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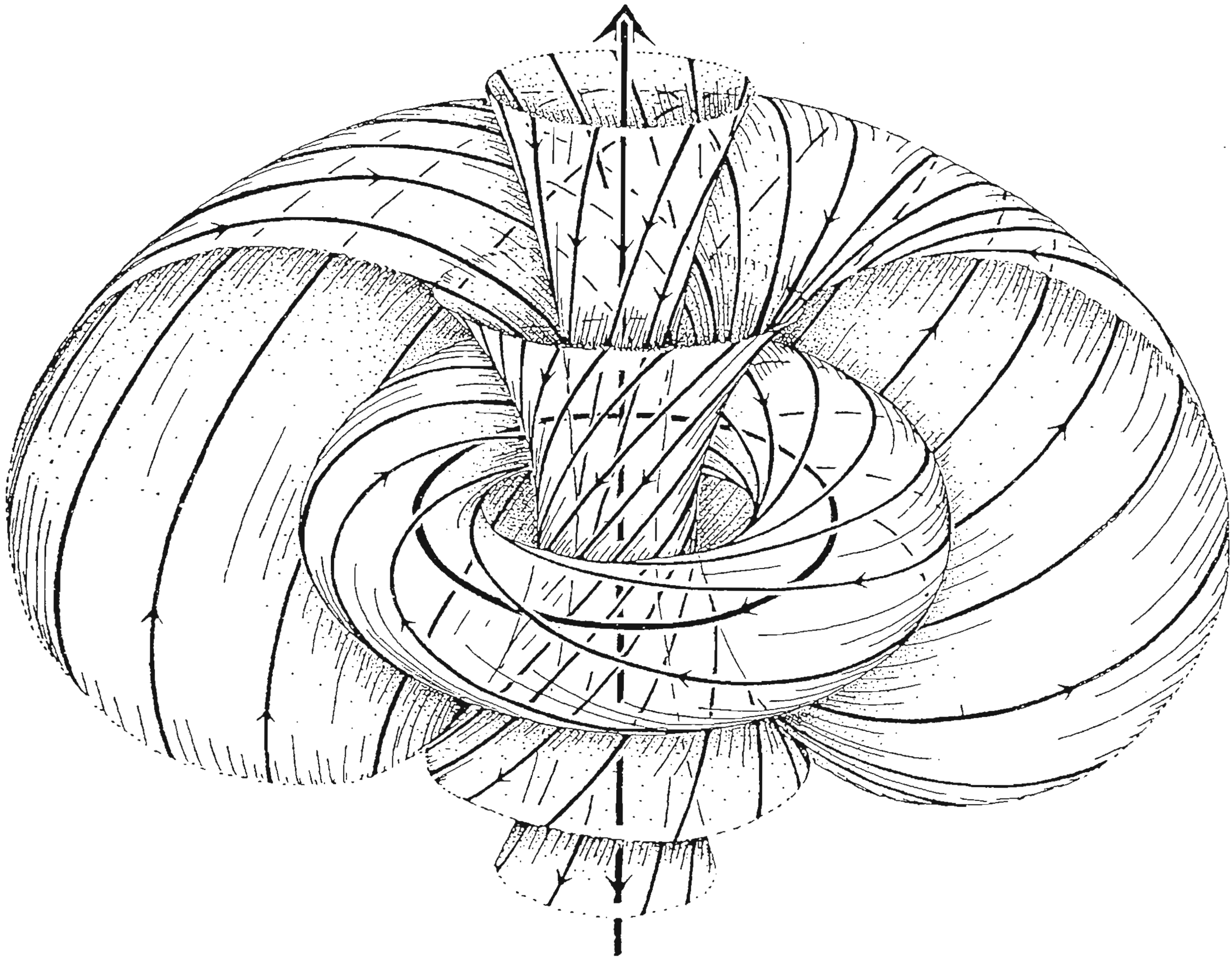

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The Twistor Space for a Strongly Asymptotically Flat Vacuum

by Roger Penrose

In TN 43, I introduced certain ideas concerning the geometrical nature of a curved twistor space $\mathcal{T}$, in terms of which the structure of an arbitrary "strongly asymptotically flat analytic vacuum" (SAFAV) space-time $\mathcal{M}$ ought to be encoded. Some suggestions were provided for the means whereby $\mathcal{M}$ might be constructed from $\mathcal{T}$. ($\mathcal{M}$ is SAFAV if it is an analytic vacuum with an analytic $\mathscr{I}^+$ and an analytic i^+. Thus, it represents a free gravitational wave which dies out in time in all directions, with no remnant in the form of a black hole or other source.)

The main purpose of this note is to provide a solution to the converse problem whereby $\mathcal{T}$ may be constructed from $\mathcal{M}$. Some more information concerning the curious geometrical nature of $\mathcal{T}$ will be given, and how it can be constructed from free holomorphic data. I shall show that the geometry of $\mathcal{M}$ (near $\mathscr{I}^+$) is indeed encoded in the structure of $\mathcal{T} = \mathcal{T}^+$ where $\mathcal{T}^+$ is constructed from free holomorphic data. As far as the construction of $\mathcal{M}$ from $\mathcal{T}^+$ is concerned, there remain many uncertainties and conjectural issues, some of which will be indicated here.

Let X be an arbitrary (finite) point of $\mathcal{M}$. I shall show, first, how to construct a twistor space $\mathcal{T}_X$, with respect to X, which encodes the information of both the SD and ASD parts of the gravitational field. Let $\mathcal{C}_X$ [respectively, $\mathbb{C}\mathcal{C}_X$] be the [complex] light cone of X, swept out by the [complex] light rays through X in $\mathcal{M}$ [resp. $\mathbb{C}\mathcal{M}$] [A complex "thickening" $\mathbb{C}\mathcal{M}$ of $\mathcal{M}$ is sufficient for this.] These rays are then the generators of $\mathcal{C}_X$ [or of $\mathbb{C}\mathcal{C}_X$]. We can construct the projective hypersurface twistor space $\mathbb{P}\mathcal{T}_X$ of $\mathcal{C}_X$ as the space of <u>α-lines</u> on $\mathbb{C}\mathcal{C}_X$, where an α-line is a complex curve with tangent vectors $o^A \pi^{A'}$, the generators of $\mathbb{C}\ell_x$ having tangent vectors $o^A \bar{o}^{A'}$, and where $\pi_{A'}$ is propagated according to the "shear-free" equation

$$\pi^{A'} \pi^{B'} \nabla_{0B'} \pi_{A'} = 0, \text{ i.e. } \pi^{B'} \nabla_{0B'} \pi_{A'} \propto \pi_{A'}.$$

The lower index "0" refers to contraction with o^A (and "0′" with $\bar{o}^{A'}$, cf. below).

