The Twistor Space for a Strongly Asymptotically Flat Vacuum

by Roger Penrose

In TN 43, I introduced certain ideas concerning the geometrical nature of a curved twistor space $\mathcal{T}$, in terms of which the structure of an arbitrary "strongly asymptotically flat analytic vacuum" (SAFAV) space-time $\mathcal{M}$ ought to be encoded. Some suggestions were provided for the means whereby $\mathcal{M}$ might be constructed from $\mathcal{T}$. ($\mathcal{M}$ is SAFAV if it is an analytic vacuum with an analytic $\mathscr{I}^+$ and an analytic i^+. Thus, it represents a free gravitational wave which dies out in time in all directions, with no remnant in the form of a black hole or other source.)

The main purpose of this note is to provide a solution to the converse problem whereby $\mathcal{T}$ may be constructed from $\mathcal{M}$. Some more information concerning the curious geometrical nature of $\mathcal{T}$ will be given, and how it can be constructed from free holomorphic data. I shall show that the geometry of $\mathcal{M}$ (near $\mathscr{I}^+$) is indeed encoded in the structure of $\mathcal{T}=\mathcal{T}^+$ where $\mathcal{T}^+$ is constructed from free holomorphic data. As far as the construction of $\mathcal{M}$ from $\mathcal{T}^+$ is concerned, there remain many uncertainties and conjectural issues, some of which will be indicated here.

Let X be an arbitrary (finite) point of $\mathcal{M}$. I shall show, first, how to construct a twistor space $\mathcal{T}_X$, with respect to X, which encodes the information of both the SD and ASD parts of the gravitational field. Let $\mathcal{C}_X$ [respectively, $\mathbb{C}\mathcal{C}_X$] be the [complex] light cone of X, swept out by the [complex] light rays through X in $\mathcal{M}$ [resp. $\mathbb{C}\mathcal{M}$] [A complex "thickening" $\mathbb{C}\mathcal{M}$ of $\mathcal{M}$ is sufficient for this.] These rays are then the generators of $\mathcal{C}_X$ [or of $\mathbb{C}\mathcal{C}_X$]. We can construct the projective hypersurface twistor space $\mathbb{P}\mathcal{T}_X$ of $\mathcal{C}_X$ as the space of $\underline{\alpha\text{-lines}}$ on $\mathbb{C}\mathcal{C}_X$, where an α-line is a complex curve with tangent vectors $o^A \pi^{A'}$, the generators of $\mathbb{C}\ell_x$ having tangent vectors $o^A \bar{o}^{A'}$, and where $\pi_{A'}$ is propagated according to the "shear-free" equation

$$\pi^{A'} \pi^{B'} \nabla_{0B'} \pi_{A'} = 0, \text{ i.e. } \pi^{B'} \nabla_{0B'} \pi_{A'} \propto \pi_{A'}.$$

The lower index "0" refers to contraction with o^A (and "0′" with $\bar{o}^{A'}$, cf. below).

The space $\mathbb{P}\mathcal{T}_x$ encodes the ASD part of the initial null-data for the gravitational field, but not the SD part (L.J. Mason, D.Phil thesis, Oxford). The idea is to encode the (remaining) SD part in terms of the way in which $\mathcal{T}_x$ sits as a kind of bundle over $\mathbb{P}\mathcal{T}_x$. The standard procedure for constructing the non-projective hypersurface twistor space from $\mathbb{P}\mathcal{T}_x$ would have been to have the proportionality constant vanish in the above equation, i.e. $\pi^{B'} \nabla_{0B'} \pi_{A'} = 0$ and to use the scaling for $\pi_{A'}$ (constant along α-lines) as the scaling for the twistor. However, this kills off the SD information. Instead, I propose that $\pi_{A'}$ be propagated along the α-line according to

$$\pi^{B'} \nabla_{0B'} \pi_{A'} = K \, \pi_{A'} (\pi_{0'})^{-5} \, \text{Þ}_{\ell} \, \tilde{\Psi}_{0'0'0'0'} \qquad \text{(A)}$$

where $\tilde{\Psi}_{A'B'C'D'}$ is the SD gravitational field (i.e. equal to the Weyl spinor $\tilde{\Psi}_{A'B'C'D'}$ but scaled under conformal rescalings $\hat{g}_{ab} = \Omega^2 g_{ab}$, $\hat{\varepsilon}_{AB} = \Omega \varepsilon_{AB}$, $\hat{\varepsilon}_{A'B'} = \Omega \varepsilon_{A'B'}$, according to

