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Twistors and Legendre Transformations

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Abstract

We discuss how twistors arise in a natural manner via Legendre transformations on a contact bundle over space-time and its restriction to null infinity. In section II this is shown to give two different twistor formulations of the good cut equation and the dual good cut equation. In section III, the framework is 'homogenized' and applied to certain questions arising in connection with Penrose's googly programme.

I. Flat Space Twistors and Legendre Transformations

In this section we show a natural relationship—via a Legendre transformation—between twistor space and certain contact spaces (specifically, the projective cotangent bundle over (1) Minkowski space and (2) over Scri). In the latter case it can be generalized to non-flat spaces and to asymptotic twistors. The correspondence between space-time and twistor space can be studied by means of the double fibration of the space of projective null covectors over both Minkowski space and twistor space. When this bundle of null covectors over Minkowski space is restricted to a Cauchy surface it has the structure of a contact manifold. These fibrations will be seen to be related by a Legendre transform.

We begin with the canonical one-form on the cotangent bundle over Minkowski space, M, treated projectively, leading to a contact form on the seven dimensional contact manifold, PT^*M

$$\alpha = p_a dx^a \sim \alpha = \lambda p_a dx^a. \tag{1}$$

For ease of presentation it is perhaps best to work with a representative form ($p_0 = 1$) so that the contact coordinates are (t, x^i, p_i) and

$$\alpha = dt + p_i dx^i. \tag{2}$$

Suppose a null surface (Arnold's "big" wavefront; a solution to the eikonal equation, $g^{ab}\partial_a S \partial_b S = 0$) is given as $S(x^a) = 0$. We can solve this for $t = -T(x^i)$, so that the normal, $p_a = (1, \partial_i T(x^i))$ is a null vector, $p_a p^a = 0$. The projective cotangent bundle on M has the structure of the first jet bundle over a space-like hypersurface coordinatized by x^i with the fiber coordinates (t, p_i) so that on the cross-section $(t, p_i) = (-T(x^i), \partial_i T(x^i))$ we have that $\alpha = 0$. This defines the cross section as a Legendrian submanifold (three dimensional) of the contact manifold (the first jet bundle over the x^i). $T(x^i)$ is an example of what Arnold calls a "generating function" for the Legendrian submanifold.

What we will do is give the *same* contact space (with the same contact form) a different fibration, that is, we will represent it as the first jet bundle over a different base space. As an illustration of this procedure, equation (2) could be rewritten, via a *Legendre transformation*, as

$$\alpha = d(t + p_x x + p_y y + p_z z) - x dp_x - y dp_y - z dp_z = dg - x dp_x - y dp_y - z dp_z \tag{3}$$

with $g(p_x, p_x, p_y) = t + p_x x + p_y y + p_z z$. This same contact form could be considered as arising from the firs jet bundle over the base (p_x, p_y, p_z) with fiber coordinates (g, x, y, z). If a Legendrian submanifold is given b a generating function $t = -T(x^i)$, it can alternatively be given by the new generating function

$$g = G(p_x, p_y, p_z) = -T(x, y, z) + p_x x + p_y y + p_z z$$

where $x^i = x^i(p_x, p_y, p_z)$ are obtained by inverting $p_i = \partial_i T(x, y, z)$. In terms of the new generating functio we have that the Legendrian submanifold is given by

$$\begin{aligned} x &= \partial_{p_x} G, \qquad y = \partial_{p_y} G, \qquad z = \partial_{p_z} G. \\ g &= -T(x, y, z) + p_x x + p_y y + p_z z. \end{aligned}$$

This in effect transforms from a fibration over position space to a fibration over momentum space.