The space $\mathbb{P}\mathcal{T}_x$ encodes the ASD part of the initial null-data for the gravitational field, but not the SD part (L.J. Mason, D.Phil thesis, Oxford). The idea is to encode the (remaining) SD part in terms of the way in which $\mathcal{T}_x$ sits as a kind of bundle over $\mathbb{P}\mathcal{T}_x$. The standard procedure for constructing the non-projective hypersurface twistor space from $\mathbb{P}\mathcal{T}_x$ would have been to have the proportionality constant vanish in the above equation, i.e. $\pi^{B'} \nabla_{0B'} \pi_{A'} = 0$ and to use the scaling for $\pi_{A'}$ (constant along α-lines) as the scaling for the twistor. However, this kills off the SD information. Instead, I propose that $\pi_{A'}$ be propagated along the α-line according to

$$\pi^{B'} \nabla_{0B'} \pi_{A'} = K \pi_{A'} (\pi_{0'})^{-5} Þ_e \tilde{\Psi}_{0'0'0'0'} \quad (A)$$

where $\tilde{\Psi}_{A'B'C'D'}$ is the SD gravitational field (i.e. equal to the Weyl spinor $\tilde{\Psi}_{A'B'C'D'}$ but scaled under conformal rescalings $\hat{g}_{ab} = \Omega^2 g_{ab}$, $\hat{\varepsilon}_{AB} = \Omega \varepsilon_{AB}$, $\hat{\varepsilon}_{A'B'} = \Omega \varepsilon_{A'B'}$, according to

$$\hat{\Psi}_{A'B'C'D'} = \Omega^{-1} \tilde{\Psi}_{A'B'C'D'}$$

where $Þ_e$ is the conformally invariant "thorn" operator, defined (in Penrose & Rindler (1984) *Spinors and Space-Time* vol. 1 (C.U.P.) p. 395) by

$$Þ_e = \nabla_{00'} - n\bar{\varepsilon} - (n+1)\bar{\rho}$$

(in spin-coefficient notation), acting on a $\{0, n\}$-scalar of conformal weight $-n-1$, and where K is some pure-number numerical constant whose exact value has yet to be determined. This propagation law is [a] conformally invariant and [b] independent of the o^A or $\bar{o}^{A'}$ scalings. The different solutions of this equation, for a given x-line, constitute the 1-dimensional fibre of $\mathcal{T}_X$ over a point of $\mathbb{P}\mathcal{T}_X$.

This is the situation for a finite point X. Of particular interest, however, is the the limiting situation given by $X = i^+$. Owing to the conformal invariance of all quantities involved in Ⓐ, the definition may indeed be applied with $X = i^+$ and $\mathcal{C}_X = \mathscr{I}^+$, giving the twistor space $\mathcal{T}^+ (= \mathcal{T}_{i^+})$. (As a notational point, in accordance with normal conventions, the basis spinors $\iota^A, \bar{\iota}^{A'}$ would be most naturally used, rather than the $o^A, \bar{o}^{A'}$ of Ⓐ.)

Although the propagation law Ⓐ may seem strange, there are relations with earlier studies. Most particularly, M.G. Eastwood described a procedure (TN 14, FATT I.2.9, pp. 40, 41) which can be used to represent massless fields on an ASD background $\mathcal{A}$ (at least in the Ricci-flat case) in terms of affine bundles over $\mathcal{A}$'s projective twistor space $\mathbb{P}\mathcal{T}$. In the present situation, $\mathbb{P}\mathcal{T} = \mathbb{P}\mathcal{T}_X$, and the "ASD background $\mathcal{A}$" is the space of (relevant) holomorphic curves in $\mathbb{P}\mathcal{T}_X$. We can formulate MGE's construction as follows. Let $\underbrace{\psi_{A'B'\dots L'}}_{n}$ be a massless field of helicity $+\frac{n}{2}$ on $\mathcal{A}$

$$\psi_{A'B'C'\dots L'} = \psi_{(A'B'C'\dots L')}, \qquad \nabla^{AA'}\psi_{A'B'C'\dots L'} = 0$$

and let $\underbrace{\phi_{A'B'\dots L'M'}}_{n+1}$ be defined locally on the primed spin bundle of $\mathcal{A}$, homogeneous of degree -1 in the primed spinor $\varpi_{A'}$ ("curly pi" — the fibre coordinate), $\phi_{\dots}$ being a massless field of helicity $\frac{1}{2}(n+1)$, for constant $\varpi_{A'}$ (which is a meaningful notion when $\mathcal{A}$ is ASD and Ricci-flat):

$$\phi_{A'B'\dots L'M'} = \phi_{(A'B'\dots L'M')}, \qquad \nabla^{AA'}\phi_{A'B'C'\dots L'M'} = 0$$

where

$$\varpi^{M'}\phi_{A'B'\dots L'M'} = \psi_{A'B'\dots L'}.$$

The freedom in $\phi_{A'\cdots M'}$ is given by

$$\phi_{A'B'\cdots M'} \mapsto \phi_{A'B'\cdots M'} + \varpi_{A'}\varpi_{B'}\cdots\varpi_{M'} f \qquad (B)$$

where f is a twistor function ($\varpi^{A'}\nabla_{AA'}f = 0$), homogeneous of degree $-n-2$ ($=-6$ in the present case, where $n=4$). From the massless field equations on $\phi_{\cdots}$, we obtain

$$\varpi^{N'}\nabla_{NN'}\phi_{A'B'\cdots L'M'} = \nabla_{NM'}\psi_{A'B'\cdots L'}. \qquad (C)$$

First cohomology in f gives the fields $\psi_{\cdots}$. The relevant bundle over $\mathbb{P}\mathcal{T}$ is the space of symmetric spinors $\underbrace{\alpha_{A'B'\cdots M'}}_{n+1} = \alpha_{(A'B'\cdots M')}$ and the affine freedom is adding multiples of $\varpi_{A'}\varpi_{B'}\cdots\varpi_{M'}$. This gives an $(n+2)$-dimensional fibre, so there is a lot of "extra baggage" here, although needed for complete invariance. However, in our case, the spinor $o^{A'}$ is singled out, and we can project down to one dimension by taking the $0'0'\cdots0'$-component. This makes clearest sense when we choose a conformal scaling such that $o^{A'}$ is constant, and we obtain, from (C)

$$\varpi^{N'}\nabla_{NN'}\phi_{0'0'\cdots0'} = \nabla_{N0'}\psi_{0'\cdots0'} \qquad (D)$$

Now define

$$\pi_{A'} = \lambda\varpi_{A'}.$$

We wish to show that, for suitable λ, (D) is equivalent to an equation like (A), but for general non-negative n, which is to hold throughout any α-plane in $\mathcal{A}$ to which $\varpi^{A'}$ is taken to be tangent:

$$\pi^{B'}\nabla_{NB'}\pi_{A'} = K\,\pi_{A'}(\pi_{0'})^{-n-1}\nabla_{N0'}\psi_{0'\cdots0'}.$$

This equation is equivalent to

$$\lambda\varpi^{B'}(\nabla_{NB'}\lambda)\varpi_{A'} = K\lambda\varpi_{A'}\lambda^{-n-1}(\varpi_{0'})^{-n-1}\nabla_{N0'}\psi_{0'\cdots0'}$$

i.e. $$\frac{1}{n+2}\varpi^{N'}\nabla_{NN'}(\lambda^{n+2}) = K(\varpi_{0'})^{-n-1}\nabla_{N0'}\psi_{0'\cdots0'}$$

which is the same as (D), provided that

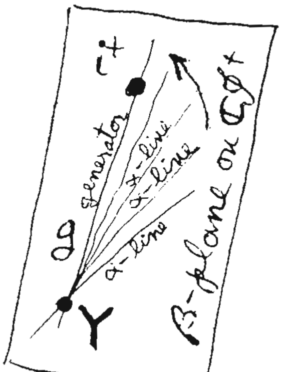
$$\lambda = \{(n+2)K(\varpi_{0'})^{-n-1}\phi_{0'\cdots0'}\}^{1/(n+2)}$$

Let us now specialize to $\mathbb{C}\mathcal{J}^+$ (and to $n=4$, for relevance to the gravitational case). The space $\mathcal{A}$ is to be Newman's $\mathcal{H}$-space (or, with normal conventions, strictly $\mathcal{H}^*$-space) which is ASD vacuum. We can identify the α-planes of $\mathcal{A}$ with the α-lines on $\mathbb{C}\mathscr{I}^+$, and the points of $\mathcal{A}$ with Newman's "good cuts", generated by α-lines, these being therefore represented as holomorphic curves in $\mathbb{P}\mathcal{T}^+$. From the above, we conclude that the bundle $\mathcal{J}^+$ over $\mathbb{P}\mathcal{T}^+$, defined by solutions of (A), gives an element of $H^1(\mathbb{P}\mathcal{T}, \mathcal{O}(-6))$ which defines a helicity $+2$ massless field Ψ on $\mathcal{A}$.