$$\hat{\Psi}_{A'B'C'D'} = \Omega^{-1} \tilde{\Psi}_{A'B'C'D'}$$

where Þ_{ℓ} is the conformally invariant "thorn" operator, defined (in Penrose & Rindler (1984) *Spinors and Space-Time* vol. 1 (C.U.P.) p. 395) by

$$\text{Þ}_{\ell} = \nabla_{00'} - n\bar{\varepsilon} - (n+1)\bar{\rho}$$

(in spin-coefficient notation), acting on a $\{0, n\}$-scalar of conformal weight $-n-1$, and where K is some pure-number numerical constant whose exact value has yet to be determined. This propagation law is [a] conformally invariant and [b] independent of the O^A or $\bar{O}^{A'}$ scalings. The different solutions of this equation, for a given x-line, constitute the 1-dimensional fibre of $\mathbb{T}_X$ over a point of $\mathbb{PT}_X$.

This is the situation for a finite point X. Of particular interest, however, is the the limiting situation given by $X = i^+$. Owing to the conformal invariance of all quantities involved in (A), the definition may indeed be applied with $X = i^+$ and $\mathscr{C}_X = \mathscr{I}^+$, giving the twistor space $\mathbb{T}^+ (= \mathbb{T}_{i^+})$. (As a notational point, in accordance with normal conventions, the basis spinors $\iota^A, \bar{\iota}^{A'}$ would be most naturally used, rather than the $O^A, \bar{O}^{A'}$ of (A).)

Although the propagation law (A) may seem strange, there are relations with earlier studies. Most particularly, M.G.Eastwood described a procedure (TN 14, FATT I.2.9, pp.40,41) which can be used to represent massless fields on an ASD background $\mathcal{A}$ (at least in the Ricci-flat case) in terms of affine bundles over $\mathcal{A}$'s projective twistor space $\mathbb{PT}$. In the present situation, $\mathbb{PT} = \mathbb{PT}_X$, and the "ASD background $\mathcal{A}$" is the space of (relevant) holomorphic curves in $\mathbb{PT}_X$. We can formulate MGE's construction as follows. Let $\underbrace{\psi_{A'B'\ldots L'}}_{n}$ be a massless field of helicity $+\frac{n}{2}$ on $\mathcal{A}$

$$\psi_{A'B'C'\ldots L'} = \psi_{(A'B'C'\ldots L')}, \qquad \nabla^{AA'}\psi_{A'B'C'\ldots L'} = 0$$

and let $\underbrace{\phi_{A'B'\ldots L'M'}}_{n+1}$ be defined locally on the primed spin bundle of $\mathcal{A}$, homogeneous of degree -1 in the primed spinor $\varpi_{A'}$ ("curly pi" – the fibre coordinate), $\phi_{\ldots}$ being a massless field of helicity $\frac{1}{2}(n+1)$, for constant $\varpi_{A'}$ (which is a meaningful notion when $\mathcal{A}$ is ASD and Ricci-flat):

$$\phi_{A'B'\ldots L'M'} = \phi_{(A'B'\ldots L'M')}, \qquad \nabla^{AA'}\phi_{A'B'C'\ldots L'M'} = 0$$

where

$$\varpi^{M'}\phi_{A'B'\ldots L'M'} = \psi_{A'B'\ldots L'}.$$

The freedom in $\phi_{A'\cdots M'}$ is given by

$$\phi_{A'B'\cdots M'} \mapsto \phi_{A'B'\cdots M'} + \varpi_{A'}\varpi_{B'}\cdots\varpi_{M'} f \qquad \text{(B)}$$

where f is a twistor function ($\varpi^{A'}\nabla_{AA'}f=0$), homogeneous of degree $-n-2$ ($=-6$ in the present case, where $n=4$). From the massless field equations on $\phi_{\cdots}$, we obtain

$$\varpi^{N'}\nabla_{NN'}\phi_{A'B'\cdots L'M'} = \nabla_{NM'}\psi_{A'B'\cdots L'}. \qquad \text{(C)}$$