We now turn to a fibration over twistor space using a different Legendre transformation. We first rewrite equation (1) making the null character of p_a explicit:

$$\alpha = \mu_A \pi_{B'} dx^{AB'} \tag{4}$$

which can be rewritten as

$$\alpha = \mu_A d(\pi_{B'} x^{AB'}) - \mu_A x^{AB'} d\pi_{B'}. \tag{5}$$

Defining $\omega^A = ix^{AB'}\pi_{B'}$, the contact form becomes

$$\alpha = -i\{\mu_A d\omega^A - i\mu_A x^{AB'} d\pi_{B'}\} = -i(\mu_A d\omega^A + \lambda^{A'} d\pi_{B'}\} = -iW_\alpha dZ^\alpha \tag{6}$$

and we see that *all* the differentials in equation (6) are differentials of the twistor $Z^\alpha = (\omega^A, \pi_{B'})$, i.e., the contact coordinates, for the base, are $(\omega^A,\ \pi_{B'})$ and the fiber coordinates, treated projectively, are the dual twistors, $W_\alpha =(\mu_A,\ \lambda^{A'} = -i\mu_A x^{AA'})$. From this point of view one again sees that twistors and dual twistors are canonically conjugate to each other.

A Legendrian submanifold can be constructed from a generating function that is an arbitrary homogeneous function *of degree zero* on twistor space

$$G(\omega^A, \pi_{B'}) = constant$$

The Legendrian submanifold thus is given by $W_\alpha = \partial_{Z^\alpha} G$ or

$$\begin{aligned} \mu_A &= \partial_{\omega^A} G, \\ \lambda^{A'} &= \partial_{\pi_{A'}} G. \end{aligned} \tag{7}$$

The vanishing of $G(Z^\alpha)$ defines a null 3-surface in Minkowski space by the 3-parameter set of lines in the twistor space that are tangent to the two-surface $G(Z^\alpha) = 0$. [See R. Penrose, TN #42 for 4 other things one can do with such a two-surface]. Note that, with an arbitrary $G(Z^\alpha)$, by inverting the Legendre transformation one automatically obtains an arbitrary solution of the eikonal equation. One can see this explicitly by first writing $\lambda^{A'} = -i\mu_A x^{AA'}$, from equation (7), as

$$\partial_{\pi_{A'}} G = -ix^{AA'} \partial_{\omega^A} G \tag{8}$$

which tell us two things: (1) it can be thought of as an equation defining $\pi_{B'} = \pi_{B'}(x^{AA'})$ implicitly and (2) by multiplying by $\pi_{A'}$ we see from the condition $Z^\alpha W_\alpha = 0$ and Euler's theorem, that G must be homogeneous of degree zero (requiring that $Z^\alpha W_\alpha = 0$ even where $G \neq 0$). Now by taking the gradient of $G(x^{AA'}\pi_{A'}, \pi_{B'})$ treating $\pi_{B'}$ as a function of $x^{AA'}$, via (8), and using (8) explicitly we have that

$$\partial_{AA'} G = i\mu_A \pi_{A'}$$

is a null vector and hence that $G(x^{AA'}\pi_{A'}(x^{BB'}), \pi_{B'}(x^{BB'}))$ satisfies the eikonal equation.

II. Asymptotic Twistors and Legendre Transformations

We now consider the projective cotangent bundle over Scri, $\mathfrak{I}^+$. We first let Scri, ($\mathfrak{I} = \mathbb{R}\mathrm{x}\mathbb{S}^2$), have the standard Bondi type coordinates, $(u, \zeta, \overline{\zeta})$ with the projectivized "momenta" $(1, p, \overline{p})$ and then complexify, letting $\overline{\zeta} \rightsquigarrow \widetilde{\zeta}$ and $\overline{p} \rightsquigarrow \widetilde{p}$, independent of, respectively, ζ and p.

The contact form on $PT^*\mathfrak{I}^+$ is

$$\alpha = du - pd\zeta - \widetilde{p}d\widetilde{\zeta} \tag{9}$$

which can be rewritten as

$$\begin{aligned} \alpha &= d(u - \widetilde{p}\widetilde{\zeta}) - pd\zeta + \widetilde{\zeta}d\widetilde{p} \qquad (10) \\ &= dg - pd\zeta + \widetilde{\zeta}d\widetilde{p} \qquad (11) \end{aligned}$$

with $g = u - \widetilde{p}\widetilde{\zeta}$. This can be thought of as defining the first jet bundle over the space coordinatised by $(\zeta, \widetilde{p})$ with fibers parametrized by

$$(g,\ p = \partial_\zeta g,\ \widetilde{\zeta} = -\partial_{\widetilde{p}} g)\,. \tag{12}$$

