Exceptional hyperKähler reductions

In our joint paper (Kobak and Swann 1993), Proposition 1.3, we pointed out that the orbifold $G_2/SO(4)/\mathbb{Z}_3$ can be viewed as the $U(1)$ quaternion-Kähler reduction of the Grassman space $\mathbb{Gr}_4(\mathbb{R}^7)$:

$$G_2/SO(4)/\mathbb{Z}_3 = \mathbb{Gr}_4(\mathbb{R}^7)/\!/\!/U(1) \qquad (*)$$

where the $U(1)$ action on $\mathbb{Gr}_4(\mathbb{R}^7)$ comes from the following action on $\mathbb{R}^7 \equiv \mathbb{R}\oplus\mathbb{C}^3$: $U(1)$ acts trivially on $\mathbb{R}$ and acts as the multiplicative group of unit complex numbers on $\mathbb{C}^3$. In this way yet again the exceptional group G_2 occurs naturally in a completely classical setup. This construction was stumbled upon by chance when one of the authors analysed various $U(1)$ actions on spaces $\mathbb{R}^k \times \mathbb{C}^l$ and realised that in the $\mathbb{R}\oplus\mathbb{C}^3$ case the co-associative 4-planes in $\mathbb{R}^7$ happen to sit in the zero locus of the $U(1)$-moment map $\mu\colon \mathbb{Gr}_4(\mathbb{R}^7) \to \mathcal{G}\otimes\mathfrak{u}(1)$ (here $\mathcal{G}$ denotes the orthogonal complement of the tautological 4-plane bundle over $\mathbb{Gr}_4(\mathbb{R}^7)$). The set of co-associative 4-planes is, of course, the exceptional quaternion-Kähler symmetric space $G_2/SO(4)$. As the moment map μ is $U(1)$-invariant, we have

$$U(1)\cdot G_2/SO(4) \subset \mu^{-1}(0)$$

and spotting that it is in fact an equality does the trick: we get Formula $(*)$ since $U(1)$ acts on $U(1)\cdot G_2/SO(4)$ with stabiliser $\mathbb{Z}_3$.

Our aim here is to show how the reduction $(*)$ fits the general construction developed by Kobak and Swann (1995) for classical groups. We will use the following general facts:

1) Rather than quaternion-Kähler reductions of quaternion-Kähler symmetric spaces one can consider hyperKähler reductions of the corresponding nilpotent orbits:

$$\begin{array}{ccc}
\mathcal{N} & \xrightarrow{\;H_G\;} & \mathcal{N}/\!/\!/G \\
\downarrow & & \downarrow \\
\mathbb{P}\mathcal{N} & & \mathbb{P}(\mathcal{N}/\!/\!/G) \\
\downarrow & & \downarrow \\
M & \xrightarrow{\;Q_G\;} & M/\!/\!/G
\end{array} \qquad (**)$$

The arrow "$\xrightarrow{Q_G}$" denotes the functor of quaternion-Kähler reduction by the group G, and "$\xrightarrow{H_G}$" is the corresponding hyperKähler reduction (defined as in Swann 1991). We recall that the projective bundles in the middle row of (**) are Simon Salamon's "quaternionic" twistor spaces (Salamon 1982).

2) The hyperKähler reduction of a hyperKähler manifold M by a product $G \times K$ of two Lie groups can be taken in steps (for details see Kobak and Swann 1995, Lemma 3.2).

$$\begin{array}{ccc} & M & \\ {}^{H_G}\swarrow & & \searrow^{H_{G\times K}} \\ M/\!\!/\!\!/G & \xrightarrow{H_K} & M/\!\!/\!\!/(G\times K) \end{array}$$