First cohomology in f gives the fields $\psi_{\cdots}$. The relevant bundle over $\mathbb{P}\mathcal{J}$ is the space of symmetric spinors $\underbrace{\alpha_{A'B'\cdots M'}}_{n+1} = \alpha_{(A'B'\cdots M')}$ and the affine freedom is adding multiples of $\varpi_{A'}\varpi_{B'}\cdots\varpi_{M'}$. This gives an $(n+2)$-dimensional fibre, ~~so~~ there is a lot of "extra baggage" here, although needed for complete invariance. However, in our case, the spinor $o^{A'}$ is singled out, and we can project down to one dimension by taking the $0'0'\cdots 0'$-component. This makes clearest sense when we choose a conformal scaling such that $o^{A'}$ is constant, and we obtain, from (C)

$$\varpi^{N'}\nabla_{NN'}\phi_{0'0'\cdots 0'} = \nabla_{N0'}\psi_{0'\cdots 0'} \qquad \text{(D)}$$

Now define

$$\pi_{A'} = \lambda\varpi_{A'}.$$

We wish to show that, for suitable λ, (D) is equivalent to an equation like (A), but for general non-negative n, which is to hold throughout any α-plane in $\mathcal{A}$ to which $\varpi^{A'}$ is taken to be tangent:

$$\pi^{B'}\nabla_{NB'}\pi_{A'} = K\,\pi_{A'}(\pi_{0'})^{-n-1}\nabla_{N0'}\psi_{0'\cdots 0'}.$$

This equation is equivalent to

$$\lambda\varpi^{B'}(\nabla_{NB'}\lambda)\varpi_{A'} = K\lambda\varpi_{A'}\lambda^{-n-1}(\varpi_{0'})^{-n-1}\nabla_{N0'}\psi_{0'\cdots 0'}$$

i.e. $$\frac{1}{n+2}\varpi^{N'}\nabla_{NN'}(\lambda^{n+2}) = K(\varpi_{0'})^{-n-1}\nabla_{N0'}\psi_{0'\cdots 0'}$$

which is the same as (D), provided that

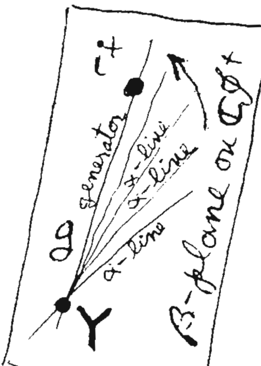
$$\lambda = \{(n+2)K(\varpi_{0'})^{-n-1}\phi_{0'\cdots 0'}\}^{1/(n+2)}$$

Let us now specialize to $\mathbb{C}\mathcal{J}^+$ (and to $n=4$, for relevance to the gravitational case). The space $\mathcal{A}$ is to be Newman's $\mathcal{H}$-space (or, with normal conventions, strictly $\mathcal{H}^*$-space) which is ASD vacuum. We can identify the α-planes of $\mathcal{A}$ with the α-lines on $\mathbb{C}\mathscr{I}^+$, and the points of $\mathcal{A}$ with Newman's "good cuts", generated by α-lines, these being therefore represented as holomorphic curves in $\mathbb{P}\mathcal{T}^+$. From the above, we conclude that the bundle $\mathcal{Y}^+$ over $\mathbb{P}\mathcal{T}^+$, defined by solutions of (A), gives an element of $H^1(\mathbb{P}\mathcal{T}, \mathcal{O}(-6))$ which defines a helicity $+2$ massless field Ψ on $\mathcal{A}$.

In fact, the radiation field of this massless field is identical with the <u>actual</u> ASD part of the gravitational radiation field in the space-time $\mathcal{M}$ (since the quantities $\not{E}_\varepsilon \Psi_{0'0'0'0'}$ agree on $\mathscr{I}^+$ and there is regularity at i^+), $\mathbb{C}\mathscr{I}^+$ being identifyable with the future null infinity of $\mathcal{A}$.

It may be noted that there is an awkwardness involved in directly computing $\Psi_{A'B'C'D'}$ at a point $Y \in \mathbb{C}\mathscr{I}^+$, from the bundle structure of $\mathcal{Y}^+$. The reason is that we would require knowledge of the entire portion of $\mathcal{Y}^+$ that constitutes a bundle over the compact holomorphic curve y that represents Y in $\mathbb{P}\mathcal{T}^+$; yet, to be compact, y must contain a "point at infinity" of $\mathbb{P}\mathcal{T}^+$, i.e. a point on $\mathbb{P}\mathcal{T}^+$'s "line I", representing i^+. In terms of $\mathbb{C}\mathscr{I}^+$, the points of y are represented by α-lines through $Y \in \mathbb{C}\mathscr{I}^+$, where the "point at infinity" G on y is represented by the generator g of $\mathbb{C}\mathscr{I}^+$ through Y. Now, there are two matters of awkwardness concerning this. The first is that the generators of $\mathbb{C}\mathscr{I}^+$ really correspond to points of the <u>blown-up</u> projective twistor space, rather than just the ordinary twistor space. The second matter is the more serious, namely that the equation (A) breaks down for a generator of $\mathbb{C}\mathscr{I}^+$, since $\pi_{A'} \propto \iota_{A'}$ for a generator, so the term "$(\pi_{0'})^{-5}$", which is actually $(\pi_{1'})^{-5}$ here, becomes infinite.

