On Extracting the Googly Information

For a quarter of a century, the most fundamental and problematic issue confronting twistor theory has been the googly problem, i.e. how do we represent the self-dual (SD) part of the gravitational field (and also gauge fields) in terms of twistor (as opposed to dual twistor) geometry. The original "non-linear graviton — i.e. leg break — construction and the Ward construction (found in, respectively, 1975 and 1976) showed how a very natural extension of flat-space twistor ideas could accommodate anti-self-dual (ASD) fields, but no way was seen how to treat the corresponding SD fields. Twistor theory has thus had 25 years of lop-sidedness; worse, the lack of a comprehensive googly construction has held up progress in almost all other areas of twistor physics. In particular, the main twistor approach to quantum field theory, namely twistor diagram theory, lacks a non-perturbative means of treating gauge interactions which could be supplied by a googly-modified Ward construction. Moreover, the twistor particle programme needed, among other things, a satisfactory way of getting at the concept of mass, so that it is not just an operator commuting with everything else. Mass, of course, is the source of gravity, so an appropriate twistor way of handling both SD and ASD aspects of gravity could supply a profound ingredient that is, at present, missing. Most importantly, there is no way that twistor theory can satisfy its original aims of supplanting the normal ideas of space-time by some appropriate non-local "quantum geometry" unless both ASD and SD parts of the gravitational field can be treated together. The hope that twistor theory might shed light on space-time singularities and, perhaps, on the quantum measurement problem, would have to be abandoned without such a development being forthcoming.

It should be remarked that the ambitwistor approach, for all its merits, cannot supply what is needed. A "non-linear graviton", as its name suggests, is to be regarded as a non-linear version of a wavefunction. An ordinary (i.e. linear) twistor wavefunction for a massless particle, of either positive or negative helicity, is a 1-function of a single twistor Z^α, not of a twistor together with a dual twistor. The canonical conjugate variable $W_\alpha = \bar{Z}_\alpha$ is taken over by the operator $-i\hbar \frac{\partial}{\partial Z^\alpha}$ rather than being considered as an independent variable. Moreover, to use a twistor description for the ASD part and a dual twistor description for the SD part makes no physical sense, since gravitons of mixed helicity (e.g. plane-polarized waves) should be as acceptable as gravitons of one or the other helicity.

In the 25 years of search for a "googly graviton", in which a 1-function of homogeneity -6 is to provide some kind of deformation of flat twistor space, the only way of doing this that has emerged was that originally described in RP, $\mathbb{T}$N 3 (1976) pp. 12–17, where the fibres in $\mathbb{T}$ which project to single points in $\mathbb{PT}$ are deformed in a curious way, where on overlaps

$$\underset{i}{Z}^\alpha = (1 + f_{(-6)ij})^{1/6} \underset{j}{Z}^\alpha,$$

giving an affine bundle of the 6^{th} power of the Euler fibres, considered as 1-dimensional vector spaces. (The "Euler fibres" are the inverse images $\mathbb{P}^{-1}x$ of points x, in the map from twistor space $\mathbb{T}$ to projective twistor space $\mathbb{PT}$. In general, I shall use the notation $\mathbb{P}^{-1}\mathcal{Q}$ to denote that part of the non-projective space that lies above a region $\mathcal{Q}$ of the projective space.) Despite the un-natural — even absurd — appearance of the above expression, it was later shown by MGE (in $\mathbb{T}$N 14) that it had some genuine relationship to a very natural-looking construction involving a 6-dimensional affine bundle over $\mathbb{PT}$.

A significant development occurred in early 1998, when it was found that twistor spaces whose Euler fibres are deformed in accordance with the above can be constructed explicitly. This construction has been described in R.P., $\mathbb{T}$N 44 (1998) pp. 1–9. Let us briefly recall this construction here. Take $\mathcal{M}$ be a either a real analytic Ricci-flat space-time with complexification $\mathbb{C}\mathcal{M}$ (a small "thickening" of $\mathcal{M}$) or else, itself, simply a complex Ricci-flat space-time. To begin, we see how to construct a curved twistor space $\mathcal{T}_x$ relative to an arbitrary point $x \in \mathcal{M}$. The projective relative twistor space $\mathbb{P}\mathcal{T}_x$ is essentially the projective hypersurface twistor space of the (if necessary, complexification $\mathbb{C}\mathcal{C}_x$ — but henceforth I shall drop the "$\mathbb{C}$" — of the) light cone $\mathcal{C}_x$ of x. Thus, each point of $\mathbb{P}\mathcal{T}_x$ corresponds to a complex curve $\mathcal{Z}$ in $\mathcal{C}_x$ called an α-curve (or twistor line) with tangent vector of the form $o^A\mu^{A'}$, with $\mu^{A'}$ propagated parallel — in the sense of proportional — to itself along $\mathcal{Z}$:

$$o^A\mu^{A'}\nabla_{AA'}\mu^{B'} \propto \mu^{B'}, \quad \text{i.e. } \mu^{A'}\mu^{B'}\nabla_{0A'}\mu_{B'}^{\ B} = 0,$$

where $o^A\tilde{o}^A$ are tangent vectors to the generators of $\mathcal{C}$ (and where lower indices 0 or $0'$ refer to contraction with o^A and $\tilde{o}^{A'}$, respectively). For the moment, let us assume $\mu_{0'} \neq 0$, so the α-curve is excluded from being a generator of $\mathcal{C}_x$. As LJM has shown (D.Phil. thesis, Oxford), if we were to take "$\mathcal{T}_x$" to be the ordinary hypersurface twistor space of $\mathcal{C}_x$ (taking scalings for $\mu_{B'}$ to be fixed by $\mu^{A'}\nabla_{0A'}\mu_{B'} = 0$) then this "twistor space" would encode only the ASD information of the (complex) gravitational field. To encode the SD gravitational information as well, we adopt the more complicated scaling, according to the twistor propagation equation:

$$\mu^{B'}\nabla_{0B'}\mu_{A'} = K\mu_{A'}(\mu_{0'})^{-5}\,\text{Þ}_{\mathcal{C}}\,\tilde{\Psi}_{0'0'0'0'}\,,$$

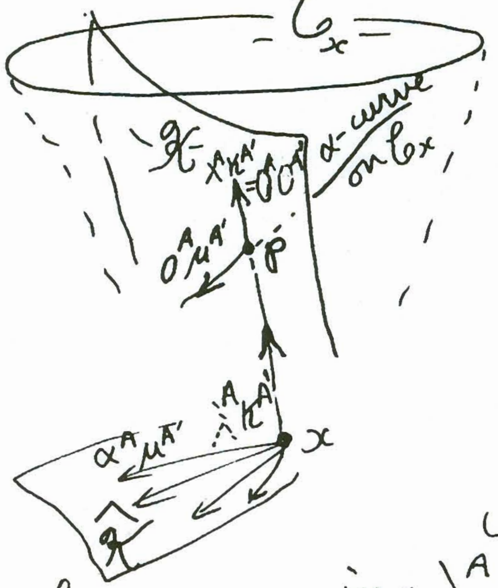
where $\text{Þ}_{\mathcal{C}}$ is the conformally invariant "thorn" of Spinors and Space-Time (RP & WR) vol. 1, p. 395 ($\text{Þ}_{\mathcal{C}} = \nabla_{00'} - 4\tilde{\varepsilon} - 5\tilde{\rho}$, in spin-coefficient notation), where $\tilde{\Psi}_{A'B'C'D'}$ is the "massless field" of helicity $+2$ that is equal to the SD Weyl spinor

$\tilde{\Psi}_{A'B'C'D'}$ when the physical metric is used, but which scales according to $\tilde{\Psi}_{A'B'C'D'} \rightsquigarrow \Omega^{-1}\tilde{\Psi}_{A'B'C'D'}$ under conformal rescaling $g_{ab} \rightsquigarrow \Omega^2 g_{ab}$ (with $\tilde{\Psi}_{A'B'C'D'} \rightsquigarrow \tilde{\Psi}_{A'B'C'D'}$, $\varepsilon_{AB} \rightsquigarrow \Omega\varepsilon_{AB}$, $\varepsilon_{A'B'} \rightsquigarrow \Omega\varepsilon_{AB}$).
The above scaling equation for $\mu_{A'}$ is then conformally invariant and independent of rescalings of o^A and $\bar{o}^{A'}$. In fact, the form of the twistor propagation equation is basically <u>fixed</u> by these invariance requirements. The quantity K is a specific numerical constant whose value should be fixed by the later considerations of this article.

