Solving Polynomial Equations with Spinors

Abstract/Disclaimer: Almost surely, everything in this article is well-known (for example, to R.P.) but it seems a story worth telling. The main observation is that spinor methods provide solutions by radicals to cubic and quartic polynomials. For polynomials of degree $\leqslant 7$, they provide a means for reducing to canonical form.

The quadratic equation

$$ax^2 + 2bx + c = 0$$

is easily solved by 'completing the square':

$$ax^2 + 2bx + c = a(x + b/a)^2 + (ac - b^2)/a.$$

Equivalently, the generic homogeneous quadratic polynomial

$$ax^2 + 2bxy + cy^2 \qquad (*)$$

in the two variables x, y may be thrown into the canonical form

$$X^2 + Y^2 = (X + iY)(X - iY)$$

by a suitable linear change of variables:

$$\left.\begin{array}{l} X = \alpha x + \beta y \\ Y = \gamma x + \delta y \end{array}\right\} \quad \text{e.g.} \quad \left.\begin{array}{ll} \alpha = \sqrt{a} & \beta = b/\sqrt{a} \\ \gamma = 0 & \delta = \sqrt{(ac - b^2)/a} \end{array}\right\} \quad \begin{array}{l} \text{if } a \neq 0 \\ \text{and } ac \neq b^2. \end{array}$$

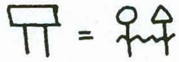
In 19 terminology, $(*)$ is a binary quadric and much effort was expended (by Cayley, Sylvester, Gordan, ... a cast of thousands...) in finding invariants, covariants, and canonical forms of binary quadrics, cubics, quartics, ..., quantics, ... The literature is vast. I can recommend two excellent texts [E,GY]. The marvelous though somewhat arcane 19 notation can be translated [P] into marvelous 20 spinor notation or even more marvelous 19/20 chemico-algebraic/twistor-bug notation [C]/[PR1].

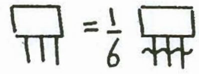
In particular,

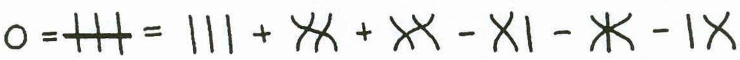
$$ax + by = \phi_A \quad \text{where} \quad \phi_0 = a\,, \quad \phi_1 = b$$

$$ax^2 + 2bxy + cy^2 = \psi_{AB} \text{ where } \psi_{00} = a, \psi_{01} = \psi_{10} = b, \psi_{11} = c$$

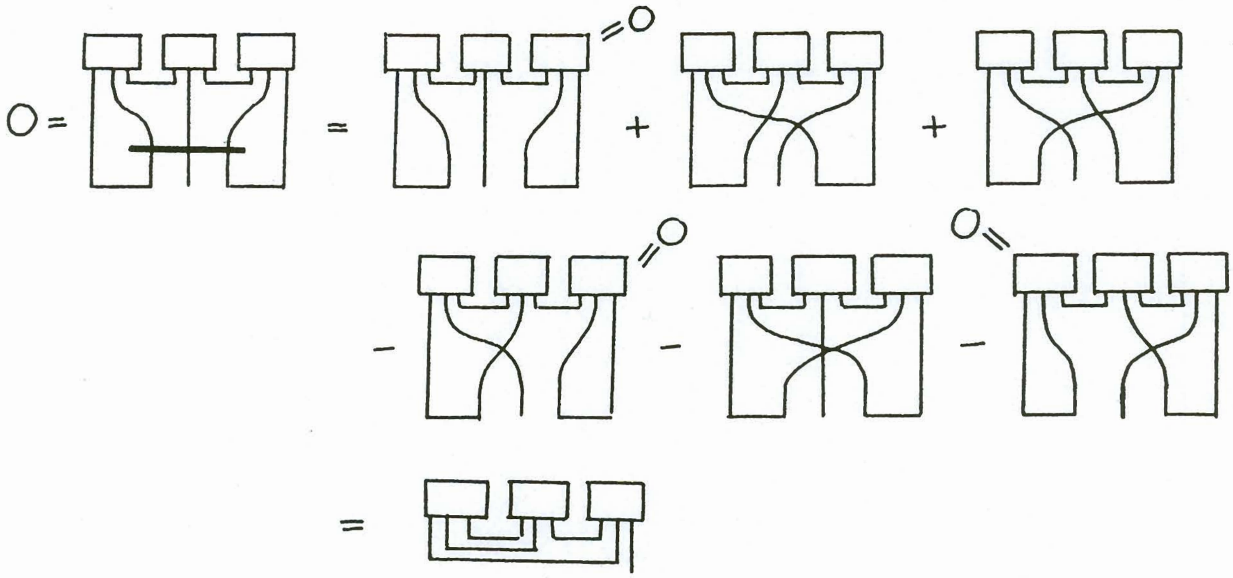
and then

$\phi_A =$  $\psi_{AB} =$  where  $= \frac{X+iY}{\sqrt{2}}$,  $= \frac{X-iY}{\sqrt{2}}$.

