi^{A'_{n+1}}\Sigma_{AA'_1A'_2...A'_{n+1}}.$$

With the chosen splitting formulae, $\Sigma_{AA'_1A'_2...A'_{n+1}} = o_{A'_1}o_{A'_2}...o_{A'_{n+1}}\nabla_{A0'}\delta\Theta$ which can be thought of as a potential for the spin $(n+2)/2$ field (the field itself is well defined only in flat space)

$$\psi_{A_1A_2...A_{n+2}} = \nabla_{A_10'}\nabla_{A_20'}...\nabla_{A_{n+2}0'}\delta\Theta$$

where

$$\delta\Theta = \frac{1}{2\pi i}\oint_\Gamma \frac{\delta f}{(\rho^{B'}o_{B'})^{n+2}}\rho_{D'}d\rho^{D'}.$$

Differentiating under the integral one shows that $\psi_{A_1A_2...A_{n+2}}$ satisfies

$$\nabla^{A_{n+2}A'_{n+2}}\psi_{A_1A_2...A_{n+2}} = C_{BCA_1(A_2}\nabla^{BA'_1}\nabla^{CA'_2}\nabla^{A_3}{}_{A'_3}...\nabla^{A_n}{}_{A'_n}\Sigma_{A_{n+1})A'_1A'_2...A'_n}{}^{A'_{n+2}}. \tag{15}$$

The last formula generalises the one given in [7] for a left-handed Rarita–Schwinger field. The Weyl spinor C_{ABCD} is present because one needs to use expressions like $\nabla_{CC'}\delta f_{AB}$. Note that the Buchdahl constraints do not appear. This can be seen by operating on (15) with $\nabla^{A_{n+1}}{}_{C'}$. The usual algebraic expression will cancel out with the RHS. (Note, however, that the definition of the field is not independent of the gauge choices as it would be in flat space.)

The notion of the recursion operator generalises to solutions of equations of type (15). We restrict ourselves to the case of ASD neutrino and Maxwell fields on an ASD background. For these two case the RHS of equation (15) vanishes and fields are gauge invariant. Define the recursion relations

$$R^*\psi_A := \nabla_{A0'}R\delta\Theta$$

for a neutrino field, and

$$R^*\psi_{AB} := \nabla_{A0'}\nabla_{B0'}R\delta\Theta$$

for a Maxwell field. It is easy to see that R^* maps solutions into solutions.

As an application consider the following example. Modify the tetrad (5) to

$$\nabla_{A0'} = \frac{\partial}{\partial x^A}, \qquad \nabla_{A1'} = \frac{\partial}{\partial w^A} + \frac{\partial \Theta_B}{\partial x^A}\frac{\partial}{\partial x_B} \tag{16}$$

where Θ_B is a pair of functions. It turns out that the second heavenly equation can be written as

$$\Box_g \Theta_A = 0, \qquad \nabla_{A0'}\Theta^A = 0. \tag{17}$$

The second condition in (17) guarantees the existence of a scalar function Θ such that $\Theta_A = \nabla_{A0'}\Theta$. The linearisations of (17) satisfy the neutrino equation

$$\nabla^A{}_{A'}\delta\Theta_A = 0. \tag{18}$$

Now one can apply R^* to generate new solutions of (18). There is a reason for rewriting (2) in the strange looking form (17). If one drops the condition $\nabla_{A0'}\Theta^A = 0$ then the tetrad (16) is not vacuum. It does however define (the most general) hyperhermitian metric and its symmetries can still be generated using R^*. A different form of equation (17) was first given by Finley and Plebański [3] in the context of 'weak heavens'. It can be given a twistorial interpretation by means of 'Twisted Photon' construction on ASD vacuum background [2].

Let us finish with the following remark. In all the consideration we assumed the vanishing of the cosmological constant. This underlies the original nonlinear graviton construction. The ASD Einstein metrics with $\Lambda \neq 0$ have a natural twistor construction [10] where the extra information about a scalar curvature is encoded into a contact structure on $\mathcal{PT}$. On the other hand Przanowski [9] reduced $\Lambda \neq 0$ case to the single second order PDE for one scalar function $u(w, \tilde{w}, z, \tilde{z})$

$$u_{w\tilde{w}}u_{z\tilde{z}} - u_{w\tilde{z}}u_{z\tilde{w}} - (2u_{w\tilde{w}} + u_w u_{\tilde{w}})e^{-u} = 0. \tag{19}$$

We remark that the linearisations of (19) satisfy

$$(\Box_g + 4\Lambda)\delta u = 0$$

and the recursion relations (4) are still valid. It should be possible to derive Przanowski's result from the structure of a curved twistor space.

Thanks to George Sparling.

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An integral formula in General Relativity

Jörg Frauendiener
Institut für Theoretische Astrophysik,
Universität Tübingen,
Auf der Morgenstelle 10,
D-72076 Tübingen,
Germany

In this note I want to present an integral formula which is valid in a general relativistic space-time M. In particular, given a hypersurface Σ embedded in M which can be foliated by two-dimensional space-like compact surfaces S_s, $s \in \mathbb{R}$, then one can relate the rate of change of the integral over S_s of the divergence of outgoing light rays to the geometric (convexity and curvature) properties of S_s and the energy-momentum content of M.

