Combinatorics and field theory

For any given sequence of integers there exists a quantum field theory whose Feynman rules produce that sequence. An example is illustrated for the Stirling numbers. The method employed here offers a new direction in combinatorics and graph theory.

In quantum field theory graphs are used to represent the terms in a diagrammatic perturbation expansion, whereby one associates with each graph a numerical amplitude. It is known [1] that there is an elegant formal construction for the set of all graphs in terms of the exponential of a derivative operator. For example, $Z(\epsilon, g) = \exp(\frac{1}{2!}\epsilon d^2/dx^2)\exp(\frac{1}{4!}gx^4)|_{x=0}$ is the generating function for the set of all 4-vertex vacuum diagrams, connected and disconnected, in which the line amplitude is ϵ and the vertex amplitude is g. Here, for simplicity, we consider field theories in zero-dimensional space so that Feynman integrals become trivial and are merely the product of the line amplitudes.

In analogy with statistical mechanics we refer to the sum of all vacuum diagrams as the *partition function.* The power series expansion of a generic partition function contains both connected and disconnected graphs. If we are interested only in connected graphs, then we consider instead the *free energy* $F = -\ln Z$; the coefficients of the power series expansion of F represent the sum of the symmetry numbers of just the connected graphs [2], where the symmetry number of a graph is defined as the reciprocal of the number of ways in which the graph can be turned into itself by permuting the lines or vertices. These are the basic rules for graphs in field theories. Using these rules, we would like to find field theories whose diagrammatic expansions correspond to graphs that are meaningful in combinatorics. A simple but intriguing example is the field theory of partitions.

Field theory of partitions

The partition of an integer n is the set of all distinct ways to represent n as a sum of positive integers smaller than or equal to n. The number of elements in the partition of n is designated P_n. Defining $P_0 = 1$, the first few P_n are $1,\ 1,\ 2,\ 3,\ 5,\ 7,\ 11, \cdots$. In the case of partitioning of labelled objects, the number of partitions is given by the Bell numbers $\{B_n\} = 1,\ 1,\ 2,\ 5,\ 15,\ 52,\ 203, \cdots$. The labelled partitions can also be grouped into classes characterised by the Stirling numbers $S(n,k)$, which count the number of partitions of n labelled objects into k groups [3]. Specifically, we have $S(1,1) = 1$, $S(2,1) = S(2,2) = 1$, $S(3,1) = 1$, $S(3,2) = 3$, $S(3,3) = 1$, and so on. Clearly, if we sum $S(n,k)$ over k, we recover the Bell numbers: $B_n = \sum_k S(n,k)$.

There is an elementary field theory whose Feynman rules produce precisely the labelled partitions into groups [4]. This is given by the potential energy $V(x) = g(e^x - 1)$ and the *line insertion operator* $D = \epsilon d/dx$ for a grouping variable g and a line amplitude ϵ:

$$Z(\epsilon, g) = \exp\left(\epsilon\frac{d}{dx}\right)\exp\left[g(e^x-1)\right]\Big|_{x=0} = \sum_{n=0}^{\infty}\frac{\epsilon^n}{n!}\left(\sum_{k=1}^{n} S(n,k)g^k\right).$$

If we set $g = 1$, then this reduces to the field theory of labelled partitions (the Bell numbers). Note that the line insertion operator in the above example is given by a first order derivative operator d/dx. Graphically, such propagators correspond to lines having only one end. This is a strange kind of line; ordinarily, a line has two ends. However, in the field theory, we can have generalised lines having multiple ends.

Generalised partitions

Consider a general field theory that includes all n-point vertices as well as all generalised lines having m ends. This leads to the expression

$$Z(L,V) \;=\; \exp\left(\epsilon \sum_{m=1}^{\infty} \frac{L_m}{m!}\frac{d^m}{dx^m}\right) \exp\left(g \sum_{n=1}^{\infty} \frac{V_n}{n!} x^n\right)\Bigg|_{x=0} ,$$

which represents the set of all vacuum diagrams, connected and disconnected, constructed from n-point vertices whose amplitudes are V_n and generalised lines having m legs whose amplitudes are L_m. If we expand this expression as a formal series in powers of ϵ, then the coefficient of ϵ^n is the sum of the symmetry numbers of all graphs having n lines, and if we expand this expression as a series in powers of g, then the coefficient of g^n is the sum of the symmetry numbers of all graphs having n vertices. We note that in general these formal power series are divergent because the number of graphs grows like a factorial.

In particular, if we set $g = 1$ and $L_m = V_n = 1$, then we obtain the following beautiful result on generalised partitions:

$$Z(\epsilon) \;=\; \exp\left(e^{\epsilon \frac{d}{dx}} - 1\right) \exp\left(e^{x} - 1\right)\Bigg|_{x=0} \;=\; \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} B_n^2 \,.$$

The graphs contributing to this expression, up to order ϵ^4, are shown in Fig. 1 below.

