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# The abstract twistor space of the Schwarzschild space-time

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Solutions are obtained to the abstract twistor structure equations of the Schwarzschild space-time. An abstract twistor ideal is built and its equations solved for a three-parameter set of abstract twistor surfaces. The limit of vanishing mass and the relation with hypersurface twistors are discussed.

## 1 Introduction

It has been shown recently that, in vacuum spacetimes, the Penrose obstruction to the existence of twistor surfaces can be removed by suitably enlarging the standard Grassmann algebra of differential forms. The key idea is to develop an abstract differential ideal from the tetrad factorization equation[1] $\theta^{AB'} = \alpha^A q^{B'}$. Here $\alpha^A$ is a spinor valued one-form.

In this work we show how to implement these ideas in the context of a physical spacetime and we explicitly solve for the two-dimensional integral manifolds. We obtain a three-parameter set of twistor surfaces. Our result provides the first example of a twistor construction in a nontrivial spacetime.

## 2 Schwarzschild space-time

We begin with the Schwarzschild metric in Eddington coordinates:

$$g = 2dudr + \left(1 - \frac{2M}{r}\right)du^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2, \tag{2.1}$$

where the null co-ordinate $u$ is defined in terms of the Schwarzschild time $t$ by $u \equiv t - r - 2M\ln(r - 2M)$. We choose a standard null tetrad, $\theta^{CD'}$:

$$\begin{aligned} \theta^{00'} &= l = du, & \theta^{10'} &= -m = \frac{r}{\sqrt{2}}(d\theta + i\sin\theta d\phi), \\ \theta^{11'} &= n = dr + \left(\frac{r-2M}{2r}\right)du, & \theta^{01'} &= -m' = \frac{r}{\sqrt{2}}(d\theta - i\sin\theta d\phi), \end{aligned} \tag{2.2}$$

such that $g = 2(ln - mm')$. In the real space-time $m'$ is the complex conjugate of $m$. For twistor surfaces to exist, however, one has to complexify the space-time. Thus the coordinates $u$, $r$, $\theta$ and $\phi$ are considered to be complex. The exterior derivatives of the tetrad forms are:

$$\begin{aligned} dl &= 0\,, \quad dn = -\frac{M}{r^2} l_\wedge n, \\ dm &= -\frac{1}{r} m_\wedge n - \frac{1}{2r}\left(1 - \frac{2M}{r}\right) l_\wedge m + \frac{\cot\theta}{r\sqrt{2}} m_\wedge m', \\ dm' &= -\frac{1}{r} m'_\wedge n - \frac{1}{2r}\left(1 - \frac{2M}{r}\right) l_\wedge m' - \frac{\cot\theta}{r\sqrt{2}} m_\wedge m'. \end{aligned} \tag{2.3}$$

For a spin basis $\zeta_A$ adapted to this tetrad, the $SL(2,\mathbb{C})$ connection one-forms defining the covariant derivatives $D\zeta_A = \Gamma^B_A \zeta_B$ are as follows [2]:

$$\begin{aligned} -\Gamma_0{}^1 = \Gamma_{00} &= \frac{1}{r} m, \\ \Gamma_0{}^0 = \Gamma_{01} &= \frac{M}{2r^2} l + \frac{1}{2\sqrt{2}r}\cot\theta(m - m'), \\ \Gamma_1{}^0 = \Gamma_{11} &= \frac{1}{2r}(1 - \frac{2M}{r})m'. \end{aligned} \tag{2.4}$$

Then for example the covariant exterior derivative $D(s,t)$ of an unprimed spinor with components $(s,t)$ has components:

$$D(s,t) \equiv (ds + \Gamma_{01}s + \Gamma_{11}t,\ dt - \Gamma_{00}s - \Gamma_{01}t). \tag{2.5}$$

The curvature forms, such that

$$D^2(s,t) = (R_{01}s + R_{11}t,\ -R_{00}s - R_{01}t) \tag{2.6}$$

are given by the formulas:

$$\begin{aligned} R_{00} &= d\Gamma_{00} - 2\Gamma_{01\wedge}\Gamma_{00} = -\frac{M}{r^3} l_\wedge m \\ R_{01} &= d\Gamma_{01} + \Gamma_{01\wedge}\Gamma_{11} = \frac{M}{r^3}(l_\wedge n - m_\wedge m') \\ R_{11} &= d\Gamma_{11} + 2\Gamma_{11\wedge}\Gamma_{01} = \frac{M}{r^3} n_\wedge m'\ . \end{aligned} \tag{2.7}$$

The primed spin connection and curvature follows from these relations by complex conjugation.