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A cross-section of $\mathscr{I}^+$ given by $u = Z(\zeta, \tilde{\zeta})$ can be lifted to the contact bundle via $p = \partial_\zeta Z$, $\tilde{p} = \partial_{\tilde{\zeta}} Z$ and yields a two-dimensional Legendre submanifold. The same Legendrian submanifold can be described after the Legendre transformation by

$$
\begin{aligned}
g &= G(\zeta, \tilde{p}) = Z(\zeta, \tilde{\zeta}) - \tilde{p}\tilde{\zeta}, && (13)\\
p &= \partial_\zeta G, \quad \tilde{\zeta} = -\partial_{\tilde{p}} G && (14)
\end{aligned}
$$

where $\tilde{\zeta} = \tilde{P}(\zeta, \tilde{p})$ via the inversion of $\tilde{p} = \partial_{\tilde{\zeta}} Z(\zeta, \tilde{\zeta})$.

We now *define* the projective twistor space to be the space coordinatized by the functions

$$
(ig, i\tilde{p}, \zeta) \qquad (15)
$$

whose differentials appear in the contact form, or non-projectively

$$
Z^\alpha = \pi_{0'}(ig, i\tilde{p}, 1, \zeta) = (\omega^A, \pi_{B'}). \qquad (16)
$$

To see that this definition agrees with the standard one when $Z(\zeta, \tilde{\zeta})$ represents a flat-space lightcone cut, we consider the intersection of a twistor 2-surface with $\mathscr{I}^+$. It takes the form, with $\pi_{B'} = \pi_{0'}(1, \zeta) =$ constant,

$$
\begin{aligned}
u &= Z(\zeta, \tilde{\zeta}) = x^{00'} + x^{01'}\tilde{\zeta} + x^{10'}\zeta + x^{11'}\zeta\tilde{\zeta}\\
&= (x^{00'} + x^{10'}\zeta) + (x^{01'} + x^{11'}\zeta)\tilde{\zeta}\\
u &= -i\{(\omega^0/\pi_{0'}) + (\omega^1/\pi_{0'})\tilde{\zeta}\} && (17)
\end{aligned}
$$

i.e., the intersection of twistor surfaces with Scri are the straight lines in the $\zeta =$ constant plane with $\tilde{\zeta}$ a parameter along the line, and the intercept and gradient given by $-i(\omega^0, \omega^1)/\pi_{0'}$. If we perform the above mentioned Legendre transformation we have that

$$
g = G = -i\omega^0/\pi_{0'}, \quad \tilde{p} = -i\omega^1/\pi_{0'}
$$

as was claimed.

We have just shown how flat twistor space arises naturally from a Legendre transformation on the contact bundle over $\mathscr{I}^+$; we now show how this can be generalized to other forms of twistor theory. There are two distinctly different points of view one could take: (1) we can define a type of flat local twistor space over each point of the sphere, $\tilde{\zeta} \varepsilon S^2$, or (2) define a global curved twistor space associated with certain second order ODEs given on $\mathscr{I}^+$.

(1) Returning to either equation (15) or equation (16), we see that for each value of $\tilde{\zeta}$ we have a flat twistor space, i.e. a local twistor space for each $\tilde{\zeta}$. There are a wide range of applications for this observation. One of them is to reexpress the Null Surface Version of GR in this language. This is being investigated. For the present, we describe two special applications: (a) to the Leg-Break construction of anti-self-dual space-times and (b) to a googly construction of self-dual space-times.

(a) The Leg-Break construction is based on the solution of the "good cut" equation

$$
\partial_{\tilde{\zeta}}\partial_{\tilde{\zeta}} Z = \tilde{\sigma}(Z, \zeta, \tilde{\zeta})
$$

for the function $Z(x^a, \zeta, \tilde{\zeta})$ where, with regularity conditions, the x^a are four constants of integration. The Leg-Break metric can be constructed from the $Z(x^a, \zeta, \tilde{\zeta})$. The above Legendre transformation transforms the good cut equation to

$$
G_{\tilde{p}\tilde{p}} = \frac{1}{\tilde{\sigma}(G - \tilde{p}G_{\tilde{p}}, \zeta, -G_{\tilde{p}})} \qquad (18)
$$

