Extending half flat metrics
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Recently Plebaniński, Przanowski and Formański, [1], re-addressed the problem of constructing real solutions of the Einstein vacuum field equations from complex half flat metrics. In this note an investigation of a related question is outlined. When can an anti self-dual solution of Einstein's vacuum equations be extended to a solution with a connection whose anti-self dual part coincides with that of the half flat solution? The set of anti self-dual solutions which can be extended in this way is non-empty. It includes complex metrics which can be extended to real solutions (for examples see Ref [2]).

In the investigation of this question it is convenient to use a version of the Cartan structure equations in which the coframe of one-forms, $\theta^a$, is an appropriately ordered null basis. The metric is given by $ds^2 = g_{ab} \theta^a \theta^b$, where

$$g_{ab} = \begin{bmatrix} 0 & \epsilon_{AB} \\ -\epsilon_{AB} & 0 \end{bmatrix}$$

(for real solutions $\theta^0$ and $\theta^3$ are real and $\theta^1 = \theta^2$).

The first Cartan structure equations can be written

$$D\theta^a := d\theta^a - \theta^b \wedge -\Gamma^a_b = \theta^b \wedge + \Gamma^a_b$$

where $D$ is the covariant exterior derivative determined by the anti self-dual part of the connection $-\Gamma^a_b$ and $+\Gamma^a_b$ is the self dual part of the connection one-form. In this (chiral) basis

$$-\Gamma^a_b = \begin{bmatrix} \omega^A_B & 0 \\ 0 & \omega^A_B \end{bmatrix}, \quad \omega_{AB} = \omega_{BA}, \text{ and, } +\Gamma^a_b = \begin{bmatrix} 1 \omega^0_0 & 1 \omega^0_0 \\ 1 \omega^0_0 & -1 \omega^0_0 \end{bmatrix} \begin{bmatrix} 1 \omega^0_0 & 1 \omega^0_0 \\ 1 \omega^0_0 & -1 \omega^0_0 \end{bmatrix} \begin{bmatrix} 1 \omega^0_0 & 1 \omega^0_0 \\ 1 \omega^0_0 & -1 \omega^0_0 \end{bmatrix}$$

Einstein's vacuum field equations may be written

$$D^2 \theta^a = \theta^b \wedge R^a_b = 0$$
where the anti-self dual part of the curvature two-form is given by

\[ R^a_b = \begin{bmatrix} \Omega_B^A & 0 \\ 0 & \Omega_B^A \end{bmatrix}, \text{and} \Omega^A_B = d\omega^A_B + \omega^A_C \wedge \omega^C_B. \] (5)

Suppose now that \( \theta^a \) is a co-frame for a half flat but not flat, anti self-dual metric, \( g \), in a gauge in which \( \Gamma^a_b = 0 \) and write the curvature in terms of its components as \( R^a_b = \frac{1}{2} R^c_{bad} \theta^d \wedge \theta^c \). It follows from Eq. (2) that \( D\theta^a = 0 \).

Now let \( \tilde{\theta}^a \) be a co-frame for a vacuum solution \( \tilde{g} \) with connection one-forms (with non-zero curvature) \( \tilde{\Gamma}^a_b \) and \( \tilde{\Gamma}^a_b = -\Gamma^a_b \). When \( \tilde{\theta}^a \) is written in terms of its components with respect to \( \theta^a \) as

\[ \tilde{\theta}^a = \delta^a_b \theta^b, \] (6)

Eq. (2) gives

\[ D\tilde{\theta}^a \wedge \theta^b = \theta^b \wedge \tilde{\Gamma}^a_b \] (7)

and the Einstein vacuum equations then hold for \( \tilde{g} \) if and only if

\[ \delta^b_e R^a_{fde} \theta^a = 0. \] (8)

Because the curvature is anti self-dual it is a straightforward matter to see that Eq (8) can hold only for anti-self dual Weyl curvatures of type N, III and certain type I's. These are necessary conditions for the half flat metric \( g \) to be extendable to \( \tilde{g} \). By solving Eq. (8), which restricts but does not determine the possible components \( \delta^a_b \), and substituting in Eq. (7), the sufficient conditions for the extendability of \( g \) can be computed. As was mentioned above the metrics found in Ref. [2], all of which have type N or III anti-self dual Weyl curvatures, are examples which satisfy the resulting differential equations. The full solution space is being investigated.

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