This does not preclude the "points at infinity" of $\mathbb{P}\mathcal{T}^+$ from having appropriate fibres over them, in $\mathcal{T}^+$. Let G be such a point, arising as a limit of a sequence of "finite" points of $\mathbb{P}\mathcal{T}^+$. This sequence corresponds to a family of α-**lines** on $\mathbb{C}\mathscr{I}^+$, through Y, approaching the generator g.



The secret is to define the scalings for the fibre over G as being given simply by the scalings for the $\pi_{A'}$-spinor at i^+, without worrying about the propagation law (A).

The reason that the bundle over the "finite" part of $\mathbb{P}\mathcal{T}^+$ actually extends in a unique way to a bundle over the "infinite" part (provided that the blown-up version of $\mathcal{T}^+$ is used) can be seen in a theorem of Andreotti and Hill. Restricting **attention** to the β-plane on $\mathbb{C}\mathscr{I}^+$ which contains Y, we find that this corresponds to a plane $\mathbb{P}B$ in $\mathbb{P}\mathcal{T}^+$. The part B of $\mathcal{T}^+$ lying above $\mathbb{P}B$ arises as an extension to the "infinite" region of the "finite" region. Andreotti-Hill tells us that the extension of 1st cohomology (of $\mathcal{O}(-6)$ functions) can be made (uniquely), whence this applies also to the bundle B over $\mathbb{P}B$ (this being an Abelian case).

All this shows that the construction of $\mathcal{T}^+$ as given by α-lines on $\mathbb{C}\mathscr{I}^+$, scaled according to (A), does indeed encapsulate the **SD** as well as **ASD** part of the (complexified) radiation field, so the geometry of $\mathcal{M}$ is encoded in $\mathcal{T}^+$ as required. I shall conclude this article by making a few remarks about **(a)** the local structure of $\mathcal{T}^+$, **(b)** the interpretation of points of $\mathcal{M}$ in terms of $\mathcal{T}^+$ and **(c)** a reason why we may expect that the vacuum equations for $\mathcal{M}$ are automatic in this construction.

(a) The basic structure of $\mathcal{T}^+$ was given in TN 43. The twistor space $\mathcal{T}^+$ is foliated by 3-surfaces which, in turn, are foliated by curves (the "Euler curves"), these being determined locally by a 1-form ι and a 3-form θ (both holomorphic) given only up to proportionality and subject to

$$\iota \wedge d\iota = 0, \quad \iota \wedge \theta = 0.$$

Factoring $\mathcal{T}^+$ by the integral curves of θ gives $\mathbb{P}\mathcal{T}^+$; factoring $\mathcal{T}^+$ by the integral 3-surfaces of ι gives the space of β-planes on $\mathbb{C}\mathscr{I}^+$ (a $\mathbb{CP}^1$). There is also a scaling for $\mathcal{T}^+$ which can be provided by the structure

$$\Pi = d\theta \otimes \iota \qquad \text{and} \qquad \Sigma = d\theta \otimes d\theta \otimes \theta.$$

What this means is that if $\mathcal{T}^+$ is pieced together from a number of patches, we must have, on each overlap (with ι, θ on one patch $\mathcal{U}$ and ι', θ' on the other patch $\mathcal{U}'$)

$$\iota' = k\,\iota, \quad \theta' = k^2\theta, \quad d\theta' = k^{-1} d\theta$$

(with k a holomorphic scalar on $\mathcal{U} \cap \mathcal{U}'$).