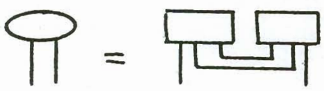
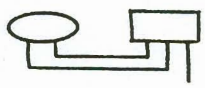
Let $\varepsilon^{AB} =$  and consider the binary cubic .
There is the spinor identity (cf. TN 1, p.1)



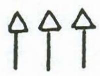
from which all other spinor identities follow. (The corresponding statement in higher dimensions is a <u>second main theorem</u> of invariant theory, essentially due to Weyl [W]—see also [G].) In particular,



In other words, the binary quadric (a covariant called the <u>Hessian</u>)

 satisfies  $= 0$.

In classical terminology,  and  are said to be <u>apolar</u>. Now we've already solved the quadratic equation . If we suppose these two zeroes to be distinct, then it follows, simply by counting dimensions, that

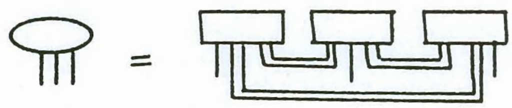
 is a linear combination of  and  .

The coefficients of this combination must be non-zero (assuming the Hessian is non-vanishing). They may be absorbed to yield the canonical form

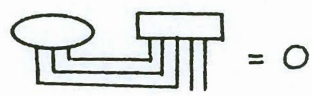
$$X^3+Y^3 \quad \left[= (X+Y)(X+\omega Y)(X+\omega^2 Y) \text{ where } \omega = e^{2\pi i/3}\right]$$

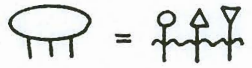
of the generic cubic. Special cases (if vanishes or has a double zero) can be similarly thrown into special canonical forms. In particular, we have <u>solved the cubic</u> with radicals.

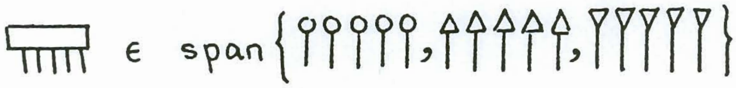
This classical canonical reduction argument applies to all binary quantics of odd degree. In the case of a quintic , for example, the cubic covariant



is apolar to , i.e.



and, having solved the cubic , it follows that

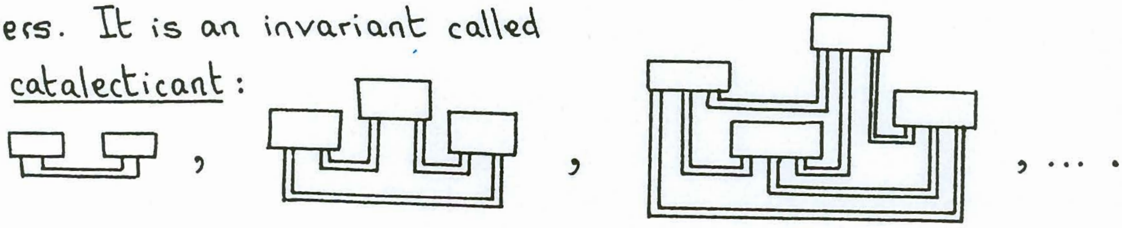


(if the roots are distinct). We have therefore written (using only radicals) the generic quintic in the form

$$X^5+Y^5+(\lambda X+\mu Y)^5.$$

Of course, this canonical form does not itself admit a solution by radicals. (This is quite different from a canonical form determined by the principal null directions of the spinor (i.e. zeroes of the polynomial) – for quintics such a reduction cannot be effected by radicals.) Septics may be similarly canonized by radicals, the canonizing apolar being a quartic.

