SOLVING POLYNOMIAL EQUATIONS WITH SPINORS

Abstract/Disclaimer: Almost surely, everything in this article is well-known (for example, to R.P.) but it seems a story worth telling. The main observation is that spinor methods provide solutions by radicals to cubic and quartic polynomials. For polynomials of degree \( \leq 7 \), they provide a means for reducing to canonical form.

The quadratic equation
\[
ax^2 + 2bx + c = 0
\]
is easily solved by ‘completing the square’:
\[
ax^2 + 2bx + c = a(x + b/a)^2 + (ac - b^2)/a.
\]
Equivalently, the generic homogeneous quadratic polynomial
\[
ax^2 + 2bxy + cy^2 \quad (*)
\]
in the two variables \( x, y \) may be thrown into the canonical form
\[
X^2 + Y^2 = (X + iY)(X - iY)
\]
by a suitable linear change of variables:
\[
\begin{align*}
X &= \alpha x + \beta y \quad \text{e.g.} \quad \alpha = \sqrt{a} \quad \beta = b/\sqrt{a} \\
Y &= \gamma x + \delta y \quad \text{if} \quad \gamma = 0 \quad \delta = \sqrt{(ac - b^2)/a} \quad \text{and} \quad ac \neq b^2.
\end{align*}
\]

In \( \text{Gr} \) terminology, (*) is a binary quadratic and much effort was expended (by Cayley, Sylvester, Gordan, ... a cast of thousands ...) in finding invariants, covariants, and canonical forms of binary quadrics, cubics, quartics, ..., quantics, .... The literature is vast. I can recommend two excellent texts [E,GY].

The marvelous though somewhat arcane \( \text{Gr} \) notation can be translated [P] into marvelous \( \text{Vo} \) spinor notation or even more marvelous \( \text{Gr}/\text{Vo} \) chemico-algebraic / twistor-bug notation [C]/[PR].
In particular,

\[ a x + b y = \phi_A \quad \text{where} \quad \phi_0 = a, \quad \phi_1 = b \]
\[ a x^2 + 2 b x y + c y^2 = \psi_{AB} \quad \text{where} \quad \psi_{00} = a, \quad \psi_{01} = \psi_{10} = b, \quad \psi_{11} = c \]

and then

\[ \phi_A = \frac{\Phi}{\sqrt{2}}, \quad \psi_{AB} = \frac{\Phi\Phi}{\sqrt{2}} \quad \text{where} \quad \Phi = \frac{X+iY}{\sqrt{2}}, \quad \frac{\Phi}{\sqrt{2}} = \frac{X-iY}{\sqrt{2}}. \]

Let \( \mathbf{e}^{AB} = \mathbf{1} \) and consider the binary cubic \( \mathbf{\Pi} = \frac{1}{6} \mathbf{1} \).

There is the spinor identity (cf. T\( ^N \) 1, p.1)

\[ 0 = \mathbf{\Pi} = \mathbf{1} + \mathbf{XX} + \mathbf{XX} - \mathbf{XI} - \mathbf{XI} - \mathbf{IX} \]

from which all other spinor identities follow. (The corresponding statement in higher dimensions is a second main theorem of invariant theory, essentially due to Weyl [W]—see also [G].) In particular,

\[ 0 = \mathbf{\Pi} = \mathbf{1} + \mathbf{XX} + \mathbf{XX} - \mathbf{XI} - \mathbf{XI} - \mathbf{IX} \]

In other words, the binary quadric (a covariant called the Hessian)

\[ \mathbf{\Pi} = \mathbf{1} + \mathbf{XX} + \mathbf{XX} - \mathbf{XI} - \mathbf{XI} - \mathbf{IX} \]

satisfies \( \mathbf{\Pi} = \mathbf{XX} = 0 \).

In classical terminology, \( \mathbf{XX} \) and \( \mathbf{1} \) are said to be apolar. Now we've already solved the quadratic equation \( \mathbf{\Pi} = \mathbf{XX} \). If we suppose these two zeroes to be distinct, then it follows, simply by counting dimensions, that

\[ \mathbf{1} \]

is a linear combination of \( \mathbf{XX} \) and \( \mathbf{XX} \).
The coefficients of this combination must be non-zero (assuming the Hessian is non-vanishing). They may be absorbed to yield the canonical form
\[ x^3 + y^3 = (x + y)(x + \omega y)(x + \omega^2 y) \] where \( \omega = e^{2\pi i/3} \)

of the generic cubic. Special cases (if \( \bigodot \) vanishes or has a double zero) can be similarly thrown into special canonical forms. In particular, we have solved the cubic with radicals.