The derivation is based on the well-known Sparling-Witten-Nester identity: given two spinor fields λ_A and μ_A on M we can define a two-form L and a three-form S by the expressions

$$L = -i\bar{\mu}_{A'}\mathrm{d}\lambda_A\theta^{AA'}, \tag{1}$$

$$S = -i\mathrm{d}\bar{\mu}_{A'}\mathrm{d}\lambda_A\theta^{AA'}, \tag{2}$$

for which we have the identity

$$\mathrm{d}L = -i\mathrm{d}\bar{\mu}_{A'}\mathrm{d}\lambda_A\theta^{AA'} - i\bar{\mu}_{A'}\mathrm{d}^2\lambda_A\theta^{AA'} = S + E. \tag{3}$$

The important fact is that the three-form E which contains the second derivatives, can be expressed entirely in terms of the Einstein tensor which, in turn, can be replaced by the energy-momentum tensor T^{ab} of the matter content of the space-time by virtue of the Einstein equation $G_{ab} = -8\pi G T_{ab}$ (for notation and conventions cf. [1]). Thus, we obtain the identity

$$dL = S + 4\pi G\, V^a T_a{}^b \Sigma_b, \tag{4}$$

where $V^a = \lambda^A \bar{\mu}^{A'}$. This is the starting point for the various proofs of positivity of mass in General Relativity. These can be obtained from (4) by integrating over a space-like hypersurface Σ and choosing the spinor fields so that they satisfy an appropriate differential equation on Σ which makes the right hand side positive definite provided the energy-momentum tensor satisfies the dominant energy condition while the left hand side reduces to the mass expression.

Instead of imposing a differential condition on the spinor fields one can just as well try to fix them geometrically and this will be pursued here. Thus, I will assume that Σ is diffeomorphic to $S \times I$, S two-dimensional and compact, I an open interval so that S_s is the image of $S \times \{s\}$ under this diffeomorphism, which is supposed to be space-like. Then there exist two unique null directions at each point of S_s orthogonal to S_s. Let l^a and n^a be null vectors along those directions. We choose the spinor fields $\lambda_A = \mu_A$ and so that the flag pole points along the same null direction, say along l^a. This fixes the spinor field up to a scaling on S_s. This can be almost fixed by noting that the light rays coming out from S_s along the chosen null direction form a null hypersurface $\mathcal{N}$. We may assume without loss of generality that the normal to $\mathcal{N}$ is a gradient and we choose the spinor field on S_s so that its flag pole agrees with that gradient on S_s. This fixes the spinor field up to the multiplication with a real constant (on S_s) and an arbitrary phase function.

Having fixed things on one surface S_s we now have to relate several surfaces. Let Z^a be a vector field on Σ with $Z^a\nabla_a s = 1$ and consider the integral

$$I(s) = \oint_{S_s} L. \tag{5}$$

We are interested in the rate of change of that integral with the parameter s which is

$$\frac{d}{ds}\oint L = \oint L_Z L = \oint i_Z dL = \oint i_Z S + \oint i_Z E \tag{6}$$

where we have used the formula for the Lie derivative of any differential form $L_Z\omega = di_Z\omega + i_Z d\omega$, Stokes' theorem and the identity (4). This formula shows that we may assume without loss of generality that Z^a is orthogonal to the surfaces S_s. A component tangent to the surfaces would correspond to applying an infinitesimal diffeomorphism to the integrand which does not change the value of the integral because there are no boundaries. Thus, we can write $Z^a = Zl^a + Z'n^a$ and we can exploit the fact that Z^a is hypersurface orthogonal in the evaluation of the integrals in (6). This is a somewhat lengthy calculation which is described in detail in [2]. It is best done using the formalism of two-dimensional Sen connections developed by L. Szabados [3] and makes use of all the assumptions above. The final result is

$$\frac{d}{ds}\oint \phi\rho\, d^2A = \oint \rho\dot{\phi}\, d^2A + Z'\phi\oint\left(\frac{\mathcal{R}}{8} - \tau\bar{\tau}\right) d^2A$$
$$+ \oint Z\phi\left(\sigma\bar{\sigma} - \rho^2\right) d^2A + 4\pi G\oint \phi\left(l^a T_a{}^b p_{bc} Z^c\right) d^2A, \tag{7}$$

where we have used the standard spin coefficients with respect to a spin frame (o^A, ι^A) which is adapted to the null directions but not specified further. p_{ab} is the volume element in the two-dimensional orthogonal complement to S_s. The function ϕ is an arbitrary positive $(-1,-1)$ weighted function on Σ (in the sense of the GHP-formalism [4]) which satisfies the equation $\eth\phi = \tau\phi$ on each two-surface S_s and $\dot{\phi} = Z^e\nabla_e\phi$.

Several points are worth mentioning:

- The term on the left hand side is the integrated divergence of the null geodesic congruence emanating from S_s.
- On the right hand side we find terms which have to do with the geometric properties of the two-surfaces, namely the scalar curvature $\mathcal{R}$ and the expression $\rho^2 - \sigma\bar{\sigma}$ which is the determinant of the extrinsic curvature associated with the normal vector l^a and which describes (part of) the convexity properties of S_s.