## 3 The abstract twistor ideal

The abstract twistor structure associated with this space-time is by definition the differential ideal generated by the twistor factorization equations:

$$l = q\beta, \quad m = -p\beta, \quad m' = -q\alpha, \quad n = p\alpha. \tag{3.1}$$

Here $p$ and $q$ are the 0-form components of $q^{A'}$, while $\alpha$ and $\beta$ are the 1-form components of $\alpha^A$. In vacuo it is consistent to write the forms $\alpha$ and $\beta$ as covariant exterior derivatives:

$$(\alpha, \beta) = D(s,t)\ . \tag{3.2}$$

Then the variables $s$ and $t$ serve as parameters for the twistor surface, which is necessarily completely null, since evidently the metric vanishes when equations (3.1) hold. Also when equations (3.1) hold, the connection forms simplify:

$$\begin{aligned} \Gamma_{00} &= -\frac{p}{r}\beta, \\ \Gamma_{01} &= \frac{M}{2r^2}q\beta + \frac{\cot\theta}{2\sqrt{2}r}(q\alpha - p\beta), \\ \Gamma_{11} &= -\frac{q}{2r}\Big(1 - \frac{2M}{r}\Big)\alpha. \end{aligned} \tag{3.3}$$

The unprimed curvature forms of Eqs. (2.7) are now zero. In particular, by Eq. (3.2), we have $D(\alpha,\beta) = 0$, whence $d\alpha$ and $d\beta$ are easily calculated. Define the two-form $z \equiv \alpha_\wedge\beta$. Using equation (3.1), we list the exterior derivatives of the tetrad and of the forms $\alpha$, $\beta$ and $z$:

$$\begin{aligned}
dl &= 0, \qquad dm = -\frac{p}{r}\Big(p+\frac{q}{\sqrt{2}}\cot\theta\Big)z, \\
dn &= \frac{M}{r^2}pqz, \quad dm' = -\frac{q}{2r}\Big[q\Big(1-\frac{2M}{r}\Big) - p\sqrt{2}\cot\theta\Big]z, \qquad (3.4) \\
d\alpha &= \frac{1}{2r}\Big[q\Big(1-\frac{M}{r}\Big)-\frac{p}{\sqrt{2}}\cot\theta\Big]z, \\
d\beta &= \frac{1}{2r}\Big[2p+\frac{q}{\sqrt{2}}\cot\theta\Big]z, \\
dz &= 0. \qquad (3.5)
\end{aligned}$$

We next take the exterior derivatives of Eqs (3.1), and get the following equations involving $dp$ and $dq$:

$$dp_\wedge\alpha = \frac{p}{2r}\Big[q\Big(1-3\frac{M}{r}\Big)-\frac{p}{\sqrt{2}}\cot\theta\Big]\beta_\wedge\alpha \qquad (3.6)$$

$$dp_\wedge\beta = \frac{pq}{2\sqrt{2}r}\cot\theta\ \alpha_\wedge\beta \qquad (3.7)$$

$$dq_\wedge\alpha = \frac{q}{2r}\Big(\frac{p}{\sqrt{2}}\cot\theta+\frac{qM}{r}\Big)\beta_\wedge\alpha \qquad (3.8)$$

$$dq_\wedge\beta = -\frac{q}{2r}\Big(2p+\frac{q}{\sqrt{2}}\cot\theta\Big)\alpha_\wedge\beta\ . \qquad (3.9)$$

