

## Classical fields as statistical states

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We present a rough outline for an idea that characterises the observed, macroscopic realisation of the electromagnetic field in terms of a probability distribution on the underlying quantum electrodynamic state space.

In this note we sketch out some tentative thoughts on the relation between microscopic and macroscopic states in quantum theory. The idea is that classical fields (e.g., electromagnetism, or weak-field gravity) should be thought of as *statistical* states. Our starting point is to take the view that the physical world is inherently quantum mechanical. The problem is thus not how to quantise a given classical theory, but rather how to classicalise the quantum theory appropriately at large scales. Starting with a general multi-particle quantum system characterised by a large state space and a myriad of associated observables, our task is to specify those states of the system that correspond, in some reasonable sense, to the classical macroscopic configurations observed in practice.

This is an interesting question, because it ties in with some of the major open issues in quantum theory that may be of relevance to practical considerations. Even in the case of electromagnetism, it has to be appreciated that there is no general agreement as to what constitutes the precise relation between *microscopic* and *macroscopic* realisations of the electromagnetic field, despite the fact that classical and quantum electrodynamics are both well developed theories. Sometimes it is suggested that the coherent states of electromagnetism correspond to classical electrodynamic fields, but the arguments supporting this idea are not entirely convincing. Coherent states, to be sure, are in one-to-one correspondence with classical solutions of Maxwell's equations—in particular, to nonsingular, normalisable solutions. These states also have the property that they saturate the uncertainty lower bounds for measurements of the field operator. However, there is no explanation for why this should be a 'natural' configuration for the electromagnetic state space. Since coherent states are pure *quantum* states, we are left to wonder if it is possible that these states could *remain* pure on a macroscopic scale. This is questionable.

To put the matter another way, we expect a pure state to have low entropy by any reasonable definition, whereas for the quasi-stable nature of a classical field configuration, a high-entropy state would be the more plausible candidate. Then we could invoke some form of the second law of thermodynamics to explain the natural occurrence of such configurations.

Now let us try to build up a model along these lines in more precise terms. Suppose we consider a complex Hilbert space  $\mathcal{H}$ , for which we denote the associated multi-particle bosonic Fock space  $\mathcal{F}$ . We use Greek indices for elements of  $\mathcal{H}$ , and Roman indices for elements of  $\mathcal{F}$ . Thus, if  $\xi^\alpha \in \mathcal{H}$  and  $\eta_\alpha \in \mathcal{H}^*$  (the dual space), then for their inner product we write  $\eta_\alpha \xi^\alpha$ . Likewise, if  $\Psi^a \in \mathcal{F}$  and  $\Phi_a \in \mathcal{F}^*$ , then we can form the inner product  $\Phi_a \Psi^a$ . An element  $\Psi^a$  of  $\mathcal{F}$  is given, more explicitly, by a normalisable set of symmetric tensors in the space  $(\mathbb{C}, \mathcal{H}, \mathcal{H} \otimes \mathcal{H}, \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}, \dots)$  given by  $\Psi^a = \{\psi, \psi^\alpha, \psi^{(\alpha\beta)}, \psi^{(\alpha\beta\gamma)}, \dots\}$  with the inner product

$$\Phi_a \Psi^a = \phi\psi + \phi_\alpha \psi^\alpha + \phi_{\alpha\beta} \psi^{\alpha\beta} + \phi_{\alpha\beta\gamma} \psi^{\alpha\beta\gamma} + \dots \quad (1)$$

We have in mind, in particular, the case where  $\mathcal{H}$  is the Hilbert space of positive-frequency

square-integrable solutions of Maxwell's equations, equipped with the usual gauge independent Hermitian inner product. Then  $\mathcal{F}$  is the multi-particle photon space of quantum electrodynamics, and the general element of  $\mathcal{F}$  determines a superposition of states consisting of various numbers of photons, where the photon number is the rank of the corresponding tensor. In addition we require a specification of the electromagnetic field operators. The creation operator  $C_{\alpha b}^a$  is a map from  $\mathcal{F}$  to  $\mathcal{F} \otimes \mathcal{H}^*$  and the annihilation operator  $A_b^{\alpha a}$  is a map from  $\mathcal{F}$  to  $\mathcal{F} \otimes \mathcal{H}$ . They satisfy the commutation relations  $C_{\beta b}^a A_c^{\alpha b} - A_b^{\alpha a} C_{\beta c}^b = \delta_{\beta}^{\alpha} \delta_c^a$ . The specific actions of  $C$  and  $A$  on  $\mathcal{F}$  (see Geroch 1971) are not required here.