In fact, the radiation field of this massless field is identical with the actual ASD part of the gravitational radiation field in the space-time M (since the quantities $\bar{E}_6\, \Psi_{0'0'0'0'}$ agree on $\mathscr{I}^+$ and there is regularity at i^+), $\mathbb{C}\mathscr{I}^+$ being identifyable with the future null infinity of $\mathcal{A}$.

It may be noted that there is an awkwardness involved in directly computing $\Psi_{A'B'C'D'}$ at a point $Y \in \mathbb{C}\mathscr{I}^+$, from the bundle structure of $\mathcal{J}^+$. The reason is that we would require knowledge of the entire portion of $\mathcal{J}^+$ that constitutes a bundle over the compact holomorphic curve y that represents Y in $\mathbb{P}\mathcal{T}^+$; yet, to be compact, y must contain a "point at infinity" of $\mathbb{P}\mathcal{T}^+$, i.e. a point on $\mathbb{P}\mathcal{T}^+$'s "line I", representing i^+. In terms of $\mathbb{C}\mathscr{I}^+$, the points of y are represented by α-lines through $Y \in \mathbb{C}\mathscr{I}^+$, where the "point at infinity" G on y is represented by the generator g of $\mathbb{C}\mathscr{I}^+$ through Y. Now, there are two matters of awkwardness concerning this. The first is that the generators of $\mathbb{C}\mathscr{I}^+$ really correspond to points of the blown-up projective twistor space, rather than just the ordinary twistor space. The second matter is the more serious, namely that the equation (A) breaks down for a generator of $\mathbb{C}\mathscr{I}^+$, since $\pi_{A'} \propto \iota_{A'}$ for a generator, so the term "$(\pi_{0'})^{-5}$", which is actually $(\pi_{1'})^{-5}$ here, becomes infinite.

This does not preclude the "points at infinity" of $\mathbb{P}\mathcal{T}^+$ from having appropriate fibres over them, in $\mathcal{T}^+$. Let G be such a point, arising as a limit of a sequence of "finite" points of $\mathbb{P}\mathcal{T}^+$. This sequence corresponds to a family of α-**lines** on $\mathbb{C}\mathscr{I}^+$, through Y, approaching the generator g.



The secret is to define the scalings for the fibre over G as being given simply by the scalings for the $\pi_{A'}$-spinor <u>at i^+</u>, without worrying about the propagation law Ⓐ.

The reason that the bundle over the "finite" part of $\mathbb{P}\mathcal{T}^+$ actually extends in a unique way to a bundle over the "infinite" part (provided that the blown-up version of $\mathcal{T}^+$ is used) can be seen in a theorem of Andreotti and Hill. Restricting **attention** to the β-plane on $\mathbb{C}\mathscr{I}^+$ which contains Y, we find that this corresponds to a plane $\mathbb{P}B$ in $\mathbb{P}\mathcal{T}^+$. The part B of $\mathcal{T}^+$ lying above $\mathbb{P}B$ arises as an extension to the "infinite" region of the "finite" region. Andreotti-Hill tells us that the extension of 1st cohomology (of $\mathcal{O}(-6)$ functions) can be made (uniquely), whence this applies also to the bundle B over $\mathbb{P}B$ (this being an Abelian case).

All this shows that the construction of $\mathcal{T}^+$ as given by α-lines on $\mathbb{C}\mathscr{I}^+$, scaled according to Ⓐ, does indeed encapsulate the SD as well as ASD part of the (complexified) radiation field, so the geometry of $\mathcal{M}$ is encoded in $\mathcal{T}^+$ as required. I shall conclude this article by making a few remarks about **(a)** the local structure of $\mathcal{T}^+$, **(b)** the interpretation of points of $\mathcal{M}$ in terms of $\mathcal{T}^+$ and **(c)** a reason why we may expect that the vacuum equations for $\mathcal{M}$ are automatic in this construction.

(a) The basic structure of $\mathcal{T}^+$ was given in TN43. The twistor space $\mathcal{T}^+$ is foliated by 3-surfaces which, in turn, are foliated by curves (the "Euler curves"), these being determined locally by a 1-form ι and a 3-form θ (both holomorphic) given only up to proportionality and subject to

$$\iota \wedge d\iota = 0, \quad \iota \wedge \theta = 0.$$

Factoring $\mathcal{T}^+$ by the integral curves of θ gives $\mathbb{P}\mathcal{T}^+$; factoring $\mathcal{T}^+$ by the integral 3-surfaces of ι gives the space of β-planes on $\mathbb{C}\mathcal{I}^+$ (a $\mathbb{CP}^1$). There is also a scaling for $\mathcal{T}^+$ which can be provided by the structure

$$\Pi = d\theta \otimes \iota \qquad \text{and} \qquad \Sigma = d\theta \otimes d\theta \otimes \theta.$$

What this means is that if $\mathcal{T}^+$ is pieced together from a number of patches, we must have, on each overlap (with ι, θ on one patch $\mathcal{U}$ and ι', θ' on the other patch $\mathcal{U}'$)

$$\iota' = k\,\iota, \quad \theta' = k^2\theta, \quad d\theta' = k^{-1}d\theta$$

(with k a holomorphic scalar on $\mathcal{U} \cap \mathcal{U}'$).

There is also to be a condition between θ and ι which must hold on each patch

$$d\theta \otimes \iota = -2\theta \odot d\iota \qquad \text{(E)}$$

where the bilinear operation $\odot$, between an n-form and a 2-form is defined from

$$\eta \odot (dp \wedge dq) = \eta \wedge dp \otimes dq - \eta \wedge dq \otimes dp.$$

The equation (E) ensures the various homogeneity relations (the second being automatic)

$$\pounds_{\Upsilon}\iota = 2\iota \qquad \text{and} \qquad \pounds_{\Upsilon}\theta = 4\theta$$

where the locally defined Euler operator Υ is given by $\Upsilon = 4\theta \div d\theta$ (which means $da \wedge \theta = \frac{1}{4}\Upsilon(a)\, d\theta$, for any scalar a).

We find, on $\mathcal{U} \cap \mathcal{U}'$,

$$\Upsilon' = k^3 \Upsilon$$

where

$$\Upsilon(k) = 2k^{-1} - 2k \text{ ; equivalently } \Upsilon'(k^{-1}) = 2k^2 - 2k^{-1}.$$

If z is an "ordinary" scaling parameter along an Euler curve, so that

$$\Upsilon(z) = z,$$

then

$$k^3 = 1 - F z^{-6}$$

with F constant along Euler curves, i.e.

$$k^3 = 1 - f_{-6}(Z^\alpha)$$

with f a twistor function defined in $\mathcal{U} \cap \mathcal{U}$, of degree -6 with respect to $\mathcal{U}$.