We note that the different scalings for $\mu_{A'}$, for a given α-curve, constitute a 1-dimensional family. This family provides the fibre (Euler curve) $\mathbb{P}^{-1}q$ in $\mathcal{T}_x$ that lies above the point q in $\mathbb{P}\mathcal{T}_x$ that corresponds to this α-curve. If we take two points on the same α-curve, then the different scalings of $\mu_{A'}$ at one point are related to those at the other by a factor of the form $(1+f_{-6})^{\frac{1}{6}}$, as above, so the bundle structure of $\mathcal{T}_x$ over $\mathbb{P}\mathcal{T}_x$ indeed encodes the information of a 1-function of homogeneity degree -6 (in fact, an ordinary linear 1-function).

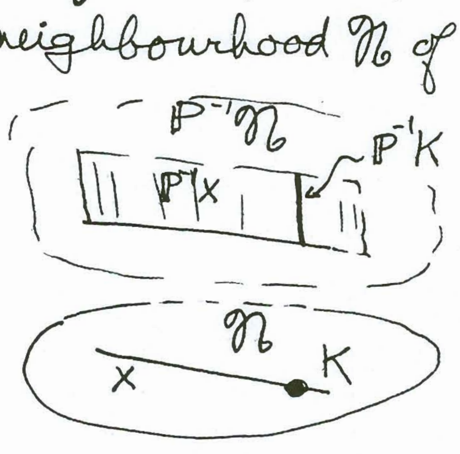
Up to this point, I have excluded the <u>generators</u> of $(\mathbb{C})\mathcal{C}_x$ as counting as "α-lines". If those were to be included in the definition of $\mathbb{P}\mathcal{T}_x$, then we would obtain a "blown-up" projective twistor space for which the point x itself would be represented as a <u>quadric</u> in $\mathbb{P}\mathcal{T}_x$, rather than as a projective line. There is an important shift in emphasis, however, in <u>this</u> article. The significance of the blown-up twistor <u>space</u> is (at least temporarily) being downgraded in importance. To complete the definition of $\mathbb{P}\mathcal{T}_x$, and subsequently of $\mathcal{T}_x$ itself, I propose, in effect, to "blow down" this quadric to obtain an actual projective line X in $\mathbb{P}\mathcal{T}_x$ to represent x. LJM has provided a procedure for doing this using an actual blow-down, but it is perhaps neater to use the following, which obtains $\mathbb{P}\mathcal{T}_x$ (and indeed $\mathcal{T}_x$) directly.

Consider a particular primed spinor $\kappa^{A'}(\neq 0)$ at x and the SD 2-plane element there consisting of tangent vectors of the form $\lambda^A\kappa^{A'}$, for varying λ^A. Construct the "α-like" 2-surface $\mathcal{X}$ swept out by the (null) geodesics through x having these tangent vectors at x, and along each such null geodesic, we parallel-propagate the (co-)tangent vector $\lambda_A\kappa_{A'}$. (If $\mathcal{M}$ were ASD then $\mathcal{X}$ would indeed be an α-surface in $\mathcal{M}$, but in general the SD condition will fail at most points of $\mathcal{X}$ owing to the presence of SD Weyl curvature.) It is important to note that $\mathcal{X}$ is completely smooth (holomorphic) in some open neighbourhood $\mathcal{W}$ of x, this being an instance of the regularity of the exponential map at x. Consider the bundle of non-zero primed spinors $\mu_{A'}$ at points of $\mathcal{W}\cap\mathcal{X}$ with $\mu_{A'}\kappa^{A'}\neq 0$. This, also, is a holomorphic manifold. Moreover, at points away from x, it is biholomorphic to an open neighbourhood of $\mathcal{T}_x$ because, locally at each point p of $\mathcal{W}\cap\mathcal{X}$, we can regard p as a point of an α-curve on $\mathcal{C}_x$ meeting $\mathcal{X}$ at p, where we carry $\mu_{A'}$ away from p according to the twistor propagation equation. What is particularly noteworthy is that we can now apply this at the point x itself; here (and only here)



do we obtain (locally) a non-uniqueness for the "α-curve" (actually a generator of $\mathcal{C}_x$) meeting $\mathcal{K}$. In fact, the entire α-like plane $\hat{\mathcal{K}}$ through x, with SD 2-plane element defined by $\mu^{A'}$ at x, corresponds to the single choice of $\mu_{A'}$ at x, with tangents of the form $\lambda^A\mu^{A'}$ at x for varying λ^A. Thus, the "blow-down" is automatically performed, giving us $\mathbb{P}\mathcal{T}_x$ directly as a smooth (holomorphic) 3-manifold locally. This also gives us $\mathcal{T}_x$ directly, but we must bear in mind that the scaling in the twistor propagation equation becomes <u>singular</u> (in the presence of SD Weyl curvature) owing to the fact that $\mu_{0'} = 0$. Here, the α-curves are simply generators of $\mathcal{C}_x$ and the relevant scaling for each corresponding twistor is determined by its value of $\mu_{A'}$ at x, without propagation.

This construction gives a description of that part $\mathbb{P}^{-1}\mathcal{N}$ of $\mathcal{T}_x$ lying above a neighbourhood $\mathcal{N}$ of some portion of the line $X \subset \mathbb{P}\mathcal{T}_x$ representing x, this portion merely having to exclude the point K of X which represents the α-like plane $\mathcal{K}$ itself. By choosing two different (non-proportional) values of $\kappa^{A'}$, we can cover the <u>whole</u> of a neighbourhood of X with two patches. Most points of $\mathcal{T}_x$ (in this neighbourhood of $\mathbb{P}^{-1}X$) have descriptions in each patch which are directly holomorphically



related to each other. The only issue needing further consideration is whether this holomorphicity extends to the points of $\mathbb{P}^{-1}X$. (That it extends to the points of X in $\mathbb{P}\mathcal{T}_x$ follows from Hartogs-type theorems.) It seems that it does, but further clarification on this point is needed.

To pass from $\mathcal{T}_x$ to a canonical twistor space $\mathcal{T}$, let us assume that M is strongly asymptotically flat in the sense that it has an (analytic) future null infinity $\mathscr{I}^+$ and a regular point i^+ at future timelike infinity — so that no material sources or black holes are present in the remote future. We then define $\mathcal{T} = \mathcal{T}_{i^+} (= \mathcal{T}^+)$, taking advantage of the conformal invariance properties of the construction. (It is usual to use spinors $\iota_A, \bar{\iota}_{A'}$ in place of $o_A, \bar{o}_{A'}$, when $\mathcal{C}_x = \mathscr{I}^+$, so this substitution is to be made in the twistor propagation equation.) It is to be expected that the more general situation when there are material sources or black holes in the remote future, there will be some appropriate way of incorporating this information into the structure of $\mathcal{T}$. This is an issue to be addressed by future considerations.