The binary quantics of even degree are more difficult. There is an obstruction to writing a $2n$-tic as a sum of n perfect $2n^{\text{th}}$ powers. It is an invariant called the <u>catalecticant</u>:



For quartics, however, we can use the geometric link between spinors and tensors to effect a canonization. This is converse to the more usual use of spinors in classifying tensors as typified by the Weyl tensor in terms of the principal null directions of its associated Weyl spinors (see [PR2, Chapter 8] for a comprehensive discussion (including an optional Maxwell field) and a similar treatment of the Ricci tensor). Introduce the symmetric spinor x^{AB} as coördinates x^{00}, $x^{01}=x^{10}$, x^{11} on $\mathbb{C}^3$. Then

$$\varepsilon_{AC}\varepsilon_{BD}x^{AB}x^{CD}=2\left[x^{00}x^{11}-(x^{01})^2\right]=\left[\frac{x^{00}+x^{11}}{\sqrt{2}}\right]^2+\left[\frac{ix^{00}-ix^{11}}{\sqrt{2}}\right]^2+\left[\sqrt{2}\,ix^{01}\right]^2$$

is evidently preserved by $SL(2,\mathbb{C})$, hence the double cover $SL(2,\mathbb{C})\xrightarrow{2:1}SO(3,\mathbb{C})$. As matrices,

$$\begin{pmatrix}\hat{x}^0\\ \hat{x}^1\end{pmatrix}=\underbrace{\begin{pmatrix}a & b\\ c & d\end{pmatrix}}_{\in SL(2,\mathbb{C})}\begin{pmatrix}x^0\\ x^1\end{pmatrix}\quad\Rightarrow\quad\begin{pmatrix}\hat{x}^{00}\\ \hat{x}^{01}\\ \hat{x}^{11}\end{pmatrix}=\underbrace{\begin{pmatrix}a^2 & 2ab & b^2\\ ac & ad+bc & bd\\ c^2 & 2cd & d^2\end{pmatrix}}_{\in SO(3,\mathbb{C})}\begin{pmatrix}x^{00}\\ x^{01}\\ x^{11}\end{pmatrix}\qquad (**)$$

Notice that lifting from $SO(3,\mathbb{C})$ to $SL(2,\mathbb{C})$ may be effected by $\sqrt{\ }$. A binary quartic ϕ_{ABCD} may be regarded, equivalently, as a trace-free symmetric form $\phi_{ABCD}x^{AB}x^{CD}$ on $\mathbb{C}^3$ (cf. [PR2, §8.3]). The generic symmetric form on $\mathbb{C}^3$ may be complex orthogonally diagonalized — the argument follows the real version save for the possibility of null eigenvectors which must be treated separately. Furthermore, the characteristic polynomial is a cubic which we already know how to solve (it is not a covariant but is, rather, an ordinary cubic polynomial whose coefficients are invariants). Therefore, we can lift to an explicitly computable $SL(2,\mathbb{C})$ change of coördinates so that, in the new coördinates:

$$\phi_{ABCD}x^{AB}x^{CD}=\lambda\left[\frac{x^{00}+x^{11}}{\sqrt{2}}\right]^2+\mu\left[\frac{ix^{00}-ix^{11}}{\sqrt{2}}\right]^2+\nu\left[\sqrt{2}\,ix^{01}\right]^2\quad\text{where }\lambda+\mu+\nu=0$$

$$=\frac{\lambda-\mu}{2}(x^{00})^2+(\lambda+\mu)\left[x^{00}x^{11}+2(x^{01})^2\right]+\frac{\lambda-\mu}{2}(x^{11})^2$$

Thus,
$$\phi_{ABCD}=\frac{\lambda-\mu}{2}x^4+3(\lambda+\mu)x^2y^2+\frac{\lambda-\mu}{2}y^4.$$

Provided that $\lambda \neq \mu$, a further coördinate rescaling gives

$$X^4 + 6m X^2 Y^2 + Y^4,$$

the canonical form obtained in [E] (by a different method which assumes that the quartic can be solved). As a quadratic in X^2 and Y^2, it is easily solved. As usual, there are some degenerate cases which must be treated separately but, in any case, we have now <u>solved the quartic</u> by radicals. Of course, these procedures follow the 16 formulae of del Ferro, Fontano, and Ferrari or, equivalently, the solutions derived from Galois theory [S].

<u>E.g.</u> We can use these procedures to find the zeroes of

$$\boxed{14x^4 + 68x^3 + 115x^2 + 76x + 14}$$

The corresponding trace-free symmetric form $\phi_{ABCD}x^{AB}x^{CD}$ is

$$14(x^{00})^2 + 68x^{00}x^{01} + 115/3\,(x^{00}x^{11} + 2(x^{01})^2) + 76x^{01}x^{11} + 14(x^{11})^2 \quad (***)$$

$$= \left(\frac{x^{00}+x^{11}}{\sqrt{2}},\ \frac{ix^{00}-ix^{11}}{\sqrt{2}},\ \sqrt{2}\,ix^{01}\right)\underbrace{\begin{pmatrix} 199/6 & 0 & -36i \\ 0 & 31/6 & 2 \\ -36i & 2 & -115/3 \end{pmatrix}}_{M}\begin{pmatrix} \frac{x^{00}+x^{11}}{\sqrt{2}} \\ \frac{ix^{00}-ix^{11}}{\sqrt{2}} \\ \sqrt{2}\,ix^{01} \end{pmatrix}.$$