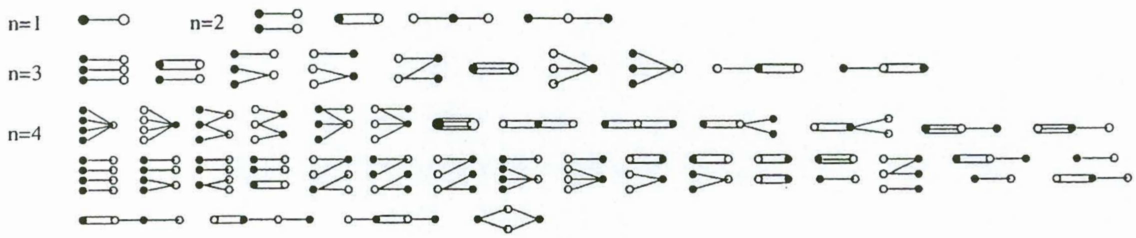

FIG. 1. Graphs in a theory whose Feynman rules allow for n-point vertices ($n = 1,\ 2,\ 3, \cdots$) and m-legged lines ($m = 1,\ 2,\ 3, \cdots$). If the vertex amplitudes are all unity and the m-legged line amplitude is ϵ^m, then the generating function $Z(\epsilon)$ for the graphs has a Taylor expansion for which the coefficient of ϵ^n is $B_n^2/n!$. Thus, the number of labelled graphs of order n is the square of the nth Bell number.

It is interesting that the graphs of generalised lines in the above figure have formal resemblance to the graphs in twistor diagrams. These generalised lines are, however, quite natural in the context of quantum field theory. For example, the strong-coupling expansion of the Lagrangian for quantum chromodynamics are known to involve these multilegged propagators. Note that the number of these graphs form a sequence $1,\ 1,\ 4,\ 10,\ 33, \cdots$. We do not know how to generate this sequence, however, if we label the vertices in Fig. 1, then the number of graphs for generalised labelled partitions are given by the square of the Bell numbers.

Topology numbers

We have illustrated how the Feynman rules in quantum field theory can naturally be associated with ideas in combinatorics. The structures we introduced above are, however, quite primitive in the sense that we worked only in zero-dimensional space, and we have not introduced Fermion lines, which give rise to directed graphs. Despite such simplicity in the underlying field theory, the corresponding graphical expansions are already quite intricate. This suggests that, by introduction of additional structures (or perhaps even without such extensions) we might be able to obtain partition functions that would generate unknown, crucially important integer sequences such as topology numbers or partially ordered sets (posets).

Although we do not know how this might be achieved, in the following we would like to sketch the line of thinking involved in such problems. Here, let us consider the labelled topologies. The n-th topology number can be represented by unlabelled transitive graphs of n nodes. By transitivity, we mean if α and β are related and β and γ are related, then α and γ also have to be related. Some of the graphs are shown in Fig. 2.

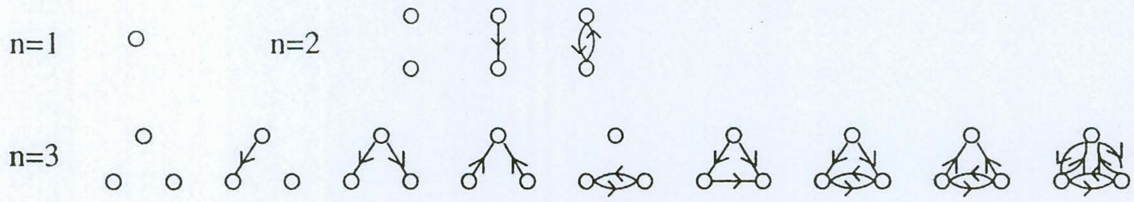

FIG. 2. Connected and disconnected graphs representing topology numbers: 1, 1, 3, 9, 33, 139, 718, 4535, $\cdots$. Only eight terms of the sequence are known.

Since disconnected graphs can be obtained by exponentiating the connected ones in the sense noted above, let us consider only the connected ones (free energy). The labelled connected topologies are then given by the sum of the symmetry numbers of the connected transitive graphs. Because we are interested in the symmetry numbers, all the transformations of the graph that preserve the symmetry numbers are allowed. Thus, we may simplify the above graphs in the following manner. The first step is to replace the double lines in Fig. 2 by single 'Bosonic' lines. The connected graphs obtained after this replacement are shown in Fig. 3.