(b) Let $X \in \mathcal{M}$ [or $\mathbb{C}\mathcal{M}$], where $\mathcal{C}_X$ [or $\mathbb{C}\mathcal{C}_X$] meets $\mathscr{I}^+$ in a reasonable cross-section (what Newman calls a "light-cone cut"). There is an open set's worth of α-lines on $\mathbb{C}\mathcal{C}_X$ which meet $\mathbb{C}\mathscr{I}^+$. At the intersection point of such an α-line z_X with $\mathbb{C}\mathscr{I}^+$, we can continue with an α-line z^+ on $\mathbb{C}\mathscr{I}^+$ in a unique way, where at the intersecton of z_X with z^+ the $\pi_{A'}$-spinors are the same for each (so $o^A\pi^{A'}$ is tangent to z_X there, and $\iota^A\pi^{A'}$ is tangent to z^+). This establishes "large" open regions of $\mathcal{T}_X$ and $\mathcal{T}^+$ identified with one another. Now, the vertex X of $\mathcal{C}_X$ provides a kind of "blow-up" of $\mathbb{P}\mathcal{T}^+$ which is somewhat similar to the canonical blow-up that is provided by the vertex i^+ of $\mathscr{I}^+$. (Actually, these vertices also define "blow-downs", but the two are not compatible, with one another.) Accordingly, the "gluing" of $\mathcal{T}_X$ on to $\mathcal{T}^+$ gives a kind of "surgery" applied to $\mathcal{T}^+$. This notion of <u>surgery</u> replaces the "holomorphic curve" definition of a "space-time point" that characterizes the original "leg-break" non-linear graviton construction.

The precise characterization of these surgeries, so that the required 4-parameter family of space-time points arises, thereby providing a construction of the space-time $\mathcal{M}$, is presently under consideration.

(c) Analogous to the (locally defined) 1-form ι, which has a characteristic relation with the point i^+ and the associated canonical blow-up of $\mathbb{P}\mathcal{T}^+$, is a (locally defined) 1-form ξ having a characteristic relation with the point X and the associated surgery provided by $\mathbb{P}\mathcal{T}_X$. In each patch we have

$$\xi \wedge d\xi = 0, \quad \xi \wedge \theta = 0, \quad d\theta \otimes \xi = -2\theta \otimes d\xi,$$

the scaling $\xi' = k\xi$ accompanying $\iota' = k\iota$ on $\mathcal{U} \cap \mathcal{U}'$.

If we move from a point X to a neighbouring point $X + \mathring{X}$, we express (so it seems) the metric separation $g = g_{ab}\mathring{X}^a\mathring{X}^b$ by

$$d\xi \wedge d\mathring{\xi} = -\tfrac{1}{2} g \, d\theta.$$

There is some reason to expect that this metric is necessarily vacuum because the quantity

$$d d_Z \xi$$

defined on the product space $\mathcal{T}^+ \times \mathcal{M}$ seems to be the Sparling 3-form. (Here d_Z stands for what d stood for before, namely "d" on $\mathcal{T}^+ \times \mathcal{M}$ with X constant.) The vanishing of the exterior derivative of the Sparling 3-form is equivalent to the vacuum equations, but here $d(dd_Z\xi) = 0$ would be automatic. Much more work is required in order to see if this really works.

Thanks to many, especially ETN and LJM.

— Roger

Extending half flat metrics

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Recently Plebański, Przanowski and Formański, [1], re-addressed the problem of constructing real solutions of the Einstein vacuum field equations from complex half flat metrics. In this note an investigation of a related question is outlined. When can an anti self-dual solution of Einstein's vacuum equations be extended to a solution with a connection whose anti-self dual part coincides with that of the half flat solution? The set of anti self-dual solutions which can be extended in this way is non-empty. It includes complex metrics which can be extended to real solutions (for examples see Ref [2]).

In the investigation of this question it is convenient to use a version of the Cartan structure equations in which the coframe of one-forms, θ^a, is an appropriately ordered null basis. The metric is given by $ds^2 = g_{ab}\theta^a\theta^b$, where

$$g_{ab} = \begin{bmatrix} 0 & \epsilon_{AB} \\ -\epsilon_{AB} & 0 \end{bmatrix} \tag{1}$$

(for real solutions θ^0 and θ^3 are real and $\theta^1 = \overline{\theta^2}$).
The first Cartan structure equations can be written

$$D\theta^a := d\theta^a - \theta^b \wedge {}^- \Gamma^a_b = \theta^b \wedge {}^+ \Gamma^a_b \tag{2}$$

where D is the covariant exterior derivative determined by the anti self-dual part of the connection ${}^-\Gamma^a_b$ and ${}^+\Gamma^a_b$ is the self dual part of the connection one-form. In this (chiral) basis

$${}^-\Gamma^a_b = \begin{bmatrix} \omega^A_B & 0 \\ o & \omega^A_B \end{bmatrix}, \ \omega_{AB} = \omega_{BA}, \text{ and }, \ {}^+\Gamma^a_b = \begin{bmatrix} 1\omega^{0'}_{0'} & 1\omega^{0'}_{1'} \\ 1\omega^{1'}_{0'} & -1\omega^{0'}_{0'} \end{bmatrix}. \tag{3}$$

Einstein's vacuum field equations may be written

$$D^2\theta^a = \theta^b \wedge R^a_b = 0 \tag{4}$$

where the anti-self dual part of the curvature two-form is given by

$$R^a_b = \begin{bmatrix} \Omega^A_B & 0 \\ 0 & \Omega^A_B \end{bmatrix}, and, \Omega^A_B = d\omega^A_B + \omega^A_C \wedge \omega^C_B. \tag{5}$$

Suppose now that θ^a is a co-frame for a half flat but not flat, anti self-dual metric, g, in a gauge in which ${}^+\Gamma^a_b = 0$ and write the curvature in terms of its components as $R^a_b = \frac{1}{2}R^a_{bcd}\theta^c \wedge \theta^d$. It follows from Eq. (2) that $D\theta^a = 0$.

Now let $\widehat{\theta^a}$ be a co-frame for a vacuum solution $\widehat{g}$ with connection one-forms (with non-zero curvature) $\widehat{{}^+\Gamma^a_b}$ and $\widehat{{}^-\Gamma^a_b} = {}^-\Gamma^a_b$. When $\widehat{\theta^a}$ is written in terms of its components with respect to θ^a as

$$\widehat{\theta^a} = \vartheta^a_b\theta^b, \tag{6}$$

Eq.(2) gives

$$D\vartheta^a_b \wedge \theta^b = \theta^b \wedge \widehat{{}^+\Gamma^a_b} \tag{7}$$

and the Einstein vacuum equations then hold for $\widehat{g}$ if and only if

$$\vartheta^f_b R^a_{fcd}\varepsilon^{bcde} = 0. \tag{8}$$

Because the curvature is anti self-dual it is a straightforward matter to see that Eq (8) can hold only for anti-self dual Weyl curvatures of type N, III and certain type I's. These are necessary conditions for the half flat metric g to be extendable to $\widehat{g}$. By solving Eq. (8), which restricts but does not determine the possible components ϑ^a_b, and substituting in Eq. (7), the sufficient conditions for the extendability of g can be computed. As was mentioned above the metrics found in Ref. [2], all of which have type N or III anti-self dual Weyl curvatures, are examples which satisfy the resulting differential equations. The full solution space is being investigated.

Thanks to Pawel Nurowski for helpful discussions and for bringing the work of Ref. [1] to my attention.

1. J.F.Plebański, M. Przanowski and S. Formański, *Linear superposition of two type-N nonlinear gravitons*, preprint 1998

2. D.C.Robinson, *Some Real and Complex Solutions of Einstein's Equations*, Gen. Rel. & Grav. **19**, 7, 693 1987

The abstract twistor space of the Schwarzschild space-time

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Solutions are obtained to the abstract twistor structure equations of the Schwarzschild space-time. An abstract twistor ideal is built and its equations solved for a three-parameter set of abstract twistor surfaces. The limit of vanishing mass and the relation with hypersurface twistors are discussed.

1 Introduction

It has been shown recently that, in vacuum spacetimes, the Penrose obstruction to the existence of twistor surfaces can be removed by suitably enlarging the standard Grassmann algebra of differential forms. The key idea is to develop an abstract differential ideal from the tetrad factorization equation[1] $\theta^{AB'} = \alpha^A q^{B'}$. Here α^A is a spinor valued one-form.

