32

Real Lorentzian metrics from complex, half flat solutions

David Robinson
Mathematics Department, King's College London,
Strand, London WC2R 2LS

In the 1960's Roger Penrose showed that all real Lorentzian solutions of Einstein's vacuum field equations could, in the approximation where fields are linearized about the Minkowski solution, be constructed from complex half flat solutions, [1]. The extent to which this result can be extended to the full non-linear theory is unknown. Real Lorentzian solutions have been constructed from complex half flat ones but they are all algebraically special and quite simple: see for example, [2] and references therein. The aim of this note is to outline some methods of constructing real Lorentzian metrics from complex half-flat ones. Natural ways to construct, locally, real 2- and 3-differential forms (which can be extended to n- forms, $2 \preceq n \preceq 8$) by using half flat solutions will be exhibited. These real forms, on a complex four manifold M, will be used to construct real metrics on real four manifolds, N, cf also [3]. Hence real metrics on N will be constructed, locally, from complex half flat metrics on M.

By using the type of formalism developed by Plebanski, complex half flat metrics on M can be represented, in terms of a basis of complex co-frames $\chi^{\mathbf{A}A'}$, by a line element

$$ds^2 = \epsilon_{\mathbf{AB}}\epsilon_{A'B'}\chi^{\mathbf{A}A'} \otimes \chi^{\mathbf{B}B'} \tag{1}$$

and the Cartan structure equations

$$d\chi^{\mathbf{A}A'} - \chi^{\mathbf{B}A'} \wedge \omega^{\mathbf{A}}_{\mathbf{B}} - \chi^{\mathbf{A}B'} \wedge \varpi^{A'}_{B'} = 0, \tag{2}$$

$$\Omega^{\mathbf{A}}_{\mathbf{B}} \equiv d\Gamma^{\mathbf{A}}_{\mathbf{B}} + \Gamma^{\mathbf{A}}_{\mathbf{C}} \wedge \Gamma^{\mathbf{C}}_{\mathbf{B}} = \frac{1}{2}\Psi^{\mathbf{A}}_{\mathbf{BCD}}\chi^{\mathbf{C}}_{A'} \wedge \chi^{\mathbf{D}A'} \tag{3}$$

$$d\varpi^{A'}_{B'} + \varpi^{A'}_{C'} \wedge \varpi^{C'}_{B'} = 0. \tag{4}$$

Here, the anti-self dual and self-dual components of the Levi-Civita spin connection are given, respectively, by $\omega^{\mathbf{A}}_{\mathbf{B}}$, with Weyl spinor $\Psi^{\mathbf{A}}_{\mathbf{BCD}}$, and the flat $\varpi^{A'}_{B'}$. Upper case Latin bold, un-primed, and Latin normal, primed, indices represent transformation properties under the structure groups SL(2,C)$_L$ and SL(2,C)$_R$ respectively. Upper case Latin indices sum and range over 0 and 1.

It follows from the above equations that

$$\Omega^{\mathbf{A}}_{\mathbf{B}} \wedge \chi^{\mathbf{B}B'} = 0. \tag{5}$$

It is useful to recall at this point that equations like Eqs. (2) and (5) constitute a simple representation of the Einstein vacuum condition.

Real forms on M, related to half-flat geometries, can be constructed by using the co-frame $\chi^{\mathbf{A}A'}$ and its complex conjugate, $\bar{\chi}^{AA'}$. First consider the real 2-forms $\sigma^{\mathbf{a}}$ (i.e. the Hermitian matrix-valued 2-form $\sigma^{\mathbf{AA'}}$) defined by

$$\sigma^{\mathbf{AA'}} = i\chi^{\mathbf{A}A'} \wedge \bar{\chi}^{\mathbf{A}A'}\kappa_{AA'}. \tag{6}$$

(Lower case Latin indices represent transformation properties under SO(1,3)$_L$ = $\{SL(2,C)_L \times c.c.\}/\mathbb{Z}_2$ and range and sum over 1 to 4.)