Solving these equations, we get

$$\begin{aligned}
dp &= \frac{pq}{2\sqrt{2}r}\cot\theta\ \alpha + \frac{p}{2r}\Big[q\Big(1-\frac{3M}{r}\Big)-\frac{p}{\sqrt{2}}\cot\theta\Big]\beta + \frac{\lambda}{\sqrt{2}}z \\
dq &= -\frac{q}{2r}\Big(2p+\frac{q}{\sqrt{2}}\cot\theta\Big)\alpha + \frac{q}{2r}\Big(q\frac{M}{r}+\frac{p}{\sqrt{2}}\cot\theta\Big)\beta + \frac{\mu}{\sqrt{2}}z\ . \qquad (3.10)
\end{aligned}$$

Here the quantities $\lambda$ and $\mu$ are -1-forms needed here to establish consistency of the later equations.

Assuming the existence of the twistor surfaces, we simplify by noting that on the twistor surface the function on the spin bundle defined by the Killing spinor of the Schwarzschild solution is constant. In the present language, this is the condition:

$$rpq = C, \quad dC = 0. \qquad (3.11)$$

Taking now the exterior derivative of Eq. (3.10), we obtain

$$d\lambda_\wedge z + 3\sqrt{2}CM\frac{p}{r^4}z = 0 \ . \tag{3.12}$$

Defining

$$\omega \equiv -\frac{r^4\lambda}{3\sqrt{2}CpM} \tag{3.13}$$

Eq. (3.12) reads $(d\omega - 1)_\wedge z = 0$ . This allows us to truncate the system by imposing the equation

$$d\omega = 1 \ . \tag{3.14}$$

Thereby our ideal is complete.

## 4 Solving the ideal

By (2.2) and (3.1), we may express

$$d\theta = -\frac{m+m'}{\sqrt{2}r} = \frac{1}{\sqrt{2}r}(q\alpha + p\beta) = \frac{1}{\sqrt{2}r}\Big(\frac{p}{q}l + \frac{q}{p}n\Big) \ . \tag{4.1}$$

Using also (3.11) and (2.2) again,

$$d\theta = \frac{p^2}{\sqrt{2}C}du + \frac{C}{\sqrt{2}r^2p^2}\Big[dr + \frac{1}{2}\Big(1 - \frac{2M}{r}\Big)du\Big] \ . \tag{4.2}$$

In a similar manner, we obtain

$$\begin{aligned} dp &= \frac{C}{2\sqrt{2}r^2p}\cot\theta\Big[dr + \frac{1}{2}\Big(1 - \frac{2M}{r}\Big)du\Big] \\ &\quad + \frac{1}{2}\Big[\frac{p}{r}\Big(1 - 3\frac{M}{r}\Big) - \frac{p^3}{\sqrt{2}C}\cot\theta\Big]du - 3\frac{pM}{r^3}\omega_\wedge dr_\wedge du \ . \end{aligned} \tag{4.3}$$

Multiplying by $2p$ and introducing the new variable $Q$ via the substitution

$$p^2 = \exp\Big\{Q - \frac{3M\omega_\wedge du}{r^2}\Big\} = e^Q\Big(1 - \frac{3M\omega_\wedge du}{r^2}\Big) \ , \tag{4.4}$$

we obtain the equations

$$d\theta = \frac{e^Q}{\sqrt{2}C}du + \frac{Ce^{-Q}}{\sqrt{2}r^2}\Big[dr - \frac{3M}{r^2}\omega_\wedge dr_\wedge du + \Big(\frac{1}{2} - \frac{M}{r}\Big)du\Big] \tag{4.5}$$

$$dQ = \frac{Ce^{-Q}\cot\theta}{\sqrt{2}r^2}\Big[dr - \frac{3M}{r^2}\omega_\wedge dr_\wedge du + \Big(\frac{1}{2} - \frac{M}{r}\Big)du\Big] + \Big(\frac{1}{r} - \frac{e^Q\cot\theta}{\sqrt{2}C}\Big)du \;. \tag{4.6}$$