In this work we show how to implement these ideas in the context of a physical spacetime and we explicitly solve for the two-dimensional integral manifolds. We obtain a three-parameter set of twistor surfaces. Our result provides the first example of a twistor construction in a nontrivial spacetime.

2 Schwarzschild space-time

We begin with the Schwarzschild metric in Eddington coordinates:

$$g = 2dudr + \left(1 - \frac{2M}{r}\right)du^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2, \tag{2.1}$$

where the null co-ordinate u is defined in terms of the Schwarzschild time t by $u \equiv t - r - 2M \ln(r - 2M)$. We choose a standard null tetrad, $\theta^{CD'}$:

$$\begin{aligned} \theta^{00'} &= l = du, & \theta^{10'} &= -m = \frac{r}{\sqrt{2}}(d\theta + i\sin\theta d\phi), \\ \theta^{11'} &= n = dr + \left(\frac{r-2M}{2r}\right)du, & \theta^{01'} &= -m' = \frac{r}{\sqrt{2}}(d\theta - i\sin\theta d\phi), \end{aligned} \tag{2.2}$$

such that $g = 2(ln - mm')$. In the real space-time m' is the complex conjugate of m. For twistor surfaces to exist, however, one has to complexify the space-time. Thus the coordinates u, r, θ and ϕ are considered to be complex. The exterior derivatives of the tetrad forms are:

$$\begin{aligned} dl &= 0\,, \quad dn = -\frac{M}{r^2} l_\wedge n, \\ dm &= -\frac{1}{r} m_\wedge n - \frac{1}{2r}\left(1 - \frac{2M}{r}\right) l_\wedge m + \frac{\cot\theta}{r\sqrt{2}} m_\wedge m', \\ dm' &= -\frac{1}{r} m'_\wedge n - \frac{1}{2r}\left(1 - \frac{2M}{r}\right) l_\wedge m' - \frac{\cot\theta}{r\sqrt{2}} m_\wedge m'. \end{aligned} \tag{2.3}$$

For a spin basis ζ_A adapted to this tetrad, the $SL(2,\mathbb{C})$ connection one-forms defining the covariant derivatives $D\zeta_A = \Gamma_A^B \zeta_B$ are as follows [2]:

$$\begin{aligned} -\Gamma_0{}^1 = \Gamma_{00} &= \frac{1}{r} m, \\ \Gamma_0{}^0 = \Gamma_{01} &= \frac{M}{2r^2} l + \frac{1}{2\sqrt{2}r} \cot\theta (m - m'), \\ \Gamma_1{}^0 = \Gamma_{11} &= \frac{1}{2r}(1 - \frac{2M}{r}) m'. \end{aligned} \tag{2.4}$$

Then for example the covariant exterior derivative $D(s,t)$ of an unprimed spinor with components (s,t) has components:

$$D(s,t) \equiv (ds + \Gamma_{01}s + \Gamma_{11}t,\ dt - \Gamma_{00}s - \Gamma_{01}t). \tag{2.5}$$

The curvature forms, such that

$$D^2(s,t) = (R_{01}s + R_{11}t,\ -R_{00}s - R_{01}t) \tag{2.6}$$

are given by the formulas:

$$\begin{aligned} R_{00} &= d\Gamma_{00} - 2\Gamma_{01\wedge}\Gamma_{00} = -\frac{M}{r^3} l_\wedge m \\ R_{01} &= d\Gamma_{01} + \Gamma_{01\wedge}\Gamma_{11} = \frac{M}{r^3}(l_\wedge n - m_\wedge m') \\ R_{11} &= d\Gamma_{11} + 2\Gamma_{11\wedge}\Gamma_{01} = \frac{M}{r^3} n_\wedge m' \ . \end{aligned} \tag{2.7}$$

The primed spin connection and curvature follows from these relations by complex conjugation.

3 The abstract twistor ideal

The abstract twistor structure associated with this space-time is by definition the differential ideal generated by the twistor factorization equations:

$$l = q\beta, \quad m = -p\beta, \quad m' = -q\alpha, \quad n = p\alpha. \tag{3.1}$$

Here p and q are the 0-form components of $q^{A'}$, while α and β are the 1-form components of α^A. In vacuo it is consistent to write the forms α and β as covariant exterior derivatives:

$$(\alpha, \beta) = D(s,t) \ . \tag{3.2}$$

Then the variables s and t serve as parameters for the twistor surface, which is necessarily completely null, since evidently the metric vanishes when equations (3.1) hold. Also when equations (3.1) hold, the connection forms simplify:

$$\begin{aligned} \Gamma_{00} &= -\frac{p}{r}\beta, \\ \Gamma_{01} &= \frac{M}{2r^2}q\beta + \frac{\cot\theta}{2\sqrt{2}r}(q\alpha - p\beta), \\ \Gamma_{11} &= -\frac{q}{2r}\Big(1 - \frac{2M}{r}\Big)\alpha. \end{aligned} \tag{3.3}$$

The unprimed curvature forms of Eqs. (2.7) are now zero. In particular, by Eq. (3.2), we have $D(\alpha,\beta) = 0$, whence $d\alpha$ and $d\beta$ are easily calculated. Define the two-form $z \equiv \alpha_\wedge\beta$. Using equation (3.1), we list the exterior derivatives of the tetrad and of the forms α, β and z:

$$\begin{aligned} dl &= 0, \qquad dm = -\frac{p}{r}\Big(p + \frac{q}{\sqrt{2}}\cot\theta\Big)z, \\ dn &= \frac{M}{r^2}pqz, \quad dm' = -\frac{q}{2r}\Big[q\Big(1-\frac{2M}{r}\Big) - p\sqrt{2}\cot\theta\Big]z, \qquad (3.4) \\ d\alpha &= \frac{1}{2r}\Big[q\Big(1-\frac{M}{r}\Big) - \frac{p}{\sqrt{2}}\cot\theta\Big]z, \\ d\beta &= \frac{1}{2r}\Big[2p + \frac{q}{\sqrt{2}}\cot\theta\Big]z, \\ dz &= 0. \qquad (3.5) \end{aligned}$$

We next take the exterior derivatives of Eqs (3.1), and get the following equations involving dp and dq:

$$dp_\wedge\alpha = \frac{p}{2r}\Big[q\Big(1-3\frac{M}{r}\Big) - \frac{p}{\sqrt{2}}\cot\theta\Big]\beta_\wedge\alpha \qquad (3.6)$$

$$dp_\wedge\beta = \frac{pq}{2\sqrt{2}r}\cot\theta\ \alpha_\wedge\beta \qquad (3.7)$$

$$dq_\wedge\alpha = \frac{q}{2r}\Big(\frac{p}{\sqrt{2}}\cot\theta + \frac{qM}{r}\Big)\beta_\wedge\alpha \qquad (3.8)$$

$$dq_\wedge\beta = -\frac{q}{2r}\Big(2p + \frac{q}{\sqrt{2}}\cot\theta\Big)\alpha_\wedge\beta\ . \qquad (3.9)$$

Solving these equations, we get

$$\begin{aligned} dp &= \frac{pq}{2\sqrt{2}r}\cot\theta\ \alpha + \frac{p}{2r}\Big[q\Big(1-\frac{3M}{r}\Big) - \frac{p}{\sqrt{2}}\cot\theta\Big]\beta + \frac{\lambda}{\sqrt{2}}z \\ dq &= -\frac{q}{2r}\Big(2p + \frac{q}{\sqrt{2}}\cot\theta\Big)\alpha + \frac{q}{2r}\Big(q\frac{M}{r} + \frac{p}{\sqrt{2}}\cot\theta\Big)\beta + \frac{\mu}{\sqrt{2}}z\ . \qquad (3.10) \end{aligned}$$

Here the quantities λ and μ are -1-forms needed here to establish consistency of the later equations.