Here the Hermitian matrix-valued function $\kappa_{AA'}$ is covariantly constant with respect to the real flat so(1,3)$_R$-valued connection represented by $\varpi^a_b \leftrightarrow \delta^A_B \varpi^{A'}_{B'} + \delta^{A'}_{B'} \bar{\varpi}^A_B$, that is

$$d\kappa_{AA'} - \kappa_{AB'} \wedge \varpi^{B'}_{A'} - \kappa_{BA'} \wedge \bar{\varpi}^B_A = 0, \tag{7}$$

and so the $\kappa_{AA'}$ label a four-parameter family of 2-forms $\sigma^{\mathbf{AA'}}$. These 2-forms are compatible with the real so(1,3)$_L$-valued connection $\omega^{\mathbf{a}}_{\mathbf{b}} \leftrightarrow \delta^{\mathbf{A}}_{\mathbf{B}} \bar{\omega}^{\mathbf{A'}}_{\mathbf{B'}} + \delta^{\mathbf{A'}}_{\mathbf{B'}} \omega^{\mathbf{A}}_{\mathbf{B}}$, since it follows from the above equations that the covariant exterior derivative of $\sigma^{\mathbf{a}}$, with respect to the latter connection, is zero i.e.

$$D\sigma^{\mathbf{a}} \equiv d\sigma^{\mathbf{a}} + \sigma^{\mathbf{b}} \wedge \omega^{\mathbf{a}}_{\mathbf{b}} = 0. \tag{8}$$

Furthermore, it also follows from the above that

$$\Omega^{\mathbf{A}}_{\mathbf{B}} \wedge \sigma^{\mathbf{BA'}} = 0 \tag{9}$$

and similarly for the complex conjugate equation. Note that from Eq.(8), it follows that, for any n-3 forms $\pi_{\mathbf{a}}$, the n-form $D\pi_{\mathbf{a}} \wedge \sigma^{\mathbf{a}}$ is closed. In a similar way, p-forms, with $3 \preceq p \preceq 8$ which satisfy equations like (8) and (9), can be constructed. All these equations can be pulled back to sub-manifolds of M to define geometries on them, in particular on four dimensional real submanifolds, N. Here only the 2-form equations above and the similar equations for 3-forms will be considered. The latter can be obtained and written in the following way. Let $\rho^{\mathbf{a}} \leftrightarrow \rho^{\mathbf{AA'}}$ be real 3-forms on M, defined by

$$\rho^{\mathbf{AA'}} = i\chi^{\mathbf{A}A'} \wedge \bar{\chi}^{A\mathbf{A'}} \wedge r_{AA'} \tag{10}$$

where $r_{AA'}$ is taken to be a Hermitian matrix valued 1-form which has zero exterior covariant derivative with respect to the flat so(1,3)$_R$ -valued connection ϖ^a_b, that is

$$dr_{AA'} + r_{BA'} \wedge \overline{\varpi}^B_A + r_{AB'} \wedge \varpi^{B'}_{A'} = 0. \tag{11}$$

By choice of gauge, $r_{AA'}$ can be chosen to be closed. It follows, once again from the above equations that the real 3-forms are compatible with the real so(1,3)$_L$-valued connection $\omega^{\mathbf{a}}_{\mathbf{b}}$, and

$$D\rho^{\mathbf{a}} \equiv d\rho^{\mathbf{a}} - \rho^{\mathbf{b}} \wedge \omega^{\mathbf{a}}_{\mathbf{b}} = 0 \;\; ; \;\; \Omega^{\mathbf{A}}_{\mathbf{B}} \wedge \rho^{\mathbf{BA'}} = 0. \tag{12}$$

Now constructions leading to 4-geometries, involving the above 2-forms, $\sigma^{\mathbf{a}}$, and then the above 3- forms,$\rho^{\mathbf{a}}$, will be considered.

First the 2- forms $\sigma^{\mathbf{a}}$ will be used to construct real 1-forms, $\varsigma^{\mathbf{a}}$, on M. When these 1-forms can be pulled back to define a real co-frame, $\theta^{\mathbf{a}}$,on a real four manifold N, locally embedded in M, they determine a Lorentz metric on N

$$ds^2 = \eta_{\mathbf{ab}}\theta^{\mathbf{a}} \otimes \theta^{\mathbf{b}}, \tag{13}$$

with Levi-Civita connection the pull-back of $\omega^{\mathbf{a}}_{\mathbf{b}}$, and anti-self dual curvature two-form the pull-back of $\Omega^{\mathbf{A}}_{\mathbf{B}}$. The properties of such metrics will be determined by the original half-flat metric, the embedding map and $\kappa_{AA'}$. Secondly, the real 3-forms $\rho^{\mathbf{a}}$ will be used to construct real 1-forms, $\theta^{\mathbf{a}}$, and a Lorentzian metric on N in a different way. (Alternative procedures, using similar ideas are possible. In particular some apply when the initial complex metric is defined on a real four manifold.)