A comment is pertinent, at this stage, concerning the choice of $\mathscr{I}^+$ for the definition of $\mathcal{T}$, rather than $\mathscr{I}^-$. We know from the work of Friedrich that a broad ("generic") class of vacuum space-times exists satisfying the above condition of strong asymptotic flatness. But we do not yet know, for sure, whether there are any such vacuum spacetimes, apart from the trivial case of Minkowski space for which there is also an analytic $\mathscr{I}^-$. Indeed, there is a school of thought according to which regularity at $\mathscr{I}^-$ and at $\mathscr{I}^+$ would be essentially incompatible, in general. Be that as it may, a choice has to be made, in the present attempt at a twistor description of general (asymptotically flat) vacuum space-times, as to whether to base this on $\mathscr{I}^+$ or $\mathscr{I}^-$. Given this choice, $\mathscr{I}^+$ is definitely to be preferred. In realistic situations involving gravitationally radiating systems we would expect this radiation to be retarded. The role of $\mathscr{I}^-$ would be merely for specifying this condition of retardedness, and for this a $\mathscr{I}^-$ with a low order of differentiability would provide an amply adequate framework. On the other hand, $\mathscr{I}^+$ would contain all the interesting information about the nature of the radiation, mass loss, angular momentum, charges, NP constants, etc., these things requiring a much more refined differentiability structure for $\mathscr{I}^+$.

In relation to all this, we must bear in mind that one of the (distant?) aims of twistor theory is to provide an appropriate marriage between general relativity and quantum mechanics according to which the rules of quantum mechanics will have to be modified — in addition to anticipated modifications that space-time structure will have to undergo at (say) the Planck-length scale. On many occasions I have expressed the view that this marriage ought to provide a solution to the measurement problem of quantum mechanics (see, most particularly, R.P. in <u>Mathematical Physics 2000</u>, Eds. A. Fokas, A. Grigoryan, T. Kibble, B. Zegarlinski (2000, Imperial College Press) pp. 266-282 and R.P. (1996) <u>Gen. Rel. Grav.</u> **28** pp. 581-600) and that it must possess a fundamental <u>time asymmetry</u> (cf. R.P. in TN 22 (1986) pp. 1-3). The main reason to expect a time-asymmetry in the correct quantum/gravity marriage is the gross time-asymmetry in space-time singularity structure that we find in Nature: future-type singularities appear to be "generic" whereas in the past (the big bang) we find a singularity that is extraordinarily constrained (to a part in at least $10^{10^{123}}$), whence the 2^{nd} law of thermodynamics and the enormously great uniformity of the early universe arise. An explanation of the singularity structure that we find in the universe ought presumably to be a consequence of the correct quantum/gravity.

This suggests that we ought to seek a twistor description of general relativity that is fundamentally time-asymmetric. Accordingly, if some form of time-asymmetry indeed appears to be forced on us, as here, we should not try to resist it, for it may actually be no bad thing. For 25 years, the twistor approach to general relativity has suffered from a manifest chiral asymmetry (whence the googly problem) whereas it is time-asymmetry that would appear to be the more evident physical requirement (and we recall that time-asymmetry is a feature of quantum state reduction as much as of singularity structure (cf. above).

The space $\mathcal{T}$ has a local structure that is given by a 1-form ι up to proportionality and a 3-form Θ up to proportionality subject to

$$\iota \wedge d\iota = 0\,, \quad \iota \wedge \Theta = 0$$

so that ι defines a foliation by 3-manifolds (except where $\iota = 0$, which occurs only on $\mathbb{P}^{-1}I$, where the line $I \subset \mathbb{P}\mathcal{T}$ represents i^+) and Θ defines a foliation of these 3-manifolds (and hence of $\mathcal{T}$ itself) by a family of curves — the Euler curves. There is a further structure, which restricts the allowed scalings for ι and Θ, that can be specified as the two quantities

$$\Pi = d\Theta \otimes \iota \quad \text{and} \quad \Sigma = d\Theta \otimes d\Theta \otimes \Theta$$

being given, where we also demand that

$$\Pi = -2\Theta \oslash d\iota.$$

(The bilinear operator $\oslash$, acting between an n-form α and a 2-form is defined by $\alpha \oslash (dp \wedge dq) = (\alpha \wedge dp) \otimes dq - (\alpha \wedge dq) \otimes dp$.)

The invariance of Π and Σ demands that on an overlap between two patches $\mathcal{U}$ and $\hat{\mathcal{U}}$ of $\mathcal{T}$, on which different scalings for ι and Θ are chosen, we have

$$\hat{\iota} = k\iota\,, \quad \hat{\Theta} = k^2\Theta\,, \quad d\hat{\Theta} = k^{-1}d\Theta.$$

The equality between the two expressions for Π tells us that ι (and $d\iota$) have homogeneity degree 2 with respect to the Euler operator $\Upsilon = \Theta \div (\frac{1}{4}d\Theta)$ (which means $\Upsilon(a)\,d\Theta = 4\,da\wedge\Theta$ for any scalar a): $\pounds_\Upsilon \iota = 2\iota$ (and $\pounds_\Upsilon d\iota = 2d\iota$). We also have (automatically) that Θ (and $d\Theta$) have homogeneity degree 4. We find that, on an overlap between patches $\mathcal{U}$ and $\hat{\mathcal{U}}$ that

$$\hat{\Upsilon} = k^3\Upsilon \quad \text{and} \quad k^3 = 1 + F z^{-6}$$

with respect to a "natural" parameter z along Euler curves satisfying $\Upsilon(z) = z$, where F is constant along Euler curves, i.e.

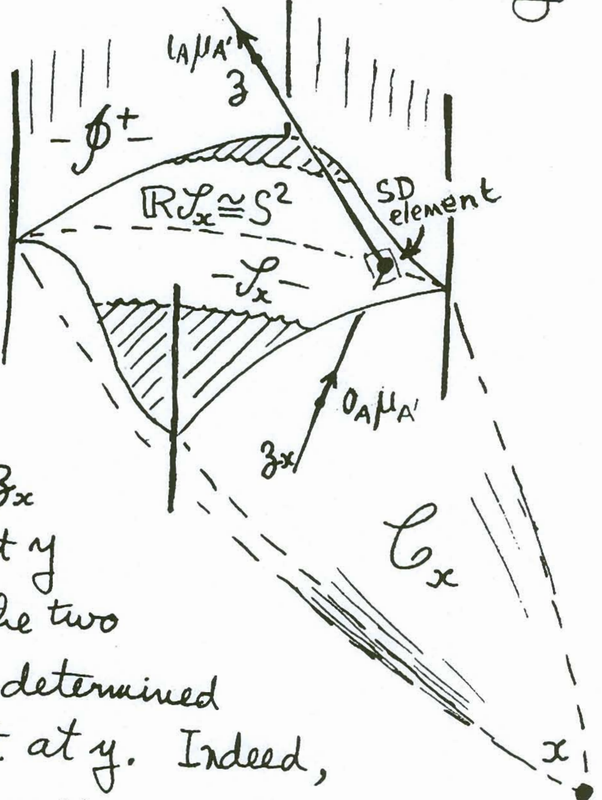
$$k^3 = 1 + f_{-6}$$

for some f_{-6} defined on $\mathcal{U}\cap\hat{\mathcal{U}}$ which is homogeneous of degree -6 with respect to Υ. This is basically the same type of behaviour as that referred to at the beginning of this article.