The pure states of quantum electrodynamic systems are not represented by elements of  $\mathcal{F}$ , but rather by points in the associated *projective* Fock space  $\Gamma$ . Let  $x$  denote a typical point in  $\Gamma$ , and  $\Psi^a(x)$  a point in the fibre above  $x$ . Here, we think of  $\mathcal{F}$  as a fibre space over  $\Gamma$ . Then the expectation of the annihilation operator  $A_b^{\alpha a}$ , conditional to a pure quantum state  $x$ , can be represented by a map  $A^\alpha(x)$  from  $\Gamma$  to  $\mathcal{H}$ , given by

$$A^\alpha(x) = \frac{\bar{\Psi}_a(x) A_b^{\alpha a} \Psi^b(x)}{\bar{\Psi}_c(x) \Psi^c(x)}. \quad (2)$$

Note that  $A^\alpha(x)$  is independent of the scale of  $\Psi^a(x)$ . Thus each quantum electrodynamic state is associated with a unique classical field, given by the map  $x \in \Gamma \rightarrow A^\alpha(x) \in \mathcal{H}$ .

Now suppose we are given a classical solution  $\xi^\alpha \in \mathcal{H}$  of Maxwell's equations, and we wish to construct a state on  $\Gamma$  to which  $\xi^\alpha$  should correspond in some natural *physical* sense. The general state on the multi-particle photon state space  $\Gamma$  is given by a *probability distribution*  $\rho(x)$  over  $\Gamma$ . Thus, if  $dx$  represents the natural volume element on  $\Gamma$  associated with the Fubini-Study metric, we have

$$\int_{\Gamma} \rho(x) dx = 1. \quad (3)$$

The associated density matrix  $\rho_b^a$ , which contains sufficient information to value the expectations of linear observables, is given by

$$\rho_b^a = \int_{\Gamma} \rho(x) \frac{\bar{\Psi}_b(x) \Psi^a(x)}{\bar{\Psi}_c(x) \Psi^c(x)} dx. \quad (4)$$

For example, the expectation of the annihilation operator can be written in the form:

$$\rho_b^a A_a^{\alpha b} = \int_{\Gamma} \rho(x) A^\alpha(x) dx. \quad (5)$$

We come to our key hypothesis. Our suggestion is that, for a given classical field configuration  $\xi^\alpha$ , the associated physical quantum electrodynamic state is given by the probability density function  $\rho(x)$  that maximises the entropy function

$$S_\rho = - \int_{\Gamma} \rho(x) \ln \rho(x) dx \quad (6)$$

over  $\Gamma$ , subject to the constraint

$$\xi^\alpha = \int_{\Gamma} \rho(x) A^\alpha(x) dx. \quad (7)$$

A standard line of argument (cf. [2]) then shows that the choice of  $\rho(x)$  that maximises  $S_\rho$  is given by a *grand canonical ensemble* on  $\Gamma$  of the form:

$$\rho(x) = \exp[-\mu_\alpha A^\alpha(x) - \bar{\mu}^\alpha C_\alpha(x)] / Z(\mu) . \quad (8)$$

Here, the partition function  $Z(\mu)$  is given by the integral

$$Z(\mu) = \int_\Gamma \exp[-\mu_\alpha A^\alpha(x) - \bar{\mu}^\alpha C_\alpha(x)] dx . \quad (9)$$

The 'chemical potential'  $\mu_\alpha \in \mathcal{H}^*$  arises as a Lagrange multiplier in the variation analysis, and is determined by the relation

$$\frac{\partial \ln Z(\mu)}{\partial \mu_\alpha} = \xi^\alpha \quad (10)$$

for the given value of  $\xi^\alpha$ . Thus the dual field  $\mu_\alpha$  is *thermodynamically conjugate* to the expectation of annihilation operator  $\xi^\alpha$ .

The manifold  $\Gamma$  is foliated by the level surfaces of  $A^\alpha(x)$ , and the grand canonical distribution  $\rho(x)$  is constant on each such surfaces. If we introduce the manifold  $\mathcal{C} \in \Gamma$  consisting of coherent states, then  $\Gamma$  can be viewed as a fibre space over  $\mathcal{C}$ , because each level surface of the expectation of the annihilation operator intersects  $\mathcal{C}$  at one point, namely, the coherent state corresponding to the given value of  $A^\alpha(x)$ . Coherent states are those points  $x$  for which the eigenvalue relation  $A_b^{\alpha a} \Psi^b(x) = \xi^\alpha \Psi^a(x)$  is satisfied for some  $\xi^\alpha$ .