Assuming the existence of the twistor surfaces, we simplify by noting that on the twistor surface the function on the spin bundle defined by the Killing spinor of the Schwarzschild solution is constant. In the present language, this is the condition:

$$rpq = C, \quad dC = 0. \qquad (3.11)$$

Taking now the exterior derivative of Eq. (3.10), we obtain

$$d\lambda_\wedge z + 3\sqrt{2}CM\frac{p}{r^4}z = 0\ . \tag{3.12}$$

Defining

$$\omega \equiv -\frac{r^4\lambda}{3\sqrt{2}CpM} \tag{3.13}$$

Eq. (3.12) reads $(d\omega - 1)_\wedge z = 0$. This allows us to truncate the system by imposing the equation

$$d\omega = 1\ . \tag{3.14}$$

Thereby our ideal is complete.

4 Solving the ideal

By (2.2) and (3.1), we may express

$$d\theta = -\frac{m+m'}{\sqrt{2}r} = \frac{1}{\sqrt{2}r}(q\alpha + p\beta) = \frac{1}{\sqrt{2}r}\Big(\frac{p}{q}l + \frac{q}{p}n\Big)\ . \tag{4.1}$$

Using also (3.11) and (2.2) again,

$$d\theta = \frac{p^2}{\sqrt{2}C}du + \frac{C}{\sqrt{2}r^2p^2}\Big[dr + \frac{1}{2}\Big(1 - \frac{2M}{r}\Big)du\Big]\ . \tag{4.2}$$

In a similar manner, we obtain

$$\begin{aligned} dp &= \frac{C}{2\sqrt{2}r^2p}\cot\theta\Big[dr + \frac{1}{2}\Big(1 - \frac{2M}{r}\Big)du\Big] \\ &+\frac{1}{2}\Big[\frac{p}{r}\Big(1 - 3\frac{M}{r}\Big) - \frac{p^3}{\sqrt{2}C}\cot\theta\Big]du - 3\frac{pM}{r^3}\omega_\wedge dr_\wedge du\ . \end{aligned} \tag{4.3}$$

Multiplying by $2p$ and introducing the new variable Q via the substitution

$$p^2 = \exp\Big\{Q - \frac{3M\omega_\wedge du}{r^2}\Big\} = e^Q\Big(1 - \frac{3M\omega_\wedge du}{r^2}\Big)\ , \tag{4.4}$$

we obtain the equations

$$d\theta = \frac{e^Q}{\sqrt{2}C}du + \frac{Ce^{-Q}}{\sqrt{2}r^2}\Big[dr - \frac{3M}{r^2}\omega_\wedge dr_\wedge du + \Big(\frac{1}{2} - \frac{M}{r}\Big)du\Big] \tag{4.5}$$

$$dQ = \frac{Ce^{-Q}\cot\theta}{\sqrt{2}r^2}\Big[dr - \frac{3M}{r^2}\omega_\wedge dr_\wedge du + \Big(\frac{1}{2} - \frac{M}{r}\Big)du\Big] + \Big(\frac{1}{r} - \frac{e^Q\cot\theta}{\sqrt{2}C}\Big)du\ . \tag{4.6}$$

Making the substitutions

$$\begin{aligned} R &= Q - \log(\sin\theta) \\ \frac{y}{\sqrt{2}} &= \cos\theta - \frac{C}{\sqrt{2}}e^{-Q}\sin\theta\frac{M}{r^3}\omega_\wedge du\ , \end{aligned} \tag{4.7}$$

we have

$$\begin{aligned} dR &= \Big(\frac{1}{r} - \frac{ye^R}{C}\Big)du \\ dy &= -\frac{e^R(2-y^2)}{2C}du - \frac{C}{r^2}e^{-R}\Big(dr + \frac{1}{2}du\Big)\ . \end{aligned} \tag{4.8}$$

This is a standard differential system. We next introduce the variable

$$x = ye^R - \frac{C}{r} \tag{4.9}$$

yielding

$$CdR = -xdu \tag{4.10}$$

$$Cdx = -\Big(\frac{1}{2}x^2 + e^{2R}\Big)du\ . \tag{4.11}$$

Put

$$w = x^2 - e^{2R}\ . \tag{4.12}$$

Then we immediately get

$$dw = (w + e^{2R})dR \tag{4.13}$$

with the general solution

$$w = e^{2R} + Be^R \tag{4.14}$$

where B is the integration constant. Then

$$x^2 = \frac{2+BS}{S^2} \tag{4.15}$$

where

$$S = e^{-R} . \tag{4.16}$$

From (4.10) we have

$$du = \frac{CdS}{(2+BS)^{1/2}} \tag{4.17}$$

which has the general solution

$$u = \frac{2C}{B}(2+BS)^{1/2} + A \tag{4.18}$$

and A is an integration constant. After backsubstitution,

$$p^2 = \frac{\sin\theta}{S}\Big[1 - \frac{3MC\omega_\wedge dS}{(2+BS)^{1/2}r^2}\Big] \tag{4.19}$$

$$\cos\theta = \frac{CS}{\sqrt{2}r} + \Big(1 + \frac{BS}{2}\Big)^{1/2} + \frac{C^2SM\omega_\wedge dS}{\sqrt{2}r^3(2+BS)^{1/2}} . \tag{4.20}$$

Finally we determine the coordinate ϕ using

$$d\phi = i\frac{m-m'}{\sqrt{2}r\sin\theta} = \frac{i}{C\sqrt{2}p^2r^2\sin\theta}\Big[\frac{C^2}{2}\Big(1-\frac{2M}{r}\Big)du + C^2dr - r^2\ p^4du\Big] . \tag{4.21}$$

Introduce the stereographic coordinate $\zeta = e^{-i\phi}\cot(\theta/2)$, substitute p and u, this yielding the remarkably simple equation

$$\frac{d\zeta}{\zeta} = -\frac{dS}{S(1+BS/2)^{1/2}} \tag{4.22}$$

with solution

$$S = \frac{8F\zeta}{B(\zeta - F)^2} \tag{4.23}$$

where F is a constant. Substituting S in Eqs. (4.19)-(4.20), we get *the abstract twistor equations* of the Schwarzschild space-time:

$$p^2 = \frac{\sin\theta}{8F\zeta r^2}\left[Br^2(\zeta - F)^2 + 12\sqrt{2}MCF\omega_\wedge d\zeta\right] \quad (4.24)$$

$$\cos\theta = \frac{8CF\zeta}{\sqrt{2}B(\zeta - F)^2 r} + \frac{\zeta + F}{\zeta - F} - 32\frac{C^2F^2M\zeta\omega_\wedge d\zeta}{B^2r^3(\zeta - F)^4} \quad (4.25)$$

$$u = 2\sqrt{2}\frac{C(\zeta + F)}{B(\zeta - F)} + A\,. \quad (4.26)$$

5 Discussion

For $M \to 0$, we obtain from here flat-space twistor calculus. Observe that we have three essential parameters in our solution: C/B, F and A. These remain independent in the limit of vanishing mass, and in that limit we find an open subset of conventional twistor space. Two independent variables are ζ and r. Some special hypersurface twistors are located, *e.g.*, on the flat $u = const.$ and $t = const.$ hypersurfaces.

6 Acknowledgments

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The Geometry of Twistor Diagrams

This note is intended not to set forth any of the mass of detailed results about twistor diagrams for first and second-order scattering processes, but to air the question of how the theory should connect with other aspects of the twistor programme.