a second order equation for $g = G(x^a, \zeta, \tilde{p})$. The point to be emphasized is that *the equation is completely given in terms of the local twistor variables and the free data $\tilde{\sigma}(G - \tilde{p}G_{\tilde{p}}, \zeta, -G_{\tilde{p}})$, also expressed in terms of the twistor variables.* (The local twistor variables can here be thought of as the flat twistor space associated to the given Bondi coordinate system as if it were built from shear free cuts in Minkowski space.) There is a more symmetric version of this. If the solution $g = G(\zeta, \tilde{p})$ is given implicitly by the function $F(g, \zeta, \tilde{p}) = 0$ then $F(g, \zeta, \tilde{p})$ satisfies the differential equation

$$
F_{\tilde{p}\tilde{p}} - 2\frac{F_{\tilde{p}}}{F_g}F_{\tilde{p}g} + \left(\frac{F_{\tilde{p}}}{F_g}\right)^2 F_{gg} - \frac{F_g}{\tilde{\sigma}(g + \tilde{p}\frac{F_{\tilde{p}}}{F_g}, \zeta, \frac{F_{\tilde{p}}}{F_g})} = 0 \qquad (19)
$$

which, unfortunately, is not a very pretty equation.

(b) A self-dual space-time can be constructed as the space of solutions to the dual good cut equation $\partial_\zeta \partial_\zeta Z = \sigma(Z, \zeta, \tilde{\zeta})$ and these solutions can, in turn, be transformed by the same Legendre transformation into an equation for $g = G(x^a, \zeta, \tilde{p})$ in the (googly) twistor space

$$G_{\tilde{p}\tilde{p}} = \frac{-(G_{\tilde{p}\zeta})^2}{\sigma(G - \tilde{p}G_{\tilde{p}}, \zeta, -G_{\tilde{p}}) - G_{\zeta\zeta}} \tag{20}$$

again an equation completely stated in terms of the local twistors which are now the canonical asymptotic twistors. This also can be easily converted into an equation for the function $H(g, \zeta, \tilde{p}) = 0$ that implicitly defines $g = G(\zeta, \tilde{p})$.

This approach to the Googly relates to that of Roger Penrose as follows; the twistor space is the flat asymptotic twistor space as used in the googly construction, and the equation above can be viewed as a differential equation for the envelope of the level surfaces of the 'googly maps' (see the next section for these). More work is still required to fully understand equations (18) and (20).

(2) An alternate view is to consider the cut function as a solution to a second order ODE which in most applications is of the form, $\partial_{\tilde{\zeta}} \partial_{\tilde{\zeta}} Z = \tilde{\sigma}(Z, \zeta, \tilde{\zeta})$. The cut, or, of more relevance, the curve obtained by fixing ζ, then depends on two parameters, the constants of integration, (α, β) (e.g., the initial position and slope at some $\tilde{\zeta} = \tilde{\zeta}_0$) and has the form $u = Z(\alpha, \beta; \zeta, \tilde{\zeta})$. We refer to the cut function *with a fixed* ζ as a twistor curve parametrized by $\tilde{\zeta}$. The space of twistor curves is parametrized by the (α, β, ζ). Below we will take the twistor variables as a specific choice of the parametrization.

We now have a two parameter family of Legendrian submanifold given, for each (α, β), by

$$\begin{aligned} u &= Z(\alpha, \beta; \zeta, \tilde{\zeta}), \\ p &= \partial_\zeta Z, \\ \tilde{p} &= \partial_{\tilde{\zeta}} Z. \end{aligned}$$