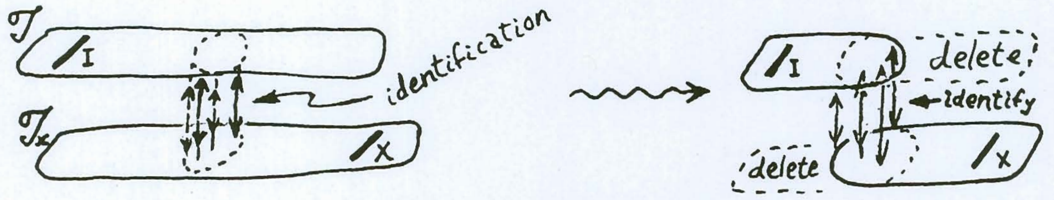
All this structure can be seen to be present in the explicit definition of $\mathcal{T}$ given above. In the flat case, we can take $\iota = \pi_{A'}d\pi^{A'} = I_{\alpha\beta}Z^\alpha dZ^\alpha$ and $\Theta = \frac{1}{6}\epsilon_{\alpha\beta\gamma\delta}Z^\alpha dZ^\beta\wedge dZ^\gamma\wedge dZ^\delta$, in standard descriptions. In the leg-break twistor space for ASD vacuums, the canonical projection from the twistor space to the $\pi_{A'}$-space collapses the 2-surfaces, along which the (simple) 2-form $d\iota$ vanishes, down to points of $\mathbb{P}\mathcal{T}$. In this case, the quantities ι and Θ are both invariant, which is stronger than merely Π and Σ being invariant.

The question that I attempt to address in this article is: how do we reconstruct the (complex) space-time $\mathcal{M}$ (or $\mathbb{C}\mathcal{M}$ for real $\mathcal{M}$ — but I am dropping the "$\mathbb{C}$") from the twistor space $\mathcal{T}$? The general way in which this is intended to work was indicated in R.P. TN 44. For any point $x \in \mathcal{M}$, close enough to i^+ (but not on $\mathscr{I}^+$), there will be an intersection $\bar{\mathscr{C}}_x \cap \mathscr{I}^+ = \mathscr{S}_x$ (what ETN and collaborators call a "light-cone cut") that encodes the point x itself in terms of information at $\mathscr{I}^+$. (I shall be concerned with complex cuts, so in the situation of a real space-time $\mathcal{M}$, the notation "$\mathscr{S}_x$" refers to a local complexification of this intersection, which would otherwise have been denoted by "$\mathbb{C}\mathscr{S}_x$".) There is a non-trivial S^2 in the complex space $\mathscr{S}_x$ (which would be the anti-celestial sphere, for a real point x, or a deformation of this for a complex x). Let us refer to this sphere as $\mathbb{R}\mathscr{S}_x$ (even in the case of complex x). When x is close enough to i^+ (as is being assumed here), then $\mathbb{R}\mathscr{S}_x$ will be non-singular. But as x moves away from i^+, we expect $\mathbb{R}\mathscr{S}_x$ to acquire caustics and crossing regions.

Let y be an arbitrary point of $\mathscr{S}_x$. A general α-curve z in $\mathscr{I}^+$, through y will have a $\mu_{A'}$-value at y (the tangent vector to z at y having the form $\iota^A\mu^{A'}$) and we use this same $\mu_{A'}$-value to continue with a ve curve z_x in $\mathscr{C}_x$ (with tangent vector to z_x at y having the form $o^A\mu^{A'}$) so that the two α-curves are tangent to (and are determined by) the same SD 2-plane element at y. Indeed,

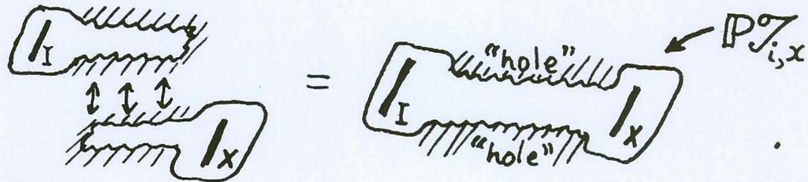


we can use this procedure to provide us with a continuation of the scaling for the twistor (scaling provided by $\mu_{A'}$) from z to z_x and thereby provide an identification between elements of $\mathbb{T}$ and elements of $\mathbb{T}_x$. This identification does not apply unambiguously to all elements of $\mathbb{T}$ and of $\mathcal{T}_x$, but only to parts of $\mathcal{T}$ represented by α-curves $\mathcal{Z}$ which meet $\mathcal{J}_x$ cleanly (by which I mean transversally in a single point), and to parts of $\mathcal{T}_x$ represented by α-curves $\mathcal{Z}_x$ which meet $\mathcal{J}$ cleanly. We may be able to extend the identification between $\mathcal{T}$ and $\mathcal{T}_x$ if we allow this correspondence not to be 1-1, but if we insist on a 1-1 relationship (for x near to i^+) we can obtain a "surgery" of $\mathcal{T}$ in which part of $\mathcal{T}_x$ (near X) is "glued" into a "hole" in $\mathcal{T}$ obtained by deleting the

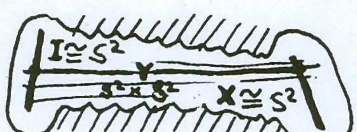


part "near X" in $\mathcal{T}$. I shall call this "glued" twistor space $\mathcal{T}_{i,x}$.

In fact this is not a proper surgery in the usual sense, as significant "holes" will survive, in $\mathbb{P}\mathcal{T}_{i,x}$, as indicated very schematically:



The diagrams are to indicate that the entire I and X have neighbourhoods (with compact closure) in $\mathbb{P}\mathcal{T}_{i,x}$, and the parts of $\mathcal{T}_{i,x}$ above these regions will have neighbourhoods (but not with compact closure) of the entire $\mathbb{P}^{-1}$I and $\mathbb{P}^{-1}$X, respectively. The "neck" joining the I-region to the X-region in $\mathbb{P}\mathcal{T}_{i,x}$ will contain a non-trivial topological S^2, in fact an S^2's worth of projective lines Y, corresponding to the points of $\mathbb{R}\mathcal{J}_x$. Each point of I and each point of X will lie on such a line Y, and this family of Ys traces out a (non-holomorphic $S^2 \times S^2$.