Making the substitutions

$$\begin{aligned} R &= Q - \log(\sin\theta) \\ \frac{y}{\sqrt{2}} &= \cos\theta - \frac{C}{\sqrt{2}}e^{-Q}\sin\theta\frac{M}{r^3}\omega_\wedge du \;, \end{aligned} \tag{4.7}$$

we have

$$\begin{aligned} dR &= \Big(\frac{1}{r} - \frac{ye^R}{C}\Big)du \\ dy &= -\frac{e^R(2-y^2)}{2C}du - \frac{C}{r^2}e^{-R}\Big(dr + \frac{1}{2}du\Big) \;. \end{aligned} \tag{4.8}$$

This is a standard differential system. We next introduce the variable

$$x = ye^R - \frac{C}{r} \tag{4.9}$$

yielding

$$CdR = -xdu \tag{4.10}$$

$$Cdx = -\Big(\frac{1}{2}x^2 + e^{2R}\Big)du \;. \tag{4.11}$$

Put

$$w = x^2 - e^{2R} \;. \tag{4.12}$$

Then we immediately get

$$dw = (w + e^{2R})dR \tag{4.13}$$

with the general solution

$$w = e^{2R} + Be^R \tag{4.14}$$

where $B$ is the integration constant. Then

$$x^2 = \frac{2+BS}{S^2} \tag{4.15}$$

where

$$S = e^{-R}\ . \tag{4.16}$$

From (4.10) we have

$$du = \frac{CdS}{(2+BS)^{1/2}} \tag{4.17}$$

which has the general solution

$$u = \frac{2C}{B}(2+BS)^{1/2} + A \tag{4.18}$$

and $A$ is an integration constant. After backsubstitution,

$$p^2 = \frac{\sin\theta}{S}\Big[1 - \frac{3MC\omega_\wedge dS}{(2+BS)^{1/2}r^2}\Big] \tag{4.19}$$

$$\cos\theta = \frac{CS}{\sqrt{2}r} + \Big(1+\frac{BS}{2}\Big)^{1/2} + \frac{C^2SM\omega_\wedge dS}{\sqrt{2}r^3(2+BS)^{1/2}}\ . \tag{4.20}$$

Finally we determine the coordinate $\phi$ using

$$d\phi = i\frac{m-m'}{\sqrt{2}r\sin\theta} = \frac{i}{C\sqrt{2}p^2r^2\sin\theta}\Big[\frac{C^2}{2}\Big(1-\frac{2M}{r}\Big)du + C^2dr - r^2\ p^4du\Big]\ . \tag{4.21}$$

Introduce the stereographic coordinate $\zeta = e^{-i\phi}\cot(\theta/2)$, substitute $p$ and $u$, this yielding the remarkably simple equation

$$\frac{d\zeta}{\zeta} = -\frac{dS}{S(1+BS/2)^{1/2}} \tag{4.22}$$

with solution

$$S = \frac{8F\zeta}{B(\zeta-F)^2} \tag{4.23}$$

where $F$ is a constant. Substituting $S$ in Eqs. (4.19)-(4.20), we get *the abstract twistor equations* of the Schwarzschild space-time:

$$p^2 = \frac{\sin\theta}{8F\zeta r^2}\left[Br^2(\zeta - F)^2 + 12\sqrt{2}MCF\omega_\wedge d\zeta\right] \quad (4.24)$$

$$\cos\theta = \frac{8CF\zeta}{\sqrt{2}B(\zeta - F)^2 r} + \frac{\zeta + F}{\zeta - F} - 32\frac{C^2F^2M\zeta\omega_\wedge d\zeta}{B^2r^3(\zeta - F)^4} \quad (4.25)$$

$$u = 2\sqrt{2}\frac{C(\zeta + F)}{B(\zeta - F)} + A\,. \quad (4.26)$$

# 5 Discussion

For $M \to 0$, we obtain from here flat-space twistor calculus. Observe that we have three essential parameters in our solution: $C/B$, $F$ and $A$. These remain independent in the limit of vanishing mass, and in that limit we find an open subset of conventional twistor space. Two independent variables are $\zeta$ and $r$. Some special hypersurface twistors are located, *e.g.*, on the flat $u = const.$ and $t = const.$ hypersurfaces.

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# References

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