The symmetric matrix M has characteristic polynomial

$$\lambda^3 - 73/12\,\lambda + 595/108$$

whose Hessian is

$$-\frac{73}{18}x^2 + \frac{595}{54}x - \frac{5329}{648} = -\frac{(438x-595-108\sqrt{3}i)(438x-595+108\sqrt{3}i)}{47304}$$

and so the characteristic polynomial may be written as

$$p(438x-595-108\sqrt{3}i)^3 + q(438x-595+108\sqrt{3}i)^3$$

for suitable p and q which are easily computed:

$$p = \frac{324 + 595\sqrt{3}\,i}{54449931456} \qquad q = \frac{324 - 595\sqrt{3}\,i}{54449931456}.$$

This gives the roots of the characteristic polynomial as $5/3, 7/6, -17/6$ and enables us complex orthogonally to diagonalize M:

$$R^t M R = \begin{bmatrix} 5/3 & 0 & 0 \\ 0 & 7/6 & 0 \\ 0 & 0 & -17/6 \end{bmatrix} \quad \text{where } R = \begin{bmatrix} 8i & 9 & 4i \\ -4 & 4i & -1 \\ 7 & -8i & 4 \end{bmatrix}.$$

Thus,

$$(***) = \frac{5}{3}\left[\frac{\hat{x}^{00}+\hat{x}^{11}}{\sqrt{2}}\right]^2 + \frac{7}{6}\left[\frac{i\hat{x}^{00}-i\hat{x}^{11}}{\sqrt{2}}\right]^2 - \frac{17}{6}\left[\sqrt{2}\,i\,\hat{x}^{01}\right]^2$$

where

$$\begin{pmatrix} \frac{\hat{x}^{00}+\hat{x}^{11}}{\sqrt{2}} \\ \frac{i\hat{x}^{00}-i\hat{x}^{11}}{\sqrt{2}} \\ \sqrt{2}\,i\,\hat{x}^{01} \end{pmatrix} = R^{-1} \begin{pmatrix} \frac{x^{00}+x^{11}}{\sqrt{2}} \\ \frac{ix^{00}-ix^{11}}{\sqrt{2}} \\ \sqrt{2}\,i\,x^{01} \end{pmatrix} \quad \text{i.e.} \quad \begin{pmatrix} \hat{x}^{00} \\ \hat{x}^{01} \\ \hat{x}^{11} \end{pmatrix} = \begin{pmatrix} -i/2 & -i & -i/2 \\ 3/2 & 4 & 5/2 \\ 9i/2 & 15i & 25i/2 \end{pmatrix} \begin{pmatrix} x^{00} \\ x^{01} \\ x^{11} \end{pmatrix}$$

see (**)

Incorporating the final rescaling gives

$$\left.\begin{aligned} X &= (1-i)(x+y)/2\sqrt{2} \\ Y &= (1+i)(3x+5y)/2\sqrt{2} \end{aligned}\right\} \text{and, sure enough,} \qquad \begin{bmatrix} (1-i)/2 & (1-i)/2 \\ 3(1+i)/2 & 5(1+i)/2 \end{bmatrix}.$$

$$14x^4 + 68x^3y + 115x^2y^2 + 76xy^3 + 14y^4 = X^4 + 34X^2Y^2 + Y^4,$$

as predicted. The zeroes of the original quartic are, thus: $(-2\pm\sqrt{2})/2,\ (-10\pm\sqrt{2})/7.$

Thanks to R.P. and N.M.J.W. (~16 years ago) and Vladimir Ezhov.

C: W.K. Clifford "Theory of Graphs" MacMillan 1881 (See Puzzle in TN 34 (1992) and [P]).

E: E.B. Elliott "Algebra of Quantics" Clarendon Press 1895.

G: A.R. Gover "The algebra of orthogonal group invariants" preprint.

GY: J.H. Grace and A. Young "Algebra of Invariants" C.U.P. 1903.

P: R. Penrose "Solution to Puzzle in TN 34" TN 35 (1992) 36-40.

PR1: PR2: } R. Penrose and W. Rindler "Spinors and Space-time" C.U.P. vol. 1 1984 vol. 2 1986.

S: I. Stewart "Galois Theory" Chapman and Hall 1973.

W: H. Weyl "Classical Groups" P.U.P. 1939.

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