After the Legendre transformation, equation (13), we have the same family of submanifolds and twistor curves, $(\zeta = const)$, but now described by

$$g = G(\alpha, \beta; \zeta, \tilde{p}) = Z(\alpha, \beta; \zeta, \tilde{\zeta}) - \tilde{p}\tilde{\zeta}, \tag{21}$$

$$\tilde{\zeta} = -\partial_{\tilde{p}} G(\alpha, \beta; \zeta, \tilde{p}) \tag{22}$$

$$p = \partial_\zeta G(\alpha, \beta; \zeta, \tilde{p}). \tag{23}$$

If we fix the $\tilde{\zeta} = \tilde{\zeta}_1$ in equation (22), the Eqs.(21) and (22) can be solved (in some region) for (α, β), i.e., $(\alpha, \beta) = (\alpha(g_1, \tilde{p}_1, \zeta; \tilde{\zeta}_1), \beta(g_1, \tilde{p}_1, \zeta; \tilde{\zeta}_1))$ which are then substituted back into Eqs.(21) and (22) yielding

$$g = G^*(g_1, \tilde{p}_1; \tilde{\zeta}_1; \zeta, \tilde{p}), \tag{24}$$

$$\tilde{\zeta} = -\partial_{\tilde{p}} G^*(g_1, \tilde{p}_1; \tilde{\zeta}_1; \zeta, \tilde{p}) \tag{25}$$

$$p = \partial_\zeta G^*(g_1, \tilde{p}_1; \tilde{\zeta}_1; \zeta, \tilde{p}). \tag{26}$$

The (projective) twistor coordinates $(g_1, \tilde{p}_1, \zeta)$ are the local coordinates for the (anti-self-dual) twistor space determined by $\tilde{\zeta} = \tilde{\zeta}_1$. Different coordinate patches (at least one extra is necessary) can be obtained by choosing different values for the fiducial $\tilde{\zeta}$, e.g., $\tilde{\zeta} = \tilde{\zeta}_2$; the overlap transformation is then given implicitly by

$$g_2 = G^*(g_1, \tilde{p}_1; \tilde{\zeta}_1; \zeta, \tilde{p}_2), \tag{27}$$

$$\tilde{\zeta}_2 = -\partial_{\tilde{p}} G^*(g_1, \tilde{p}_1; \tilde{\zeta}_1; \zeta, \tilde{p}_2). \tag{28}$$

The twistor coordinates on the two patches, $(g_1, \tilde{p}_1, \zeta)$ and $(g_2, \tilde{p}_2, \zeta)$, with the overlap transformation *define* the asymptotic twistor space and this space, in turn, parametrizes the twistor curves which rule the "good cuts" or the solution to the "good cut equation".

We thus see that the anti-self dual space-times can be described via either the local twistor point of view, i.e., from equation (18) or from the twistor space point of view, namely from equations (24),(25),(26).

III. The Homogeneous Scri Formalism and 'Googly maps'

In this section we express the correspondence between asymptotic twistor space and Scri in homogeneous coordinates (due originally to Sparling, see Eastwood & Tod 1982 and Tod 1981). We first apply the formalism to give a Lorentz invariant formulation of equations (20). We then calculate the metric of an $\mathcal{H}$-space in terms of the cut function, reformulating the calculations in Ko, Ludvigsen, Newman and Tod in terms of homogeneous coordinates. This can be compared with an expression proposed for the metric by RP in the context of his googly programme. The most basic form of this expression does not work, although the full features of RP's framework are not used so the results are inconclusive. The formula is not in any case central to the programme. A formula that does work (a transcription of the Newman integral formula adapted from one given in Tod 1979) is presented at the end. Although these calculations are inconclusive, the purpose of presenting them is to show how the homogeneous coordinate formulation allows one to do quite detailed calculations in relation to the googly with relative ease in complete generality.

We work with homogeneous coordinates on scri (with apologies for the conflict with earlier notation; the u of this section should be identified with $\pi_{0'}\bar{\pi}_0 u$ from the previous section)

$$(u, \pi_{A'}, \bar{\pi}_A) \sim (\lambda\bar{\lambda}u, \lambda\pi_{A'}, \bar{\lambda}\bar{\pi}_A).$$

These can be related to the coordinates used earlier by setting $\pi_{0'} = 1$ and $\pi_{1'} = \zeta$. This can be tied into the GHP formalism by choosing u so that (thorn)$'u = 1$. The main advantage of this formalism is that it allows one to maintain explicit Lorentz invariance.

We first set up the homogeneous versions of the correspondence between twistor space and null infinity and the dual good cut equation (see also Tod 1981 and eastwood & Tod 1982).