In our situation the nilpotent orbits which correspond to $G_2/SO(4)$ and $\mathbb{G}r_4(\mathbb{R}^7)$ are the respective minimal nonzero nilpotent orbits: $\mathcal{N}^{\mathrm{min}}_{G_2} \subset \mathfrak{g}_2^{\mathbb{C}}$ and $\mathcal{N}^{\mathrm{min}}_{SO(7)} \subset SO(7)$. The crucial ingredient is the fact that $\mathcal{N}^{\mathrm{min}}_{G_2}$ is a 3:1 branched covering of the nilpotent variety $\overline{\mathcal{N}^{\mathrm{reg}}_{SL(3)}} \subset \mathfrak{sl}(3,\mathbb{C})$. (Brylinski and Kostant (1992) have classified all such finite coverings.) It is thus enough to exhibit the regular nilpotent orbit $\mathcal{N}^{\mathrm{reg}}_{SL(3)}$ as a hyperKähler reduction of $\mathcal{N}^{\mathrm{min}}_{SO(7)}$ by $U(1)$. Now we are in the realm of classical groups and the machinery from (Kobak and Swann 1996) can be set in motion. The necessary data concerning the orbits in question is contained in the table below:

Nilpotent orbit	Jordan type	Diagram	HyperKähler reduction
$\mathcal{N}^{\mathrm{reg}}_{SL(3)}$	(3)	$0 \rightleftarrows \mathbb{C} \rightleftarrows \mathbb{C}^2 \rightleftarrows \mathbb{C}^3$	$\mathbb{H}^8 /\!\!/\!\!/ (U(1)\times U(2))$
$\mathcal{N}^{\mathrm{min}}_{SO(7)}$	$(2^2, 1^3)$	$0 \rightleftarrows \mathbb{C}^2 \rightleftarrows \mathbb{C}^7$	$\mathbb{H}^7 /\!\!/\!\!/ Sp(1)$

The construction now follows from the fact that one can factor $\mathbb{H}^8$ either by $U(1)\times U(2)$ to get $\mathcal{N}^{\mathrm{reg}}_{SL(3)}$, or, in steps, to get first $\mathbb{H}^7$ and then $\mathcal{N}^{\mathrm{min}}_{SO(7)} = \mathbb{H}^7 /\!\!/\!\!/ Sp(1)$. One needs, however, to take into account various quotients by finite groups that arise on the way. Firstly, the $U(1)$ reduction of $\mathbb{H}^8$ gives $\mathbb{H}/\mathbb{Z}_2 \times \mathbb{H}^6$ rather than $\mathbb{H}^7$. To see this note that the first two components of the diagram for $\mathcal{N}^{\mathrm{reg}}_{SL(3)}$ give rise to the diagram $0 \rightleftarrows \mathbb{C} \rightleftarrows \mathbb{C}^2$. This is the diagram for the nilpotent variety $\overline{\mathcal{N}^{\mathrm{reg}}_{SL(2)}}$ which is equal to $\mathbb{H}/\mathbb{Z}_2$ and arises in the model case of the hyperKähler–twistor–quaternion-Kähler setup:

$$\begin{array}{ccccc} \mathcal{N}^{\mathrm{reg}}_{SL(2)} & \longrightarrow & \mathbb{P}\mathcal{N}^{\mathrm{reg}}_{SL(2)} & \longrightarrow & \mathbb{HP}^1 \\ \| & & \| & & \| \\ \mathbb{H}\setminus\{0\}/\mathbb{Z}_2 & \longrightarrow & \mathbb{CP}^3 & \longrightarrow & S^4 \end{array}$$

In effect we have reductions

$$\mathbb{H}^8 \xrightarrow{H_{U(1)}} \mathbb{H}/\mathbb{Z}_2 \times \mathbb{H}^6 \xrightarrow{H_{Sp(1)}} \mathcal{N}^{\mathrm{min}}_{SO(7)}/\mathbb{Z}_2 \xrightarrow{H_{U(1)}} \mathcal{N}^{\mathrm{reg}}_{SL(3)}$$