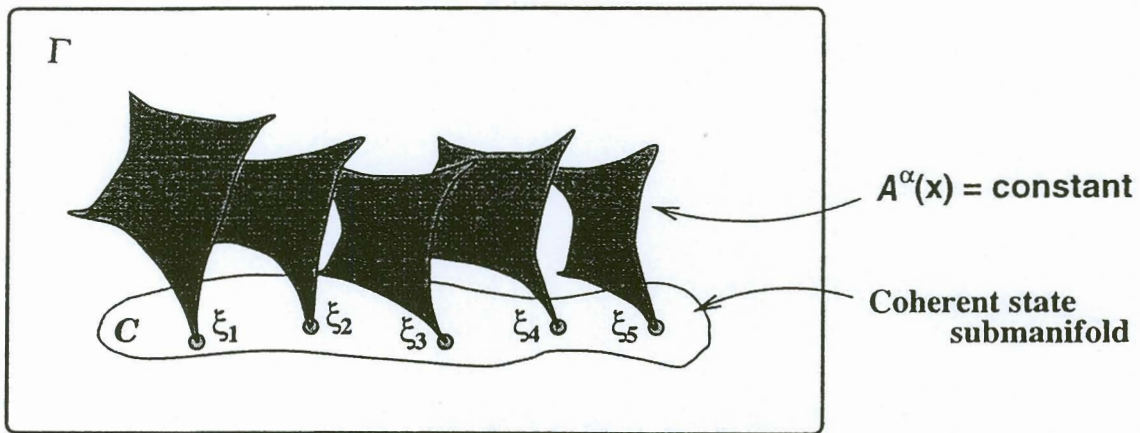


FIG. 1. Foliation of projective Fock space  $\Gamma$  by level surfaces of the expectation of the annihilation operator  $A^\alpha(x)$ . Each such surface intersects the manifold  $\mathcal{C}$  of coherent states at a point.

There are many interesting analogies that follow from the ideas suggested above, and it is tempting to take them seriously. For example, the phenomenon of classicalisation can be viewed as a consequence of the second law of thermodynamics (cf. [3]). Suppose we have a large region of essentially classical configurations (the laboratory) and a small region (the experimental region) where we create, say, a region of pure quantum state. Initially the experimental region is insulated from the rest of the laboratory, but after the shield is removed the experimental region 'decoheres' by adjusting itself to the chemical potential  $\mu_\alpha$  of the laboratory. This follows from the requirement of matching chemical potentials (in

this case, electromagnetic fields) for systems in equilibrium. More precisely, the chemical potential of the laboratory has to shift slightly to accommodate the experimental region, and as this happens the pure state of the experimental region decoheres, or classicalises, just in such a way as to match up with the chemical potential of the laboratory, so that the combined system is now characterised by a new classical state, say  $\nu_\alpha$ , which differs from the original  $\mu_\alpha$  by an essentially negligible perturbation.

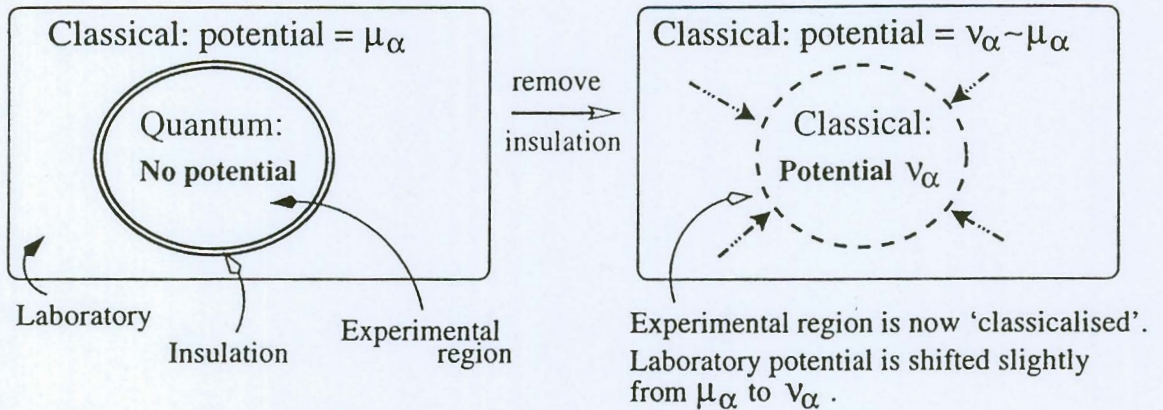


FIG. 2. Classicalisation of a quantum field. An experimental region confining a pure quantum field is insulated from the environment of the surrounding laboratory, which is permeated with a classical field of chemical potential  $\mu_\alpha$ . After the insulation is removed the pure quantum state decoheres as it comes into equilibrium with the environment, and acquires a potential  $\nu_\alpha$  equal to that of the laboratory, which has shifted very slightly in response to its interaction with the quantum field.

In the idea outlined above, we implicitly assume that there exists a dynamical mechanism leading to a kind of Boltzmann's  $H$ -theorem for quantum electromagnetism. However, such a mechanism may not exist as such in quantum physics, in which case the resulting 'equilibrium' (classical) distribution would have to be altered. For example, we might be led to the density matrix

$$\rho_b^a = \exp(-\mu_\alpha A_b^{\alpha a} - \bar{\mu}^\alpha C_{\alpha b}^a) / Q(\mu), \quad (11)$$

where the expectation  $\rho_b^a A_a^{\alpha b} = \xi^\alpha$  determines  $\mu_\alpha$  and  $Q(\mu)$  is chosen so that  $\rho_a^a = 1$ . Nevertheless, the idea of representing classical states as statistical distributions is legitimate, and it also ties in naturally with the foundations of statistical mechanics, where the aim is to account for the observed characteristics of macroscopic physics in terms of an underlying microscopic dynamics. We hope to exploit the idea further.

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2. D. C. Brody & L. P. Hughston, J. Math. Phys. **39**, 6502 (1998); **40**, 12 (1999).
3. D. C. Brody & L. P. Hughston, Proc. Roy. Soc. London A, **455** (1999), in press.

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