1. The refinement of the diagram calculus over the years has established that the fundamental element of the calculus is complex contour integration with boundaries on subspaces of form W. Z = k,

written 

We also need derivatives of this boundaries, namely simple, double, triple and quadruple poles which have the effect of restricting the integration to such subspaces: written

We also need conformal-symmetry-breaking boundaries of form

At a secondary level, we bring massive fields into the picture with poles of form

and the evaluation of the 'constant' field, the elementary state based at infinity, which corresponds to the Higgs field in the standard model:

2. There is a natural value of |k|, namely exp(— γ), where γ is Euler's constant, but the *sign* of k is not determined. It is obvious that there is also a sign ambiguity in specifying the conformal-symmetry-breaking elements. These questions of sign show up non-trivially in the evaluation of larger diagrams.
The simplest example comes from considering a chain of diagram elements:

 $= \log\left(\frac{W.Z}{k}\right)$

but

 $= \log\left(\frac{W.Z}{-k}\right)$

In the chains required for building up the massive progagator one also finds that one chain gives a term of form

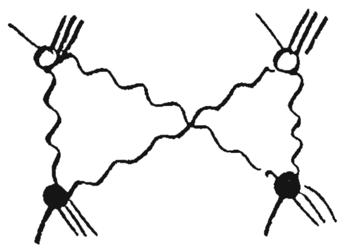
$$\frac{1}{2}\log^2\left(\frac{\begin{matrix}WY\\ \vdash\!\dashv\\ XZ\end{matrix}\; m^2}{\underset{\smile}{WY}\;\; \overset{\frown}{XZ}}\right)$$

but a longer one gives

$$\frac{1}{2}\log^2\left(\frac{\begin{matrix}WY\\ \vdash\!\dashv\\ XZ\end{matrix}\; m^2}{\underset{\smile}{WY}\;\; \overset{\frown}{XZ}}\right) - \pi^2/3$$

These differences are crucial to the problem that remains of getting exact agreement with the Feynman propagator, and seem to be related to the question of time-direction.

3. New progress has been made on the evaluation of such larger diagrams. Russ Ennis has seen how to describe in a more systematic way the building up of many-dimensional contours with boundary, and in particular has shown how to integrate the fundamental diagram



This in turn supplies a more elegant description of the integration of all 'single-box' diagrams. At the present time the parallel treatment of the 'double-box' diagram is not quite complete. This is of particular interest as it complicated enough for the sign of k to play a non-trivial role. R.E. has also seen applications of his new description to the long chains involved in the description of massive fields in which, as indicated above, signs are crucial.

4. There are now many directions in which to pursue the further extension of the diagram calculus. For instance, it is very striking how second-order gauge-theoretic interactions take on a simple form as twistor diagrams, with the many gauge-non-invariant Feynman diagrams reduced to one gauge-invariant twistor diagram. This is an area where one can foresee the emergence of a formalism with real advantages over Feynman diagram summation.

However, it seems that essential *geometrical ingredients* have emerged as fundamental, and that we now should be looking for connections with the new twistor geometry developed by R.P. for the description of curved space-time. From the point of view of the twistor programme, of course, such a unification of description is in any case the main point of pursuing the topic of twistor diagrams. We can ask:

(a) What is the generating principle which lies behind all the examples of twistor diagrams as so far evaluated?

(b) Can the non-projective twistor spaces employed be connected with the non-projective elements involved in R.P's extended twistor geometry? Can the special role of I, the line at infinity, be connected likewise?

(c) Could the values of k and mass parameters be involved in completing R.P.'s constructions, perhaps in the sense of taking some limit analogous to the regularisations of conventional QFT?

(c) In the calculus of twistor diagrams, twistor space and dual twistor space are on an equal footing, and as it stands this is goes against the 'googly' philosophy in R.P.'s programme of putting everything into twistor space. Can the dual spaces in twistor diagrams be understood through (inhomogeneous) twistor transforms which translate 'googly' constructions? As the simplest example, can the inner-product integral

i.e. $$\int_{W\cdot Z>k} f_{-4}(Z^\alpha)\, g_{-4}(W_\alpha)\, d^4W \wedge d^4Z$$

be related geometrically to the googly description of spin-1 fields?

(d) If we could make progress with any of these questions, the outstanding questions of sign and time-direction should then make sense.

Andrew Hodges

Exceptional hyperKähler reductions

In our joint paper (Kobak and Swann 1993), Proposition 1.3, we pointed out that the orbifold $G_2/SO(4)/\mathbb{Z}_3$ can be viewed as the $U(1)$ quaternion-Kähler reduction of the Grassman space $\mathbb{G}r_4(\mathbb{R}^7)$:

$$G_2/SO(4)/\mathbb{Z}_3 = \mathbb{G}r_4(\mathbb{R}^7)/\!/\!/U(1) \qquad (*)$$

where the $U(1)$ action on $\mathbb{G}r_4(\mathbb{R}^7)$ comes from the following action on $\mathbb{R}^7 \equiv \mathbb{R}\oplus\mathbb{C}^3$: $U(1)$ acts trivially on $\mathbb{R}$ and acts as the multiplicative group of unit complex numbers on $\mathbb{C}^3$. In this way yet again the exceptional group G_2 occurs naturally in a completely classical setup. This construction was stumbled upon by chance when one of the authors analysed various $U(1)$ actions on spaces $\mathbb{R}^k \times \mathbb{C}^l$ and realised that in the $\mathbb{R} \oplus \mathbb{C}^3$ case the co-associative 4-planes in $\mathbb{R}^7$ happen to sit in the zero locus of the $U(1)$-moment map $\mu\colon \mathbb{G}r_4(\mathbb{R}^7) \to \mathcal{G}\otimes\mathfrak{u}(1)$ (here $\mathcal{G}$ denotes the orthogonal complement of the tautological 4-plane bundle over $\mathbb{G}r_4(\mathbb{R}^7)$). The set of co-associative 4-planes is, of course, the exceptional quaternion-Kähler symmetric space $G_2/SO(4)$. As the moment map μ is $U(1)$-invariant, we have

$$U(1)\cdot G_2/SO(4) \subset \mu^{-1}(0)$$

and spotting that it is in fact an equality does the trick: we get Formula $(*)$ since $U(1)$ acts on $U(1)\cdot G_2/SO(4)$ with stabiliser $\mathbb{Z}_3$.

Our aim here is to show how the reduction $(*)$ fits the general construction developed by Kobak and Swann (1995) for classical groups. We will use the following general facts:

1) Rather than quaternion-Kähler reductions of quaternion-Kähler symmetric spaces one can consider hyperKähler reductions of the corresponding nilpotent orbits:

$$\begin{array}{ccc} \mathcal{N} & \xrightarrow{H_G} & \mathcal{N}/\!/\!/G \\ \downarrow & & \downarrow \\ \mathbb{P}\mathcal{N} & & \mathbb{P}(\mathcal{N}/\!/\!/G) \\ \downarrow & & \downarrow \\ M & \xrightarrow{Q_G} & M/\!/\!/G \end{array} \qquad (**)$$

The arrow "$\xrightarrow{Q_G}$" denotes the functor of quaternion-Kähler reduction by the group G, and "$\xrightarrow{H_G}$" is the corresponding hyperKähler reduction (defined as in Swann 1991). We recall that the projective bundles in the middle row of (**) are Simon Salamon's "quaternionic" twistor spaces (Salamon 1982).

