Extending half flat metrics

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Recently Plebański, Przanowski and Formański, [1], re-addressed the problem of constructing real solutions of the Einstein vacuum field equations from complex half flat metrics. In this note an investigation of a related question is outlined. When can an anti self-dual solution of Einstein's vacuum equations be extended to a solution with a connection whose anti-self dual part coincides with that of the half flat solution? The set of anti self-dual solutions which can be extended in this way is non-empty. It includes complex metrics which can be extended to real solutions (for examples see Ref [2]).

In the investigation of this question it is convenient to use a version of the Cartan structure equations in which the coframe of one-forms, θ^a, is an appropriately ordered null basis. The metric is given by $ds^2 = g_{ab}\theta^a\theta^b$, where

$$g_{ab} = \begin{bmatrix} 0 & \epsilon_{AB} \\ -\epsilon_{AB} & 0 \end{bmatrix} \tag{1}$$

(for real solutions θ^0 and θ^3 are real and $\theta^1 = \overline{\theta^2}$).
The first Cartan structure equations can be written

$$D\theta^a := d\theta^a - \theta^b \wedge^- \Gamma^a_b = \theta^b \wedge^+ \Gamma^a_b \tag{2}$$

where D is the covariant exterior derivative determined by the anti self-dual part of the connection ${}^-\Gamma^a_b$ and ${}^+\Gamma^a_b$ is the self dual part of the connection one-form. In this (chiral) basis

$${}^-\Gamma^a_b = \begin{bmatrix} \omega^A_B & 0 \\ o & \omega^A_B \end{bmatrix}, \ \omega_{AB} = \omega_{BA}, \text{ and }, \ {}^+\Gamma^a_b = \begin{bmatrix} 1\omega^{0'}_{0'} & 1\omega^{0'}_{1'} \\ 1\omega^{1'}_{0'} & -1\omega^{0'}_{0'} \end{bmatrix}. \tag{3}$$

Einstein's vacuum field equations may be written

$$D^2\theta^a = \theta^b \wedge R^a_b = 0 \tag{4}$$

where the anti-self dual part of the curvature two-form is given by

$$R^a_b = \begin{bmatrix} \Omega^A_B & 0 \\ 0 & \Omega^A_B \end{bmatrix}, and, \Omega^A_B = d\omega^A_B + \omega^A_C \wedge \omega^C_B. \tag{5}$$

Suppose now that θ^a is a co-frame for a half flat but not flat, anti self-dual metric, g, in a gauge in which ${}^+\Gamma^a_b = 0$ and write the curvature in terms of its components as $R^a_b = \frac{1}{2}R^a_{bcd}\theta^c \wedge \theta^d$. It follows from Eq. (2) that $D\theta^a = 0$.

Now let $\widehat{\theta^a}$ be a co-frame for a vacuum solution $\widehat{g}$ with connection one-forms (with non-zero curvature) $\widehat{{}^+\Gamma^a_b}$ and $\widehat{{}^-\Gamma^a_b} = {}^-\Gamma^a_b$. When $\widehat{\theta^a}$ is written in terms of its components with respect to θ^a as

$$\widehat{\theta^a} = \vartheta^a_b\theta^b, \tag{6}$$

Eq.(2) gives

$$D\vartheta^a_b \wedge \theta^b = \theta^b \wedge \widehat{{}^+\Gamma^a_b} \tag{7}$$

and the Einstein vacuum equations then hold for $\widehat{g}$ if and only if

$$\vartheta^f_b R^a_{fcd}\varepsilon^{bcde} = 0. \tag{8}$$

Because the curvature is anti self-dual it is a straightforward matter to see that Eq (8) can hold only for anti-self dual Weyl curvatures of type N, III and certain type I's. These are necessary conditions for the half flat metric g to be extendable to $\widehat{g}$. By solving Eq. (8), which restricts but does not determine the possible components ϑ^a_b, and substituting in Eq. (7), the sufficient conditions for the extendability of g can be computed. As was mentioned above the metrics found in Ref. [2], all of which have type N or III anti-self dual Weyl curvatures, are examples which satisfy the resulting differential equations. The full solution space is being investigated.

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1. J.F.Plebański, M. Przanowski and S. Formański, *Linear superposition of two type-N nonlinear gravitons*, preprint 1998

2. D.C.Robinson, *Some Real and Complex Solutions of Einstein's Equations*, Gen. Rel. & Grav. **19**, 7, 693 1987