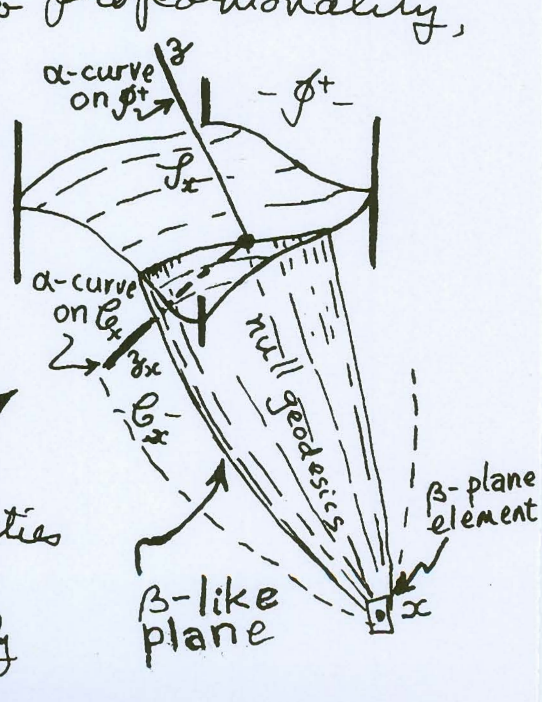


The key problem, with regard to extracting the googly information contained in $\mathcal{T}$, is in deciding which are allowable surgeries $\mathcal{T}_{i,x}$, whence the points x of the complexified) space-time $\mathcal{M}$ to be constructed are thereby determined. A secondary problem is the twistor expression for the metric of $\mathcal{M}$, once its family of points (i.e. family of cuts of $\mathscr{I}^+$) is known. I shall address this secondary question first, since its essential resolution has served to point to a new direction for resolving the first (key) problem.

A suggestion for the metric that I had tentatively suggested earlier turns out not to be correct, as the accompanying article by SF, FH, LJM, and ETN shows, but a new suggestion for the metric, developed by FH and RP from KPT's twistor formulation of ETN's remarkable original formula for the reciprocal of the metric in terms of an integral of W^{-2} over $\mathbb{R}\mathscr{I}$, where W represents the displacement of the cut defining the space-time point x from that defining a neighbouring point x' to x. To translate this into the present twistor framework, we need a 1-form ξ for $\mathcal{T}_x$, defined locally up to proportionality, in a way similar to the way that ι is defined for $\mathcal{T}$. The quantity $\Pi_x = d\Theta \otimes \xi$ is an invariant structure for $\mathcal{T}_x$ (as is Σ) and we require $\Pi_x = -2\Theta \otimes d\xi$ also, so on the overlap between patches $\mathcal{U}$ and $\hat{\mathcal{U}}$ we have

$$\hat{\xi} = k\,\xi$$

just as was the case for ι. To define ξ up to proportionality, we consider the family of β-like planes ruled by the null geodesics through x that are tangent to fixed β-plane elements (ASD plane elements) at x. In the pure googly (SD) case, these β-like planes will be actual β-planes, and their intersections with $\mathscr{I}^+$ will be β-curves on $\mathscr{I}^+$ and α-curves on $\mathcal{C}_x$. In the general case, however, none of these properties will normally hold. These β-like planes will meet $\mathscr{I}^+$ in curves that I shall refer to as $\underline{\xi\text{-curves}}$. The family



of α-curves on $\mathscr{I}^+$, or on $\mathscr{C}_x$, that meet a fixed ξ-curve will provide a 3-surface in $\mathbb{T}$, or in $\mathbb{T}_x$, whose tangent directions annihilate ξ. (In $\mathbb{PT}$ or in $\mathbb{PT}_x$ this would be a 2-surface, referred to as a "crinkly cone in $\mathbb{PT}$.)

This construction does not fix the scaling for ξ, and a reasonable-looking requirement for restricting this scale freedom (consistent with the "$\gamma+\lambda=\Theta$" of R.P. in TN43, p.3) is to demand

$$\xi \wedge dI + d\xi \wedge I = 2\Theta$$

(which implies $d\xi \wedge dI = 4d\Theta$). We note that there is an entire (holomorphic) $\mathbb{CP}^1$'s worth of ξ-curves on $\mathscr{I}_x$, even though individual ξ-curves generally encounter singularities. The freedom in the ξ-scaling — where I am assuming that the above relationship can be (and is) satisfied — is given by $\xi \mapsto (1-e^{\lambda})\xi$, where

$$d\lambda \wedge \xi \wedge I = 2\Theta.$$

This is an equation governing the propagation of λ, along each projective line Y, which has no global solution over Y (a whole $\mathbb{CP}^1$). (Note that $d\lambda$ has to become infinite at both ends, at I and X.) Thus, the scaling for ξ appears to be fixed by these considerations.

The suggested twistor expression for the metric (discussed more fully in the accompanying article by FH, LJM, and ETN) is given by

$$\frac{1}{g} = \frac{1}{8\pi^2}\oint_{S^3} \frac{\Theta}{\gamma^2} \qquad \text{where } \xi \wedge \xi' \wedge I = \gamma\Theta,$$

the metric $g = g(\delta x, \delta x)$ being evaluated at the point x, corresponding to ξ, where δx represents the infinitesimal displacement from the point x to a neighbouring point $x' = x + \delta x$, this neighbouring point corresponding to the 1-form ξ', which differs from ξ by an infinitesimal amount. (N.B. In this expression, terms in $(\delta x)^2$ play no role, although in my earlier incorrect expression "$d\xi \wedge d\xi' = -\frac{1}{2} g \, d\Theta$" these would have been the crucial terms.) Note that the scalar quantity γ scales as $\hat{\gamma} = k^2 \gamma$ from patch to patch, whence the integrand Θ/γ^2 is unchanged from patch to patch. We know, from the accompanying article by SF, FH, LJM, and ETN that this expression is correct in the ASD and SD cases. It is at least <u>conformally</u> correct in the general case, since it gives the null cones correctly.

Let us now attempt to address the key issue of fixing the cuts $\mathcal{T}_x$ to be actually light-cone cuts. My present ideas on this are still not fully formulated, but they appear to have been driven relentlessly in what I hope is the appropriate direction toward this goal (and, I hope, is not too far from it). The essential idea is to provide a condition that is to hold separately at each point Y of the cut $\mathcal{T}_x$ (or $\mathbb{R}\mathcal{T}_x$, which would be sufficient), this condition to depend upon the scalings of the relevant forms and on the -6 homogeneity functions that relate these scalings from patch to patch. The special role that these functions have that is relevant to our purposes is the following.

Let ξ, η, and ζ be 1-forms, each of homogeneity degree 2, scaling from patch to patch according to

$$\hat{\xi} = k\xi, \quad \hat{\eta} = k\eta, \quad \hat{\zeta} = k\zeta.$$

Consider the 3-form $\xi \wedge \eta \wedge \zeta$. This will have a homogeneity degree, namely 6, and must therefore be proportional to Θ. On an overlap between patches, we have

$$\hat{\xi} \wedge \hat{\eta} \wedge \hat{\zeta} - \xi \wedge \eta \wedge \zeta = (1 + f_{-6})\, \xi \wedge \eta \wedge \zeta - \xi \wedge \eta \wedge \zeta$$
$$= f_{-6}\, \xi \wedge \eta \wedge \zeta .$$

This has homogeneity degree 0, so it is defined on the projective space $\mathbb{P}\mathcal{T}$ (or $\mathbb{P}\mathcal{T}_x$ or $\mathbb{P}\mathcal{T}_{i,x}$). This space is 3-dimensional, so $d(f_{-6}\xi\wedge\eta\wedge\zeta)$ must vanish. Hence

$$d(\hat{\xi}\wedge\hat{\eta}\wedge\hat{\zeta}) = d(\xi\wedge\eta\wedge\zeta),$$

the 4-form $d(\xi\wedge\eta\wedge\zeta)$ extends globally across patches. Note that in an application of the fundamental theorem of exterior calculus (FTEC — i.e. the theorem commonly referred to as "the generalized Stokes theorem") we have to pay attention to these -6-functions when we pass from one patch to another. For example, the integral of "$d(\xi\wedge\eta\wedge\zeta)$" over a compact 4-contour need not vanish, for this reason.