In the case of the 2-forms, real one-forms, $\varsigma^{\mathbf{a}} = X \rfloor \sigma^{\mathbf{a}}$, can be constructed by contraction with a real vector field, X, on M. When the forms and vector fields satisfy appropriate conditions these one-forms will satisfy the equations on M

$$d\varsigma^{\mathbf{a}} - \varsigma^{\mathbf{b}} \wedge \omega^{\mathbf{a}}_{\mathbf{b}} = 0, \tag{14}$$

$$\Omega^{\mathbf{A}}_{\mathbf{B}} \wedge \varsigma^{\mathbf{BA'}} = 0. \tag{15}$$

A set of (non-gauge covariant) conditions on X which lead to Eqs. (14) is given by

$$\pounds_X \sigma^{\mathbf{a}} = f^{\mathbf{a}}_{\mathbf{b}}\sigma^{\mathbf{b}}, \quad X\rfloor\omega^{\mathbf{a}}_{\mathbf{b}} = -f^{\mathbf{a}}_{\mathbf{b}}, \tag{16}$$

for some real functions $f^{\mathbf{a}}_{\mathbf{b}}$. If, furthermore, X also satisfies

$$\pounds_X \omega^{\mathbf{a}}_{\mathbf{b}} = -Df^{\mathbf{a}}_{\mathbf{b}}, \tag{17}$$

Eq.(15) holds too. When the 1-forms, $\varsigma^{\mathbf{a}}$, can be pulled back to a co-frame $\theta^{\mathbf{a}}$ on N, the pull backs of Eqs.(14) and (15) imply that the metric, given by Eq.(13), satisfies the Einstein vacuum field equations. However, since $X\rfloor\varsigma^{\mathbf{a}} = 0$, the construction may in fact lead to solutions of the vacuum constraint equations on a three manifold which is a sub-manifold of N.

In the case of the 3-forms, $\rho^{\mathbf{a}}$, it is natural to construct metrics on N by using the duality of vector densities and 3-forms in four dimensions. When $\rho^{\mathbf{a}}$ can be pulled back to a basis of 3-forms on N, $\tau^{\mathbf{a}}$ say, then there exist real 1-forms $\theta^{\mathbf{a}}$ on N (and hence a Lorentz metric as in Eq.(13) above) such that $\tau^{\mathbf{a}} = \frac{1}{6}\epsilon^{\mathbf{a}}_{\mathbf{bcd}}\theta^{\mathbf{b}} \wedge \theta^{\mathbf{c}} \wedge \theta^{\mathbf{d}}$. It follows from the pull-backs to N of Eq.(12) that

$$d\theta^{\mathbf{a}} - \theta^{\mathbf{b}} \wedge \omega^{\mathbf{a}}_{\mathbf{b}} = \Theta^{\mathbf{a}}, \tag{19}$$

$$\Theta^{\mathbf{a}} = \frac{1}{2}\Theta^{\mathbf{a}}_{\mathbf{bc}}\theta^{\mathbf{b}} \wedge \theta^{\mathbf{c}}, \quad \Theta^{\mathbf{a}}_{\mathbf{bc}} = -\Theta^{\mathbf{a}}_{\mathbf{cb}}, \quad \Theta^{\mathbf{a}}_{\mathbf{ba}} = 0, \tag{20}$$

$$d\tau^{\mathbf{a}} - \tau^{\mathbf{b}} \wedge \omega^{\mathbf{a}}_{\mathbf{b}} = 0. \tag{21}$$

Eqs. (19-21) relate real metrics on N to the pull back of the connection $\omega^{\mathbf{a}}_{\mathbf{b}}$ (also written $\omega^{\mathbf{a}}_{\mathbf{b}}$), and encode, at least in part, the half-flat geometry on M. Explicit examples of all the above geometries and further investigations of the above, including the role of conformal freedom in the last construction and the relation to linearized theory, will be presented elsewhere.

References

1. Penrose R. (1965) Proc. Roy. Soc., London Ser., **A284**,159
2. Robinson D.C. (1987) Gen. Rel. & Grav., **19**, 693
   Plebański J.F., Garcia -Compean H. & Garcia-Diaz A. (1995) Class. Quantum Grav., **12**, 1093.
   Plebański J.F., Przanowski M, Formański S.(1998) *Linear superposition of two type-N non-linear gravitons*
   Robinson D.C. (1998) Twistor Newsletter **44**, 10
3. Rozga K. (1977) Rep. Math.Phys., **11**, 197
   Woodhouse N. (1977) Int. J. Theor. Phys., **16**, 663