There is also to be a condition between θ and ι which must hold on each patch

$$d\theta \otimes \iota = -2\theta \odot d\iota \qquad \text{(E)}$$

where the bilinear operation $\odot$, between an n-form and a 2-form is defined from

$$\eta \odot (dp \wedge dq) = \eta \wedge dp \otimes dq - \eta \wedge dq \otimes dp.$$

The equation (E) ensures the various homogeneity relations (the second being automatic)

$$\pounds_\Upsilon \iota = 2\iota \qquad \text{and} \qquad \pounds_\Upsilon \theta = 4\theta$$

where the locally defined Euler operator Υ is given by $\Upsilon = 4\theta \div d\theta$ (which means $da \wedge \theta = \frac{1}{4}\Upsilon(a)\, d\theta$, for any scalar a).

We find, on $\mathcal{U} \cap \mathcal{U}'$,

$$\Upsilon' = k^3 \Upsilon$$

where

$$\Upsilon(k) = 2k^{-1} - 2k\,; \text{ equivalently } \Upsilon'(k^{-1}) = 2k^2 - 2k^{-1}.$$

If z is an "ordinary" scaling parameter along an Euler curve, so that

$$\Upsilon(z) = z,$$

then

$$k^3 = 1 - F z^{-6}$$

with F constant along Euler curves, i.e.

$$k^3 = 1 - f_{-6}(Z^\alpha)$$

with f a twistor function defined in $\mathcal{U} \cap \mathcal{U}$, of degree -6 with respect to $\mathcal{U}$.

(b) Let $X \in \mathcal{M}$ [or $\mathbb{C}\mathcal{M}$], where $\mathcal{C}_X$ [or $\mathbb{C}\mathcal{C}_X$] meets $\mathscr{I}^+$ in a reasonable cross-section (what Newman calls a "light-cone cut"). There is an open set's worth of α-lines on $\mathbb{C}\mathcal{C}_X$ which meet $\mathbb{C}\mathscr{I}^+$. At the intersection point of such an α-line z_X with $\mathbb{C}\mathscr{I}^+$, we can continue with an α-line z^+ on $\mathbb{C}\mathscr{I}^+$ in a unique way, where at the intersecton of z_X with z^+ the $\pi_{A'}$-spinors are the same for each (so $o^A\pi^{A'}$ is tangent to z_X there, and $\iota^A\pi^{A'}$ is tangent to z^+). This establishes "large" open regions of $\mathcal{T}_X$ and $\mathcal{T}^+$ identified with one another. Now, the vertex X of $\mathcal{C}_X$ provides a kind of "blow-up" of $\mathbb{P}\mathcal{T}^+$ which is somewhat similar to the canonical blow-up that is provided by the vertex i^+ of $\mathscr{I}^+$. (Actually, these vertices also define "blow-downs", but the two are not compatible, with one another.) Accordingly, the "gluing" of $\mathcal{T}_X$ on to $\mathcal{T}^+$ gives a kind of "surgery" applied to $\mathcal{T}^+$. This notion of <u>surgery</u> replaces the "holomorphic curve" definition of a "space-time point" that characterizes the original "leg-break" non-linear graviton construction.

The precise characterization of these surgeries, so that the required 4-parameter family of space-time points arises, thereby providing a construction of the space-time $\mathcal{M}$, is presently under consideration.

(c) Analogous to the (locally defined) 1-form ι, which has a characteristic relation with the point i^+ and the associated canonical blow-up of $\mathbb{P}\mathscr{T}^+$, is a (locally defined) 1-form ξ having a characteristic relation with the point X and the associated surgery provided by $\mathbb{P}\mathscr{T}_X$. In each patch we have

$$\xi\wedge d\xi = 0, \quad \xi\wedge\theta = 0, \quad d\theta\otimes\xi = -2\theta\otimes d\xi,$$

the scaling $\xi' = k\xi$ accompanying $\iota' = k\iota$ on $\mathcal{U}\cap\mathcal{U}'$.

If we move from a point X to a neighbouring point $X+\mathring{X}$, we express (so it seems) the metric separation $g = g_{ab}\mathring{X}^a\mathring{X}^b$ by

$$d\xi\wedge d\mathring{\xi} = -\tfrac{1}{2}g\, d\theta.$$

There is some reason to expect that this metric is necessarily <u>vacuum</u> because the quantity

$$dd_Z\xi$$

defined on the product space $\mathscr{T}^+\times\mathcal{M}$ seems to be the <u>Sparling 3-form</u>. (Here d_Z stands for what d stood for before, namely "d" on $\mathscr{T}^+\times\mathcal{M}$ with X constant.) The vanishing of the exterior derivative of the Sparling 3-form is equivalent to the vacuum equations, but here $d(dd_Z\xi)=0$ would be automatic. Much more work is required in order to see if this really works.

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— Roger