and their composition is the reduction of $\mathbb{H}^8$ by $U(1)\times U(1)\times U(2)$. Note that this gives $\mathcal{N}^{\mathrm{reg}}_{SL(3)}$ rather than $\mathcal{N}^{\mathrm{reg}}_{SL(3)}/\mathbb{Z}_2$: we know that $\mathbb{H}^8 /\!\!/\!\!/ (U(1)\times U(2))$ is $\mathcal{N}^{\mathrm{reg}}_{SL(3)}$ and $U(2) = (U(1) \times SU(2))/\mathbb{Z}_2$ so reducing $\mathbb{H}^8$ by a 'smaller' group cannot result in a 'smaller' space. To be more precise, one can check that the $\mathbb{Z}_2$ which appears in the above group isomorphism is contained in the $U(1)$ which we use to reduce $\mathcal{N}^{\mathrm{min}}_{SO(7)}$ to $\mathcal{N}^{\mathrm{reg}}_{SL(3)}$. We can put together these reductions to form a diagram:

$$\begin{array}{ccccc}
\mathbb{H}^8 & \xrightarrow{H_{U(1)}} & \mathbb{H}/\mathbb{Z}_2 \times \mathbb{H}^6 & \xrightarrow{H_{Sp(1)}} & \mathcal{N}^{\mathrm{min}}_{SO(7)}/\mathbb{Z}_2 \\
 & \searrow & & & \downarrow H_{U(1)} \\
 & & H_{U(1)\times U(2)} & & \mathcal{N}^{\mathrm{reg}}_{SL(3)}
\end{array}$$

By considering $\mathbb{Z}_2$ coverings of $\mathbb{H}/\mathbb{Z}_2 \times \mathbb{H}^6$ and $\mathcal{N}^{\mathrm{min}}_{SO(7)}/\mathbb{Z}_2$, we get another smaller diagram which explains where the reduction $(*)$ comes from:

$$\begin{array}{ccc}
\mathbb{H}^7 & \xrightarrow{H_{Sp(1)}} & \mathcal{N}^{\mathrm{min}}_{SO(7)} \\
 & {\scriptstyle H_{U(2)}}\searrow & \downarrow H_{U(1)} \\
 & & \mathcal{N}^{\mathrm{reg}}_{SL(3)}
\end{array}$$

The following diagrams tell the quaternion-Kähler version of the story:

$$\begin{array}{ccccc}
\mathbb{HP}^7 & \xrightarrow{Q_{U(1)}} & \mathbb{HP}^6/\mathbb{Z}_2 & \xrightarrow{Q_{Sp(1)}} & \mathbb{G}\mathrm{r}_4(\mathbb{R}^7)/\mathbb{Z}_2 \\
 & \searrow & & & \downarrow Q_{U(1)} \\
 & & Q_{U(1)\times U(2)} & & G_2/SO(4)/\mathbb{Z}_3
\end{array}$$

$$\begin{array}{ccc}
\mathbb{HP}^6 & \xrightarrow{Q_{Sp(1)}} & \mathbb{G}\mathrm{r}_4(\mathbb{R}^7) \\
 & {\scriptstyle Q_{U(2)}}\searrow & \downarrow Q_{U(1)} \\
 & & G_2/SO(4)/\mathbb{Z}_3
\end{array}$$

Note that by reordering the groups one can also realise $G_2/SO(4)/\mathbb{Z}_3$ as a hyperKähler reduction of $\mathbb{G}\mathrm{r}_4(\mathbb{R}^8)$ by $U(1) \times U(1)$: we have

$$\mathbb{HP}^7 \xrightarrow{Q_{Sp(1)}} \mathbb{G}\mathrm{r}_4(\mathbb{R}^8) \xrightarrow{Q_{U(1)\times U(1)}} G_2/SO(4)/\mathbb{Z}_3.$$

By looking at Brylinski and Kostant's (1994) classification of shared nilpotent orbits one finds that there is one other exceptional quaternion-Kähler symmetric space which has finite quotients obtainable from a classical reduction, namely $F_4/Sp(3)Sp(1)$. This has quotients of degree 2 and 4 which are reductions of $\mathbb{HP}^{17}$ by $Sp(2)$ and $Sp(2) \times \mathbb{Z}_2$ (respectively).

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