We will confine ourselves to space-times with self-dual Weyl tensor and consider the (flat) asymptotic twistors (i.e. the 'googly' situation). A twistor $(\omega^A, \pi_{A'})$ determines a line in scri given by the equation $iu = \omega^A\bar{\pi}_A$. A 'dual good cut' is determined by a homogeneity (1,1) function $Z(x^a, \pi_{A'}, \bar{\pi}_A)$ by $u = Z$. In flat space $Z(x^a, \pi_{A'}, \bar{\pi}_A) = x^{AA'}\pi_{A'}\bar{\pi}_A$. See equation (17). For agreement with the previous section we have that Z and σ in this section are identified with $\bar{\pi}_0\pi_{0'}Z$ and $\bar{\pi}_0(\pi_{0'})^3\sigma$ from the last. In a self-dual space-time we have that Z satisfies the good cut equation

$$\partial^{A'}\partial^{B'}Z = \pi^{A'}\pi^{B'}\sigma\,, \qquad \partial^{A'} = \partial/\partial\pi_{A'}\,.$$

where the form of the right hand side is determined by the homogeneity of Z and $\sigma = \sigma(Z, \pi_{A'}, \bar{\pi}_A)$ is the Bondi shear. Note that although this is formally 3 equations, it is really just one in the sense that both sides are proportional to $\pi^{A'}\pi^{B'}$.

The twistor generating function for dual good cuts

Solutions to the dual good cut equation were shown to give rise to 2-surfaces in twistor space subject to equation (20) in terms of affine coordinates on twistor space. We now reformulate that equation in terms of homogeneous coordinates. This has the advantage that the equation now has explicit Lorentz invariance, but the drawback that it becomes three equations. This 2-surface will be the envelope of the level surfaces of the googly map corresponding to the dual good cut given below, but will not otherwise relate to the subsequent development.

The dual good cut determines a Legendrian 2-surface in the contact space $PT^*\mathcal{I} = PT^*\mathbb{PT}$ and this can in turn be projected down to give a 2-surface in twistor space. We can represent a 1-parameter family of such 2-surfaces by $G := G(\omega^A, \pi_{A'}) =$ constant. (One can simply just focus on $G = 0$.)

The corresponding two surface in $PT^*\mathcal{I} = PT^*\mathbb{PT}$ can be expressed first in terms of coordinates $(Z^\alpha, W_\beta) = (\omega^A, \pi_{A'}, \mu_A, \lambda^{A'})$ with $Z^\alpha W_\alpha = 0$ by $\mu_A = \partial G/\partial\omega^A$ and $\lambda^{A'} = \partial G/\partial\pi_{A'}$. We can identify $\partial Z/\partial\pi_{A'} = \partial G/\partial\pi_{A'}$ although one should be careful to note that $\partial/\partial\pi_{A'}$ has different meanings on each side of the equation, on the left holding $\bar{\pi}_A = \mu_A$ constant and on the right holding ω^A constant. To impose the dual good cut equation we need to take the $\partial/\partial\pi_{A'}$ derivative holding $\mu_A = \bar{\pi}_A$ constant. When expressed in twistor coordinates, this derivative operator becomes

$$\frac{\partial}{\partial\pi_{A'}} - \frac{2}{G_{CD}G^{CD}}G^{AB}\frac{\partial^2 G}{\partial\omega^B\partial\pi_{A'}}\frac{\partial}{\partial\omega^A} \quad \text{where} \quad G_{AB} = \frac{\partial^2 G}{\partial\omega^A\partial\omega^B}\,.$$

Thus the dual good cut equation becomes

$$\frac{\partial^2 G}{\partial\pi_{A'}\partial\pi_{B'}} - \frac{2}{G_{CD}G^{CD}}G^{AB}\frac{\partial^2 G}{\partial\omega^B\partial\pi_{A'}}\frac{\partial^2 G}{\partial\omega^A\partial\pi_{B'}} - \pi^{A'}\pi^{B'}\sigma = 0 \tag{29}$$

where $\sigma := \sigma\left(\pi_{C'}\frac{\partial G}{\partial\pi_{C'}}, \pi_{A'}, \frac{\partial G}{\partial\omega^A}\right)$. Note that, as above, this is actually one equation being proportional to $\pi^{A'}\pi^{B'}$.