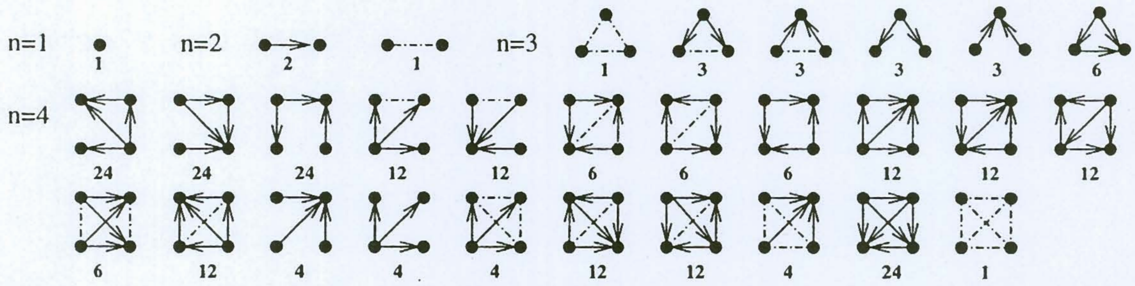

FIG. 3. Graphs representing connected topologies, after replacing the double arrows in Fig. 2 by single Bosonic lines. The numbers are the associated symmetry numbers times $n!$.

The second step is to 'squash' all the Bosonic lines, namely, any graph having n vertices joined by Bosonic lines can be squashed into a single n-vertex. The symmetry number is preserved by associating the factor of $1/n!$ to each n-vertex. The final step is then to lift the transitivity in the sense that, if α is related to β and β is related to γ, then the redundant relation joining α and γ by a 'Fermion' line (arrow) can be removed. It can easily be seen that this simplification does not alter the symmetry numbers. After these transformations, the simplified graphs appear to be identical to the connected posets. An example for $n = 4$ in Fig. 3 is shown in Fig. 4.

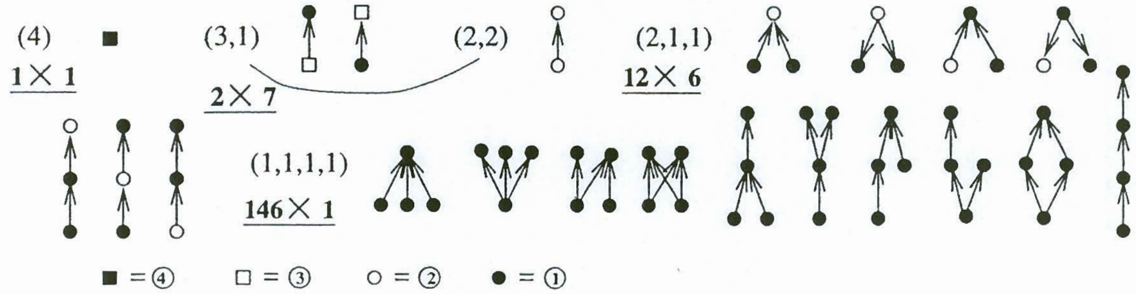

FIG. 4. The 21 graphs for $n = 4$ in Fig. 3 reduce to these graphs, which are just the connected posets. In particular, the squashing of Bosonic lines gives five partitions of 4. The corresponding Stirling numbers are 1, 7, 6, 1 respectively for one, two, three, and four groups to partition 4. The number of connected labelled posets, on the other hand, are given by 1, 2, 12, 146, and so on. Therefore, we deduce that the number of connected labelled topologies for $n = 4$ is given by $1 \times 1 + 2 \times 7 + 12 \times 6 + 146 \times 1 = 233$, without counting the symmetry numbers indicated in Fig. 3.

The number of graphs corresponding to Fig. 4, if we label the vertices, can be determined by counting the associated symmetry numbers. However, we can deduce this number without such a procedure, because the graphs in Fig. 4 are in fact precisely the connected posets, and the Stirling numbers tells us how many ways we can partition the number in the group, as stated in the figure caption. Therefore, if we let $\{d_n\} = 1,\ 2,\ 12,\ 146,\ 3060,\ 101642, \cdots$ denote connected labelled posets and $\{t_n\} = 1,\ 3,\ 19,\ 233,\ 4851,\ 158175, \cdots$ denote the connected labelled topologies, we deduce the following formula:

$$t_n = \sum_{k=1}^{n} S(n,k) d_k \ .$$

By exponentiating the foregoing arguments in the sense noted earlier, we can show that the same identity via Stirling numbers also holds for disconnected graphs, a formula known in combinatorics [3].

In the above discussion on topology numbers our line of thinking in simplifying the problem is motivated by field theoretic ideas. The challenging problem is to find a field theory whose Feynman rules produce graphs corresponding to those in Fig. 4. This may be achieved by introducing Fermions, Wick ordering, and so on. We note that only the first 14 terms in the sequences $\{d_n\}$, $\{t_n\}$ are known.

1. C. M. Bender and W. E. Caswell, J. Math. Phys. **19**, 2579 (1978).

2. C. M. Bender, *et. al.*, Phys. Rev. D **20**, 1374 (1979).

3. L. Comtet, *Advanced Combinatorics* (Reidel, Dordrecht, 1974).

4. C. M. Bender, D. C. Brody, and B. K. Meister, (submitted to JMP, 1999).

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