We wish to apply these ideas to the particular case when $\zeta = \iota$ and η is a 1-form, like ξ, but which refers to an arbitrary point y on $\mathscr{I}_x$ (or on $R\mathscr{I}_x$) rather than to x itself. We cannot now expect to use the normalization "$\eta\wedge d\iota + d\eta\wedge\iota = 2\Theta$" since the left-hand side now vanishes (because there are not enough variables in η and ι to span all three projective dimensions). In what follows, a normalization for η plays no role. We wish to obtain a condition on these forms which will amount to the fixing of one complex number per point (i.e. y) of $\mathscr{I}_x$ so, in principle, this could fix the cut of $\mathscr{I}^+$ that determines a point x. The idea is to integrate the 4-form

$$\chi = d(\xi\wedge\eta\wedge\iota),$$

which by the above discussion is global across patches, over some appropriate 4-contour with boundary, in $\mathcal{T}_{i,x}$, and then use the FTEC to convert this to an integral over the boundary, where patching issues bring the f_{-6} functions into play.

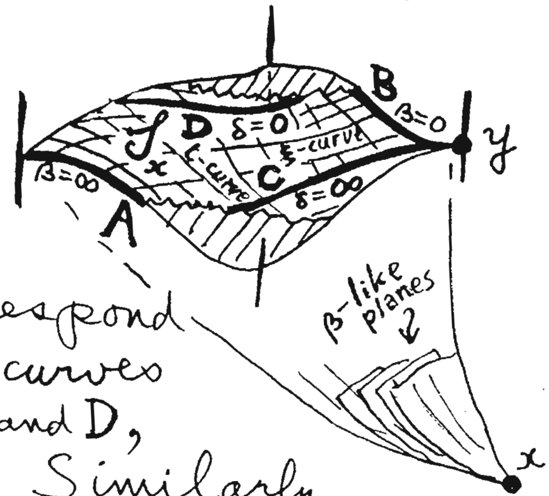
At the time of writing there are still some uncertainties as to the exact regions that would be needed. I shall first describe a prescription that is not (quite?) correct and then make some suggestions as to how this might appropriately be amended. To begin, we find scalar functions A, B, C, D which scale from patch to patch according to

$$\hat{A} = k^{\frac{1}{2}}A,\quad \hat{B} = k^{\frac{1}{2}}B,\quad \hat{C} = k^{\frac{1}{2}}C,\quad \hat{D} = k^{\frac{1}{2}}D,$$

each of A, B, C, D being homogeneous of degree 1, where

$$\iota = A\,dB - B\,dA,\quad \eta = B\,dC - C\,dB,\quad \xi = C\,dD - D\,dC.$$

We do this by noting, first, that there is a complete Riemann sphere's worth of ι-curves (ι-curves being the intersections of $\mathcal{T}_x$ with β-planes on $\mathscr{I}^+$) and of ξ-curves (from the primed spin-spaces at i^+ and at x, respectively, so the complex stereographic coordinates $\beta = B/A$, $\delta = D/C$ can be defined on $\mathcal{T}$ and on $\mathcal{T}_x$. Geometrically, $\delta = \infty$ and $\delta = 0$ are to correspond to two ξ-curves on $\mathcal{T}_x$, these curves to be where α-lines meeting C and D, respectively, become zero. Similarly $\beta = \infty$ and $\beta = 0$ correspond to two ι-curves on $\mathcal{T}_x$ where α-lines meeting A and B, respectively, become zero. We observe that ι is proportional to $d\beta$ and ξ is proportional to $d\delta$, so we can define A and C by



$$\iota = A^2\,d\beta,\quad \xi = C^2\,d\delta$$

(there being no obstruction to taking the square roots), whence we can now define

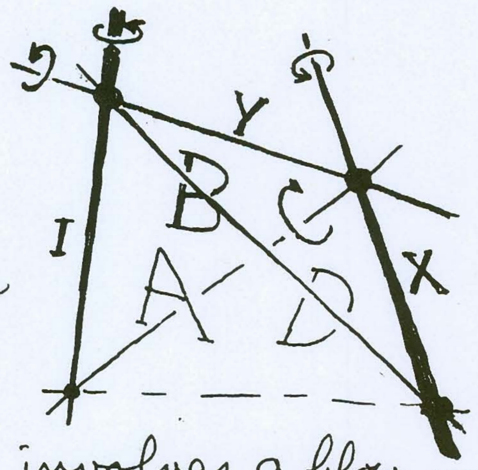
$$B = A\beta,\quad D = C\delta.$$

Each constant ratio $A:B$ gives an ι-curve and each constant ratio $D:C$ gives a ξ-curve. In the above, we can also choose the curves C ($\delta = \infty$) and B ($\beta = \infty$) to pass through the point y. Then we can take a new Riemann sphere (stereographic) coordinate γ defined by

$$\gamma = C/B$$

and we have $\eta = B^2\,d\gamma$. (This is basically the definition of η that I am adopting here. Previously, I had given merely a vague description).

The geometric role of the three Riemann sphere coordinates β, γ, δ, in the case of flat twistor space $\mathbb{PT}$, is given in the figure. Planes through I are labelled by β, through Y by γ, and through **X** by δ. The coordinate space is $S^2 \times S^2 \times S^2$ which is, of course, topologically distinct from $\mathbb{CP}^3$.

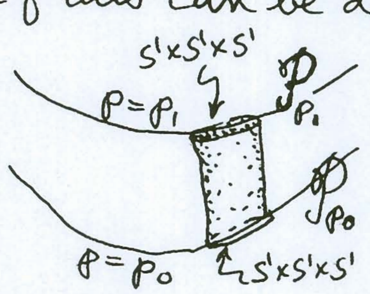


The relation to $\mathbb{CP}^3$ involves a blow-up of I and X, a "double blow-up at the points $I \cap Y$ and $X \cap Y$ and a blow-up followed by a blow-down in the opposite direction along Y (so that the line Y itself is reduced to a point but the family of planes through it provides a new line Y′). Note that this "blow-up situation" is quite different from that which motivated many earlier attempts at the googly.

As our first try for a contour arrangement, we try to integrate the quantity χ $(= d(\xi \wedge \eta \wedge \iota))$ over a 4-region bounded by two 3-surfaces $\mathcal{P}_\rho$, given by two values of ρ, where

$$AB^2C^2D = \rho .$$

If this can be done entirely in one patch, the integral of χ



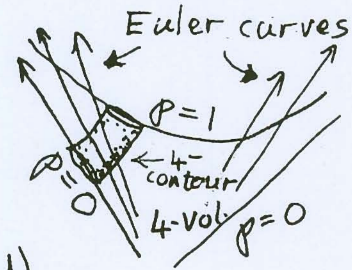
over the 4-volume (real 4-contour with boundary) would be equal to the difference between the integrals of $\xi \wedge \eta \wedge \iota$ over (a closed contour in) $\mathcal{P}_{\rho_1}$ and over (a closed 3-contour in) $\mathcal{P}_{\rho_0}$. If our contour's boundaries were some appropriate pair of $S^1 \times S^1 \times S^1$ regions, then each

$$\oint_{\mathcal{P}_\rho} \xi \wedge \eta \wedge \iota = \rho \oint_{\mathcal{P}_\rho} \frac{\xi \wedge \eta \wedge \iota}{\rho} = \rho \oint \frac{(A\,dB - B\,dA)}{AB} \wedge \frac{(B\,dC - C\,dB)}{BC} \wedge \frac{(C\,dD - D\,dC)}{CD}$$

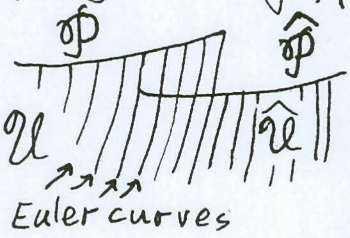
$$= \rho\,(2\pi i)^3,$$

so the integral of χ between $\mathcal{P}_{\rho_0}$ and $\mathcal{P}_{\rho_1}$ would turn out to be $(\rho_1 - \rho_0)(2\pi i)^3$.