This classical canonical reduction argument applies to all binary quantics of odd degree. In the case of a quintic, for example, the cubic covariant
\[ \bigodot = \begin{array}{c}
\end{array} \]
is apolar to \( \bigodot \), i.e.
\[ \bigodot = 0 \]
and, having solved the cubic \( \bigodot = \bigodot \), it follows that
\[ \bigodot \in \text{span}\{\begin{array}{c}
\end{array}\} \]
(if the roots are distinct). We have therefore written (using only radicals) the generic quintic in the form
\[ x^5 + y^5 + (\lambda x + \mu y)^5 \]

Of course, this canonical form does not itself admit a solution by radicals. (This is quite different from a canonical form determined by the principal null directions of the spinor (i.e. zeroes of the polynomial) - for quantics such a reduction cannot be effected by radicals.) Septics may be similarly canonized by radicals, the canonizing apolar being a quartic.

The binary quantics of even degree are more difficult. There is an obstruction to writing a 2n-tic as a sum of \( n \) perfect 2n-th powers. It is an invariant called the catalecticant:
\[ \begin{array}{c}
\end{array} \]
For quartics, however, we can use the geometric link between spinors and tensors to effect a canonization. This is converse to the more usual use of spinors in classifying tensors as typified by the Weyl tensor in terms of the principal null directions of its associated Weyl spinors (see [PR2, Chapter 8] for a comprehensive discussion (including an optional Maxwell field) and a similar treatment of the Ricci tensor). Introduce the symmetric spinor $x^{AB}$ as coordinates $x^{00}, x^{01}, x^{10}, x^{11}$ on $\mathbb{C}^3$. Then

$$
\varepsilon_{AC} \varepsilon_{BD} x^{AB} x^{CD} = 2\left[ x^{00} x'' - (x^{01})^2 \right] = \left[ \frac{x^{00} + x''}{\sqrt{2}} \right]^2 + \left[ \frac{i x^{00} - i x''}{\sqrt{2}} \right]^2 + \left[ \sqrt{2} i x^{01} \right]^2
$$

is evidently preserved by $SL(2, \mathbb{C})$, hence the double cover $SL(2, \mathbb{C}) \xrightarrow{2:1} SO(3, \mathbb{C})$. As matrices,

$$
\begin{pmatrix}
\hat{x}^0 \\
\hat{x}^1
\end{pmatrix} = \begin{pmatrix} a & b \\
c & d \end{pmatrix} 
\begin{pmatrix}
x^0 \\
x^1
\end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix}
\hat{x}^{00} \\
\hat{x}^{01} \\
\hat{x}^{10} \\
\hat{x}^{11}
\end{pmatrix} = \begin{pmatrix} a^2 & 2ab & b^2 \\
ac & adtbc & bd \\
c^2 & 2cd & d^2 \end{pmatrix} 
\begin{pmatrix}
x^{00} \\
x^{01} \\
x^{10} \\
x^{11}
\end{pmatrix}
$$

$$
\in SO(3, \mathbb{C})
$$

Notice that lifting from $SO(3, \mathbb{C})$ to $SL(2, \mathbb{C})$ may be effected by $\sqrt{2}$.

A binary quartic $\varphi_{ABCD}$ may be regarded, equivalently, as a trace-free symmetric form $\varphi_{ABCD} x^{AB} x^{CD}$ on $\mathbb{C}^3$ (cf. [PR2, §8.3]). The generic symmetric form on $\mathbb{C}^3$ may be complex orthogonally diagonalized—the argument follows the real version save for the possibility of null eigenvectors which must be treated separately. Furthermore, the characteristic polynomial is a cubic which we already know how to solve (it is not a covariant but is, rather, an ordinary cubic polynomial whose coefficients are invariants). Therefore, we can lift to an explicitly computable $SL(2, \mathbb{C})$ change of coordinates so that, in the new coordinates:

$$
\varphi_{ABCD} x^{AB} x^{CD} = \lambda \left[ \frac{x^{00} + x''}{\sqrt{2}} \right]^2 + \mu \left[ \frac{i x^{00} - i x''}{\sqrt{2}} \right]^2 + \nu \left[ \sqrt{2} i x^{01} \right]^2 \quad \text{where} \quad \lambda + \mu + \nu = 0
$$

$$
= \frac{\lambda - \mu}{2} (x'^0)^2 + (\lambda + \mu) [x'^0 x'' + 2 (x'^0)^2] + \frac{\lambda - \mu}{2} (x'')^2
$$

Thus,

$$
\varphi_{ABCD} = \frac{\lambda - \mu}{2} x'^4 + 3(\lambda + \mu) x'^2 y'^2 + \frac{\lambda - \mu}{2} y'^4.
$$
Provided that \( \lambda \neq \mu \), a further coordinate rescaling gives
\[ X^4 + 6mX^2Y^2 + Y^4, \]
the canonical form obtained in [E] (by a different method which assumes that the quartic can be solved). As a quadratic in \( X^2 \) and \( Y^2 \), it is easily solved. As usual, there are some degenerate cases which must be treated separately but, in any case, we have now solved the quartic by radicals. Of course, these procedures follow the 16 formulae of del Ferro, Fontano, and Ferrari or, equivalently, the solutions derived from Galois theory [S].