Googly maps

The dual good cut determines a map from twistor space to the unprimed spin space, $Z^\alpha \mapsto \tilde{\pi}^A$ which is given projectively by mapping the twistor to the value of $\tilde{\pi}_A$ at which its line hits the good cut. Thus $\tilde{\pi}^A$ is determined implicitly by $\omega^A\tilde{\pi}_A = iZ(x, \pi_{A'}, \tilde{\pi}_A)$. The scale of this map can be fixed by imposing

$$\tilde{\pi}^A = \omega^A - i\tilde{\partial}^A Z\,, \tag{30}$$

where $\tilde{\partial}^A = \partial/\partial\tilde{\pi}_A$. This gives $\tilde{\pi}^A$ as an implicit function of $(x, \omega^A, \pi_{A'})$. This is the homogeneous 'googly map' (as obtained by K.P.Tod in TN9). Note that although this choice of scaling is canonical, it is not clear that it is unique or indeed the 'correct' one in the context of the googly construction especially in view of RP's proposal to deform the scalings on the twistor space which would certainly require some alteration to the above formula. Nevertheless, one can see that a change in the prescribed scalings is unlikely to improve the situation described below, and the purpose of this calculation is to show firstly that this is not the likely to lead to the correct formulation, and secondly how relatively straightforward this formalism is for testing such conjectures.

Penrose recently conjectured a formula for the space-time metric in terms of these structures. He introduces a 1-form $\xi = \tilde{\pi}_A d\tilde{\pi}^A$ thought of as a 1-form on twistor space depending on the space-time point x. To obtain the metric, consider a tangent vector v at x, then introduce a small displacement operator

$$\delta = v^a \frac{\partial}{\partial x^a}\bigg|_{\omega^A, \pi_{A'}} \tag{31}$$

in the direction of v^a while keeping the twistor $(\omega^A, \pi_{A'})$ constant. Note that thus far partial x^a-derivative implicitly meant keeping $(\tilde{\pi}_A, \pi_{A'})$ constant.

Now define $\tilde{\pi}'_A \equiv \delta(\tilde{\pi}_A)$. Then RP's earlier proposal was that the metric be obtained from the formula

$$d(\tilde{\pi}'_A d\tilde{\pi}'^A)\wedge d\xi = g(v,v)\Phi\,, \qquad 24\Phi = \varepsilon_{\alpha\beta\gamma\delta}dZ^\alpha dZ^\beta dZ^\gamma dZ^\delta$$

This formula can be checked directly using the homogeneous scri formalism as follows. Firstly we need to know what the metric actually is. We have that, holding π and $\tilde{\pi}$ fixed, Z is constant on null hypersurfaces. Hence

$$g^{ab}Z_{,a}Z_{,b} = 0\,, \qquad Z_{,a} = \partial Z/\partial x^a$$

We can differentiate this with respect to π and $\tilde{\pi}$. To introduce some notation, note that if we set

$$Z^A = \tilde{\partial}^A Z, \quad Z^{A'} = \partial^{A'} Z\,, \quad Z^{AA'} = \tilde{\partial}^A\partial^{A'} Z\,,$$

then by homogeneity

$$\pi_{A'}Z^{AA'} = Z^A\,, \quad \tilde{\pi}_A Z^{AA'} = Z^{A'}\,, \qquad \tilde{\partial}^A\tilde{\partial}^B Z = \tilde{\pi}^A\tilde{\pi}^B\tilde{\Lambda}$$

where the last equation defines $\tilde{\Lambda}$ as a function of x and π and $\tilde{\pi}$. Recall also that the **dual** good cut equation implies

$$\partial^{A'}Z^{B'} = \pi^{A'}\pi^{B'}\sigma.$$

First, differentiating twice with respect to π and using the good cut equation we find that

$$g^{ab}\partial^{A'}Z_{,a}\partial^{B'}Z_{,b} = 0, \qquad \text{since} \quad g^{ab}Z_{,a}\sigma_{,b} = \dot{\sigma}g^{ab}Z_{,a}Z_{,b} = 0\,.$$