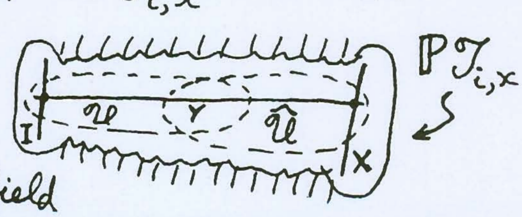
The idea is now to choose (say) $\rho_1 = 1$ and $\rho_0 = 0$, whence the entire integral should turn out to be $(2\pi i)^3$. We note that in the above displayed expression, the third $\oint$ is actually a projective twistor integral, since the integrand has homogeneity 0. When $\rho = 0$, however, this does not apply, and the contour gets squashed into a region that contains Euler curves, so that projectively it is 2-dimensional, one of the contour directions being actually <u>along</u> the Euler curves, so the entire 3-dimensional boundary integral (at that end) is essentially <u>non</u>-projective. The integrand is now zero, however, because $\xi \wedge \eta \wedge \iota$ is proportional to Θ and vanishes along the Euler direction.



Now suppose that our 4-volume needs to be covered by two patches. A complication now arises because the equation $AB^2C^2D = \rho$ is not invariant under the googly rescalings, so the 3-surface $\mathcal{P}$ $(= \mathcal{P}_1)$ "jumps" to a new 3-surface $\hat{\mathcal{P}}$, given by $\hat{A}\hat{B}^2\hat{C}^2\hat{D} = 1$, i.e. $AB^2C^2D = \frac{1}{1+f_{-6}}$. The situation is like that indicated in the diagram. How do we deal with this?



I am going to suppose that we are in a situation that is simple enough that, for a given choice of $\mathcal{Y}$, the patches can be arranged (using "cohomology freedom") so that for a neighbourhood of the line $Y \in \mathbb{P}\mathcal{T}_{i,x}$ we can use just two patches, one of which contains $I \cap Y$ and the other, $X \cap Y$. (This is certainly OK in the weak-field limit, say with an elementary state.) We can arrive

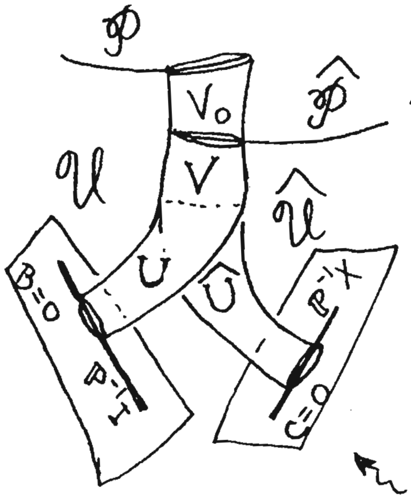


at the $\rho=0$ boundary in various different ways, but let us choose our contour as collapsing down to $\rho=0$ by means of the limit $\varepsilon\to 0$ $(\varepsilon^2=\rho)$ with

$$A=e^{2i\theta},\quad B=\varepsilon e^{-i(\theta+\phi+\psi)},\quad C=e^{i\phi},\quad D=e^{2i\psi}$$

near I (in the $\mathcal{U}$ region) and by means of $\varepsilon\to 0$ with

$$A=e^{2i\theta},\quad B=e^{i\varphi},\quad C=e^{-i(\theta+\varphi+\psi)},\quad D=e^{2i\psi}$$



near X (in the $\hat{\mathcal{U}}$ region). Note that there is a non-Hausdorfness at the $\rho=0$ boundary, in the coordinate space of (A,B,C,D), where $B=0$ and $C=0$ are not Hausdorffly separated. The χ-volumes $U, \hat{U}, V$, and V_0 are as indicated in the diagram, all boundaries lying in $B=0$, $C=0$, $\mathcal{P}$, and $\hat{\mathcal{P}}$. Applying FTEC we obtain

$$U+V+V_0=\oint_{\mathcal{P}}\xi\wedge\eta\wedge\iota$$

and

$$\hat{U}+V=\oint_{\hat{\mathcal{P}}}\hat{\xi}\wedge\hat{\eta}\wedge\hat{\iota}.$$

But we see that $\oint_{\mathcal{P}}\xi\wedge\eta\wedge\iota$ is just the same expression as $\oint_{\hat{\mathcal{P}}}\hat{\xi}\wedge\hat{\eta}\wedge\hat{\iota}$ when we use the "multiplying top and bottom by ρ" device that we employed earlier in order to evaluate the integrals (for $S^1\times S^1\times S^1$ contours) that yielded $(2\pi i)^3$. Hence

$$V_0=\hat{U}-U.$$

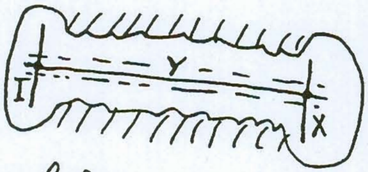
But we can also use FTEC directly to V_0, using $\chi=d(\xi\wedge\eta\wedge\iota)$ (or equivalently $\chi=d(\hat{\xi}\wedge\hat{\eta}\wedge\hat{\iota})$), whence

$$V_0=\oint_{\mathcal{P}}\xi\wedge\eta\wedge\iota-\oint_{\hat{\mathcal{P}}}\xi\wedge\eta\wedge\iota=\oint_{\hat{\mathcal{P}}}\hat{\xi}\wedge\hat{\eta}\wedge\hat{\iota}-\oint_{\hat{\mathcal{P}}}\xi\wedge\eta\wedge\iota$$

$$=\oint_{\hat{\mathcal{P}}}f_{-6}\,\xi\wedge\eta\wedge\zeta=\oint_{\mathcal{P}}f_{-6}\,\xi\wedge\eta\wedge\iota$$

the final integrands being of degree 0, so the integral is projective.

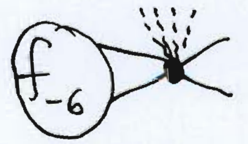
Thus, if we expect a non-zero result from this integral, then we must expect that U and $\hat{U}$ are different. This difference is a reflection of the difference between the



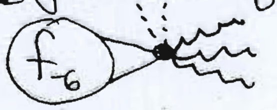
behaviour of the Y-lines (in the neighbourhood of the chosen Y-line) near I and near X. (Note that this difference is independent of how much of V is incorporated into U and $\hat{U}$. Owing to the non-Hausdorffness in coordinate space, we are really concerned with "relative cohomology" here.) For example, the family of Y lines passing through a fixed point of X will look "crinkly" rather than planar near I, in general. The "crinkliness" is a measure of the shear of the cut $\mathcal{J}_x$ at the point Y (at least, it is directly related to the shear in the pure googly case). Thus, we begin to see a connection between the "helicity +2 massless field" that is defined by the f_{-6} cohomology and the measure of shearing of the cut, which is just the kind of thing that is needed — giving the googly information in the curiously twisted Euler fibres an actual geometric content. Accordingly, we could conjecture that the condition for a correct "light-cone cut" is that, for each y, the difference between the U and the $\hat{U}$ x-volumes is indeed equal to the expression $\oint f_{-6}\,\xi\wedge\eta\wedge\iota$, and that perhaps $U+V=\hat{U}+V+V_0=(2\pi i)^3$.