2) The hyperKähler reduction of a hyperKähler manifold M by a product $G \times K$ of two Lie groups can be taken in steps (for details see Kobak and Swann 1995, Lemma 3.2).

$$\begin{array}{ccc} & M & \\ {}^{H_G}\swarrow & & \searrow^{H_{G\times K}} \\ M /\!\!/\!\!/ G & \xrightarrow{H_K} & M /\!\!/\!\!/ (G \times K) \end{array}$$

In our situation the nilpotent orbits which correspond to $G_2/SO(4)$ and $\mathrm{Gr}_4(\mathbb{R}^7)$ are the respective minimal nonzero nilpotent orbits: $\mathcal{N}^{\min}_{G_2} \subset \mathfrak{g}_2^{\mathbb{C}}$ and $\mathcal{N}^{\min}_{SO(7)} \subset SO(7)$. The crucial ingredient is the fact that $\mathcal{N}^{\min}_{G_2}$ is a 3:1 branched covering of the nilpotent variety $\overline{\mathcal{N}^{\mathrm{reg}}_{SL(3)}} \subset \mathfrak{sl}(3,\mathbb{C})$. (Brylinski and Kostant (1992) have classified all such finite coverings.) It is thus enough to exhibit the regular nilpotent orbit $\mathcal{N}^{\mathrm{reg}}_{SL(3)}$ as a hyperKähler reduction of $\mathcal{N}^{\min}_{SO(7)}$ by $U(1)$. Now we are in the realm of classical groups and the machinery from (Kobak and Swann 1996) can be set in motion. The necessary data concerning the orbits in question is contained in the table below:

Nilpotent orbit	Jordan type	Diagram	HyperKähler reduction
$\mathcal{N}^{\mathrm{reg}}_{SL(3)}$	(3)	$0 \rightleftarrows \mathbb{C} \rightleftarrows \mathbb{C}^2 \rightleftarrows \mathbb{C}^3$	$\mathbb{H}^8 /\!\!/\!\!/ (U(1) \times U(2))$
$\mathcal{N}^{\min}_{SO(7)}$	$(2^2, 1^3)$	$0 \rightleftarrows \mathbb{C}^2 \rightleftarrows \mathbb{C}^7$	$\mathbb{H}^7 /\!\!/\!\!/ Sp(1)$

The construction now follows from the fact that one can factor $\mathbb{H}^8$ either by $U(1) \times U(2)$ to get $\mathcal{N}^{\mathrm{reg}}_{SL(3)}$, or, in steps, to get first $\mathbb{H}^7$ and then $\mathcal{N}^{\min}_{SO(7)} = \mathbb{H}^7 /\!\!/\!\!/ Sp(1)$. One needs, however, to take into account various quotients by finite groups that arise on the way. Firstly, the $U(1)$ reduction of $\mathbb{H}^8$ gives $\mathbb{H}/\mathbb{Z}_2 \times \mathbb{H}^6$ rather than $\mathbb{H}^7$. To see this note that the first two components of the diagram for $\mathcal{N}^{\mathrm{reg}}_{SL(3)}$ give rise to the diagram $0 \rightleftarrows \mathbb{C} \rightleftarrows \mathbb{C}^2$. This is the diagram for the nilpotent variety $\overline{\mathcal{N}^{\mathrm{reg}}_{SL(2)}}$ which is equal to $\mathbb{H}/\mathbb{Z}_2$ and arises in the model case of the hyperKähler–twistor–quaternion-Kähler setup:

$$\begin{array}{ccccc} \mathcal{N}^{\mathrm{reg}}_{SL(2)} & \longrightarrow & \mathbb{P}\mathcal{N}^{\mathrm{reg}}_{SL(2)} & \longrightarrow & \mathbb{HP}^1 \\ \| & & \| & & \| \\ \mathbb{H} \setminus \{0\}/\mathbb{Z}_2 & \longrightarrow & \mathbb{CP}^3 & \longrightarrow & S^4 \end{array}$$

In effect we have reductions

$$\mathbb{H}^8 \xrightarrow{H_{U(1)}} \mathbb{H}/\mathbb{Z}_2 \times \mathbb{H}^6 \xrightarrow{H_{Sp(1)}} \mathcal{N}^{\text{min}}_{SO(7)}/\mathbb{Z}_2 \xrightarrow{H_{U(1)}} \mathcal{N}^{\text{reg}}_{SL(3)}$$

and their composition is the reduction of $\mathbb{H}^8$ by $U(1)\times U(1)\times U(2)$. Note that this gives $\mathcal{N}^{\text{reg}}_{SL(3)}$ rather than $\mathcal{N}^{\text{reg}}_{SL(3)}/\mathbb{Z}_2$: we know that $\mathbb{H}^8 /\!/\!/ (U(1)\times U(2))$ is $\mathcal{N}^{\text{reg}}_{SL(3)}$ and $U(2) = (U(1) \times SU(2))/\mathbb{Z}_2$ so reducing $\mathbb{H}^8$ by a 'smaller' group cannot result in a 'smaller' space. To be more precise, one can check that the $\mathbb{Z}_2$ which appears in the above group isomorphism is contained in the $U(1)$ which we use to reduce $\mathcal{N}^{\text{min}}_{SO(7)}$ to $\mathcal{N}^{\text{reg}}_{SL(3)}$. We can put together these reductions to form a diagram:

$$\begin{array}{ccccc} \mathbb{H}^8 & \xrightarrow{H_{U(1)}} & \mathbb{H}/\mathbb{Z}_2 \times \mathbb{H}^6 & \xrightarrow{H_{Sp(1)}} & \mathcal{N}^{\text{min}}_{SO(7)}/\mathbb{Z}_2 \\ & \searrow & & & \downarrow H_{U(1)} \\ & & H_{U(1)\times U(2)} & & \mathcal{N}^{\text{reg}}_{SL(3)} \end{array}$$

By considering $\mathbb{Z}_2$ coverings of $\mathbb{H}/\mathbb{Z}_2 \times \mathbb{H}^6$ and $\mathcal{N}^{\text{min}}_{SO(7)}/\mathbb{Z}_2$, we get another smaller diagram which explains where the reduction $(*)$ comes from:

$$\begin{array}{ccc} \mathbb{H}^7 & \xrightarrow{H_{Sp(1)}} & \mathcal{N}^{\text{min}}_{SO(7)} \\ & \searrow H_{U(2)} & \downarrow H_{U(1)} \\ & & \mathcal{N}^{\text{reg}}_{SL(3)} \end{array}$$

The following diagrams tell the quaternion-Kähler version of the story:

$$\begin{array}{ccccc} \mathbb{HP}^7 & \xrightarrow{Q_{U(1)}} & \mathbb{HP}^6/\mathbb{Z}_2 & \xrightarrow{Q_{Sp(1)}} & \mathbb{G}\mathrm{r}_4(\mathbb{R}^7)/\mathbb{Z}_2 \\ & \searrow & & & \downarrow Q_{U(1)} \\ & & Q_{U(1)\times U(2)} & & G_2/SO(4)/\mathbb{Z}_3 \end{array}$$

$$\begin{array}{ccc} \mathbb{HP}^6 & \xrightarrow{Q_{Sp(1)}} & \mathbb{G}\mathrm{r}_4(\mathbb{R}^7) \\ & \searrow Q_{U(2)} & \downarrow Q_{U(1)} \\ & & G_2/SO(4)/\mathbb{Z}_3 \end{array}$$

Note that by reordering the groups one can also realise $G_2/SO(4)/\mathbb{Z}_3$ as a hyperKähler reduction of $\mathbb{G}\mathrm{r}_4(\mathbb{R}^8)$ by $U(1) \times U(1)$: we have

$$\mathbb{HP}^7 \xrightarrow{Q_{Sp(1)}} \mathbb{G}\mathrm{r}_4(\mathbb{R}^8) \xrightarrow{Q_{U(1)\times U(1)}} G_2/SO(4)/\mathbb{Z}_3.$$

By looking at Brylinski and Kostant's (1994) classification of shared nilpotent orbits one finds that there is one other exceptional quaternion-Kähler symmetric space which has finite quotients obtainable from a classical reduction, namely $F_4/Sp(3)Sp(1)$. This has quotients of degree 2 and 4 which are reductions of $\mathbb{HP}^{17}$ by $Sp(2)$ and $Sp(2) \times \mathbb{Z}_2$ (respectively).

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