E.g. We can use these procedures to find the zeroes of
\[ 14x^4 + 68x^3 + 115x^2 + 76x + 14 \]
The corresponding trace-free symmetric form \( \varphi_{ABCD} x^{AB} x^{CD} \) is
\[ 14(x^0)^2 + 68x^0x^1 + 115/3 (x^0x^1 + 2(x^0)^2) + 76x^0x^1 + 14 (x^1)^2 \]
\[ = \begin{pmatrix} x^0 + x^1 \sqrt{2}, ix^0 - ix^1 \sqrt{2} \end{pmatrix} \begin{pmatrix} 193/6 & 0 & -36i \\ 0 & 3/6 & 2 \\ -36i & 2 & -115/3 \end{pmatrix} \begin{pmatrix} x^0 + x^1 \sqrt{2} \\ ix^0 - ix^1 \sqrt{2} \end{pmatrix}. \]
The symmetric matrix \( M \) has characteristic polynomial
\[ \chi^3 - \frac{73}{12} \chi + \frac{595}{108} \]
whose Hessian is
\[ -\frac{73}{18} \chi^2 + \frac{595}{54} \chi - \frac{5329}{648} = - \frac{(438\chi - 595 - 108\sqrt{3}i)(438\chi - 595 + 108\sqrt{3}i)}{47304} \]
and so the characteristic polynomial may be written as
\[ p(438\chi - 595 - 108\sqrt{3}i)^3 + q(438\chi - 595 + 108\sqrt{3}i)^3 \]
for suitable \( p \) and \( q \) which are easily computed:
This gives the roots of the characteristic polynomial as \(5/3, 7/6, -17/6\) and enables us complex orthogonally to diagonalize \(M\):

\[
R^t M R = \begin{bmatrix}
5/3 & 0 & 0 \\
0 & 7/6 & 0 \\
0 & 0 & -17/6
\end{bmatrix} \quad \text{where} \quad R = \begin{bmatrix}
8i & 9 & 4i \\
-4 & 4i & -1 \\
7 & -8i & 4
\end{bmatrix}.
\]

Thus,

\[
(*\star\star) = \frac{5}{3} \left[ \frac{\hat{x}^{oo} + \hat{x}^{11}}{\sqrt{2}} \right]^2 + \frac{7}{6} \left[ \frac{ix^{oo} - ix^{11}}{\sqrt{2}} \right]^2 - \frac{17}{6} \left[ \sqrt{2} i \hat{x}^{00} \right]^2
\]

where

\[
\begin{align*}
\frac{\hat{x}^{oo} + \hat{x}^{11}}{\sqrt{2}} &= R^{-1} \begin{pmatrix}
\hat{x}^{oo} + \hat{x}^{11} \\
ix^{oo} - ix^{11} \\
\sqrt{2} i \hat{x}^{01}
\end{pmatrix} \\
\text{i.e.} \quad \begin{pmatrix}
\hat{x}^{oo} \\
\hat{x}^{11} \\
\hat{x}^{01}
\end{pmatrix} &= \begin{pmatrix}
-i/2 & -i & -i/2 \\
3/2 & 4 & 5/2 \\
9i/2 & 15i & 25i/2
\end{pmatrix} \begin{pmatrix}
x^{oo} \\
x^{11} \\
x^{01}
\end{pmatrix}
\end{align*}
\]

Incorporating the final rescaling gives

\[
X = (1-i)(x+y)/2\sqrt{2} \quad \text{and, sure enough,} \quad \begin{pmatrix}
(1-i)/2 & (1-i)/2 \\
3(1+i)/2 & 5(1+i)/2
\end{pmatrix}
\]

\[
14x^4 + 68x^3y + 115x^2y^2 + 76xy^3 + 14y^4 = X^4 + 3\sqrt{2}Y^2 + Y^4,
\]

as predicted. The roots of the original quartic are thus: \((-2\pm\sqrt{2})/2, (-10\pm\sqrt{2})/7\).

Thanks to R.P. and N.M.J.W. (16 years ago) and Vladimir Ezhov.

C: W.K.Clifford "Theory of Graphs" MacMillan 1881 (See Puzzle in TNN 34 (1992) and [P]).
PR1: J.R.Penrose and W. Rindler "Spinors and Space-Time" C.U.P.