- The τ-terms have no immediately apparent geometric meaning. It is, however, worthwhile to mention that in the case where Σ is space-like and when the dominant energy condition holds the τ-terms and the energy-momentum terms combine with the same sign so that one could view them as some kind of gravitational contribution to the total energy.
- We have not yet fixed the scaling of the spin frame. Under special circumstances this can be specialised to further simplify the formula.
- The term involving the scalar curvature $\mathcal{R}$ of S_s integrates to a constant by the Gauß-Bonnet theorem.
- The term involving the derivative of ϕ is present because it guarantees the invariance under reparametrisation of the foliation.
- Finally, there exists a "primed" version of this integral formula corresponding to choosing the spinor fields along the other outgoing null direction.

I now want to briefly discuss some special applications for the integral formula. The first of these concerns the case when Σ is itself a null hypersurface. Then one can choose the conormal, say l_a, to be a gradient. The vector l^a is tangent to the generators of Σ and we choose s to be an affine parameter. From a starting surface S_0 we obtain a foliation by the level sets of s. Thus, we can take $\phi = 1$ and $Z^a = l^a$ and we obtain

$$\frac{d}{ds}\oint \rho\, d^2A = \oint \left(\sigma\bar{\sigma} - \rho^2\right) d^2A + 4\pi G \oint T_{ab}l^a l^b d^2A. \tag{8}$$

This equation is an integrated version of one of the optical equations which governs the focusing of light rays.

The next application is to the case when Σ approaches $\mathcal{J}$ in an asymptotically flat space-time. We use the asymptotic solution of the field equations from [5] and choose the foliation so that it agrees with the cuts of $\mathcal{J}$ defined by a Bondi retarded time coordinate u. Then we specify the spin-frame and the function ϕ appropriately and we obtain the Bondi mass loss formula. Again, this is discussed in more detail in [2]. This shows again that the focusing of light rays is closely related to the positivity of mass, a fact which has been exploited earlier [6].

The final special case is in space-times with spherical symmetry. This assumption entails a tremendous simplification in the integral formula and one finds that, for a space-like asymptotically flat hypersurface Σ, the two formulae corresponding to the two different null directions are completely equivalent to the constraint equations in that case. Furthermore, one can recover a formulation of the constraints due to O'Murchadha and Malec [7]. With this form of the constraints it is easy to prove the spherically symmetric version of the Penrose inequality.

The integral formula presented above seems to have a rather broad range of validity and some interesting special applications. In particular, in the spherically symmetric case one can obtain results on the occurrence of trapped surfaces and singularities in the domain of dependence of Σ from the special form of the constraints. It is hoped that one can derive similar results from the more general formula. There are several arbitrary pieces in the formula which need to be chosen appropriately. But exactly how, remains to be seen. Work is in progress.

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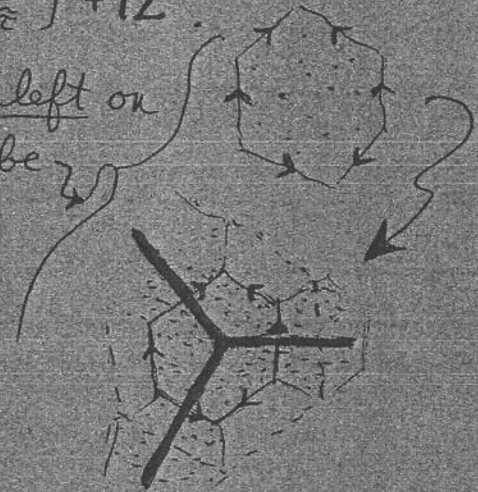
Corrections to Missprints in Solutions to TN 41 Puzzles

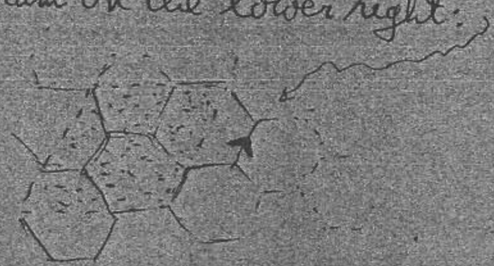
Readers may have been confused by some misprints on pp. 25, 26 of TN 42. There is clearly a "12" missing from the third displayed line on p. 25, where we should have

$$T_n = 12\left(\frac{F_n}{n}+1\right) = \frac{12}{n}\left(\frac{\tau^n-\tilde{\tau}^n}{\tau-\tilde{\tau}}\right)+12 .$$

The replacement indicated at the lower left on p. 26 also is not correct, and should be



Moreover, there is a wrong arrow on the diagram on the lower right.



TWISTOR NEWSLETTER No. 43

CONTENTS

Short contributions for TN 44 should be sent to

Maciej Dunajski
Twistor Newsletter Editor
Mathematical Institute
24–29 St. Giles'
Oxford OX1 3LB
United Kingdom
E-mail: tnews@maths.ox.ac.uk

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