Differentiating twice with respect to $\tilde{\pi}$ we get the following nine equations that, at a fixed value of π and $\tilde{\pi}$ determine the conformal structure

$$g^{ab}Z^{C(C'}{}_{,a}Z^{D')D}{}_{,b} = -\tilde{\pi}^C\tilde{\pi}^D\tilde{\Lambda}^{C',D'}\,, \quad \tilde{\Lambda}^{C',D'} = g^{ab}\partial^{(C'}\tilde{\Lambda}_{,a}Z_{,b}^{D')}\,.$$

Thus a scale for g^{ab} exists such that

$$g^{ab}Z^{CC'}{}_{,a}Z^{DD'}{}_{,b} = \varepsilon^{CD}\varepsilon^{C'D'} - \tilde{\pi}^C\tilde{\pi}^D\tilde{\Lambda}^{C',D'}$$

and this scale turns out to be the one that gives a Ricci flat space-time. One can therefore see that

$$\theta^{AA'} = Z_{,c}^{AA'}dx^c + \frac{1}{2}\tilde{\pi}^A\tilde{\Lambda}^{A'}{}_{B'}Z_{,c}^{B'}dx^c$$

is an orthonormal tetrad so that the metric can be given as

$$g_{cd} = \varepsilon_{AB}\varepsilon_{A'B'}Z_{,c}^{AA'}Z_{,d}^{BB'} + \tilde{\Lambda}_{A'B'}Z_{,c}^{A'}Z_{,d}^{B'}\,.$$

Now we compute RP's expression to see whether we get the above expression for the metric. Keeping the definition in (31) mind, it is not difficult to show that

$$d\tilde{\pi}^A = (i\Lambda\tilde{\pi}^A\tilde{\pi}_B - \varepsilon^A{}_B)(d\omega^B - iZ^{BB'}d\pi_{B'}),$$

and $d\xi := \varepsilon_{AB}d\tilde{\pi}^A \wedge d\tilde{\pi}^B = \nu_A \wedge \nu^A$, where $\nu^A = d\omega^A - iZ^{AA'}d\pi_{A'}$. Applying the displacement δ gives

$$d\xi \wedge d(\tilde{\pi}'_A d\tilde{\pi}'^A) = \nu_A \wedge \nu^A \wedge \delta\nu_A \wedge \delta\nu^A$$

where $\delta\nu^A = i(Z^{AA'}_{,a} - iZ_{,a}\tilde{\pi}^A\Lambda^{A'})d\pi_{A'}$. This yields the metric expression

$$g_{cd} = \epsilon_{AB}\epsilon_{A'B'}Z^{AA'}_{,c}Z^{BB'}_{,d} - 2iZ^{AA'}_{,(c}Z_{,d)}\tilde{\pi}_A\Lambda_{A'}.$$

Unfortunately one does not pick up the part $v^c v^d \tilde{\Lambda}_{A'B'}Z^{A'}{}_{,c}Z^{B'}{}_{,d}$ of the metric that distinguishes the metric from the flat one. It might be argued that when the scalings are introduced, the above formula changes in such a way as to pick up the missing term. This seems unlikely in view of the fact that the scalings effectively involve $\partial^3\sigma/\partial u^3$ and this is of a different character to the term that is missing. So these arguments are inconclusive and more work is required to pin down these issues (finding the correct specification of the scalings for googly maps is still an open problem). Although this particular calculation is not encouraging, this formalism has lead to the twistorial reformulation of the Newman integral formula for the metric.

Let ξ be the 1-form corresponding to a point x, $\xi' = \mathcal{L}_\delta\xi$ and $\iota = \pi_{A'}d\pi^{A'}$. Then it is a simple consequence of googly geometry that the form $\xi \wedge \xi' \wedge \iota$ has to be proportional to $\theta = \frac{1}{6}\varepsilon_{\alpha\beta\gamma\delta}Z^\alpha dZ^\beta dZ^\gamma dZ^\delta$, the standard volume 3-form on the projective twistor space

$$\xi \wedge \xi' \wedge \iota = \gamma\theta.$$

The metric is then given by the following integral:

$$\frac{16\pi^2}{g(\delta x^a, \delta x^a)} = \oint \frac{\theta}{\gamma^2}.$$

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