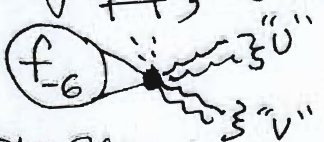
However, when we look at what has been achieved by the above a little more carefully, we find that we are far from finished. For, as things stand, we are not able to get a non-zero answer for $\oint_{\mathbb{P}} f_{-6}\,\xi\wedge\eta\wedge\iota$. The reason for this is that the integral is now projective, and we have lost the handy poles that we obtained from the AB^2C^2D in the denominator when we integrated $\xi\wedge\eta\wedge\iota$ rather than $f_{-6}\,\xi\wedge\eta\wedge\iota$. (Our integration now has the character of trying to get a non-zero answer out of a twistor diagram

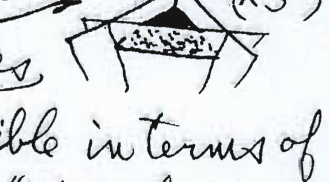
without the needed "ears" present.

in  .) The difficulty is that we need a non-zero answer both when $(AB^2C^2D)^{-1}$ is present and when it is absent.

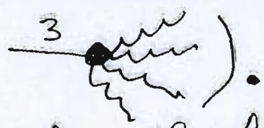
The most promising route out of this dilemma appears to be to take advantage of a curious property of boundary contours in non-projective contour integral arising in twistor diagram theory. An inhomogeneous boundary like $A=a$, for some non-zero constant a, can occur simultaneously with a pole at $A=0$, and this situation plays an important role in many basic twistor diagrams. (See articles by APH and his students in many TN articles; cf. also APH's articles in *Twistors in Mathematics and Physics*, eds Bailey & Baston (Cambridge 1990) and *The Geometric Universe*, eds. Huggett, Mason, Tod, Tsou, and Woodhouse (Oxford 1998).) Thus, instead of our f_{-6} function manifesting itself in the form 

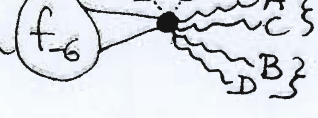
, which would evaluate the (+2)-helicity field at the specific point defined by UV, we look at expressions like 

or 

. The first of these certainly gives a non-zero integral (cf. RP in TN 1), but the second may possibly be the more appropriate here. The difference is that the first is concerned basically with "triangle areas" 

and the second, with "quadrilateral areas" 

(these latter being expressible in terms of the difference between two "triangle areas" in the particular case when U, U′, V, V′ are all linearly dependent; this is not the same contour as for the tetrahedral volume that occurs with 

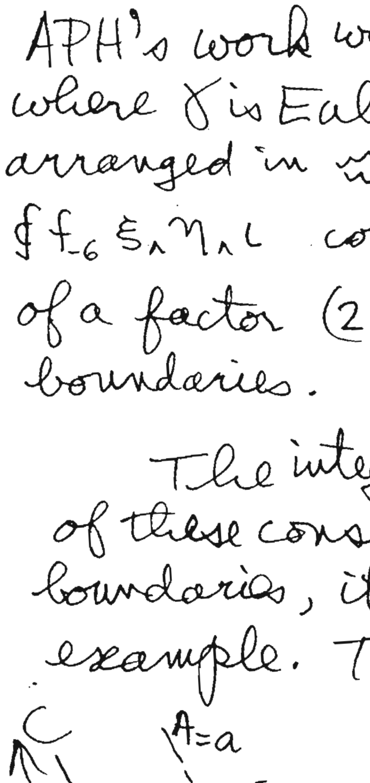
). It seems that the integral that we require has the form 

, but I have not been able to

sort out all this in any detail. It ought to have a space-time interpretation of integrals of the null-datum for the (+2)-helicity field along generators of $\mathscr{I}^+$, as is required for obtaining the asymptotic shear from this null datum (at least in the pure googly case). Something similar may also be (implicitly?) involved in relation to $\mathcal{C}_x$, but this needs further investigation.

The boundary lines $\rightsquigarrow A, \ldots, \rightsquigarrow D$ would have to be interpreted as inhomogeneous boundaries in APH's sense, namely boundaries on $A=a$, $B=b$, $C=c$, $D=d$, where a, b, c, d are numerical constants. APH's work would suggest that perhaps $a=b=c=d=e^{\gamma}$ where γ is Euler's constant. The way that the boundaries arranged in $\left.\begin{matrix}\rightsquigarrow A\\ \rightsquigarrow C\end{matrix}\right\}$ and $\left.\begin{matrix}\rightsquigarrow B\\ \rightsquigarrow D\end{matrix}\right\}$ would suggest that our $\oint f_{-6}\,\xi\wedge\eta\wedge\iota$ could be re-expressed by the incorporation of a factor $(2\pi i)^{-2}\log\left(\frac{A-a}{C-c}\right)\log\left(\frac{B-b}{D-d}\right)$ instead of using boundaries.

The integrals $\int \chi$ needs re-examination in the light of these considerations. To understand the role of these boundaries, it may be helpful to look at a 2-dimensional example. Take the two variables to be A and C, with the "surface" $\mathscr{P}$ defined by $AC=\rho$, with boundaries on $A=a$ and $C=c$. Our integral is now



$$\oint (A\,dC - C\,dA) = \oint \frac{\rho A^2 d\left(\frac{C}{A}\right)}{AC} = \rho\int_{c^2/\rho}^{\rho/a^2}\frac{d\lambda}{\lambda}$$

[over $AC=\rho$ boundary from $C=c$ to $A=a$]

$$= 2\rho\log\left(\frac{\rho}{ac}\right)$$

where $\lambda = \frac{C}{A}$. Thus, we have a log term instead of the $2\rho\pi i$ that we had before. The 4-dimensional case is presumably similar.

In the full situation we must see how to deal with the jump in the boundary location as we pass from patch to patch (since $A, \ldots, D$ rescale, as before, whereas $a, \ldots, d$ are simply fixed numbers). I have not yet been able to ascertain the plausibility and the full nature of this proposed scheme. The idea for fixing the "light-cone cuts" would be that we have analogues of the χ-volume integrals U and $\hat{U}$ that we had before, and that for each y this difference is equal to the appropriate f_{-6} integral. Moreover, we might expect that there is a condition analogous to the "$U+V = \hat{U} + V + V_0 = (2\pi i)^3$" that I suggested earlier, where the correct curved-space volumes should be equal to the corresponding flat-space ones.

Clearly a great deal needs to be done in order to ascertain the status of this (kind of) proposal. Nevertheless, I have the impression that the correct answer to the determination of the correct surgeries, giving the actual "light-cone cuts" has to follow the general lines that I have indicated above (at least in the pure googly case), although a considerable clarification of the ideas will be needed. (But even if this programme is fully successful, we must expect that the determination of the surgeries and cuts is very implicit, rather than providing an effective construction of vacuum space-times.) As a final comment, I find it gratifying that there seems to be a strong connection with twistor diagram theory. (Some other connections have been mentioned to me by APH.) I believe that there is a genuine possibility of progress with some of the obstacles mentioned at the beginning of this article.

Many thanks, especially to ETN, APH, LJM, and FH. Roger