A conjectured form of the Goldberg-Sachs theorem[1]

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The rather well-known Goldberg-Sachs theorem [3] is one of the most beautiful results in the mathematics of general relativity theory. It played a major role in the work on algebraically special solutions of Einstein's equations [5].

For the purposes of this Letter, it is convenient to formulate it as follows. Let $\mathfrak{M}$ be a set of Lorentzian, not conformally flat, manifolds (M, g) of dimension 4. For $(M, g) \in \mathfrak{M}$, let $K \subset TM$ be a null line bundle; its sections are null vector fields. Following the notation and terminology of [7], consider the following two properties of K:

(GSR) K is geodetic and shear-free;
(PND) K is a bundle of *repeated* principal null directions of the Weyl tensor C.

A Goldberg-Sachs theorem GST($\mathfrak{M}$) is a statement of the form: *if* $(M, g) \in \mathfrak{M}$, *then the conditions (GSR) and (PND) are equivalent.* Goldberg and Sachs proved the theorem for $\mathfrak{M}$ = the set of Einstein spaces, i.e. solutions of $R_{\mu\nu} = \frac{1}{4} g_{\mu\nu} R$. Shortly afterwards, Kundt and Thompson [6] and Robinson and Schild [8] pointed out that both conditions (GSR) and (PND) are conformally invariant, but the property of being an Einstein space is not. They proved a generalized Goldberg-Sachs theorem that, in a refined form, is given in §7.3 of [7] and in §7.5 of [5]. This generalized theorem involves only conformal notions, but requires a separate formulation for each degree of degeneracy of the Weyl tensor.

A conformally invariant set of space-times is

$$\mathfrak{M}_c = \{(M, g) \text{ is conformal to an Einstein space}\}$$

and GST($\mathfrak{M}_c$) is true as a consequence of the classical Goldberg-Sachs theorem. It is not easy to find a description of $\mathfrak{M}_c$ by means of tensorial or spinorial equations; see [4] for an account of the early work by Brinkman. To appreciate the difficulty of the subject, recall that Schouten (p. 314 in [9]) attributes to Brinkman the following statement: In dimension 4, if two manifolds are Ricci flat and conformal to each other, but not to a flat space, then they are isometric. Ehlers and Kundt (p. 99 in [2]) give a counterexample to this: there are *pp* waves that are conformal, but not isometric, to each other.

It is known that $\mathfrak{M}_c$ is contained in the set $\mathfrak{M}_b$ of spaces satisfying the conformally invariant *Bach equation* $B = 0$. According to the arguments due to Geroch and Horowitz, presented in [4], there are space-times satisfying the Bach equation that are not conformal to an Einstein space.

[1]During the *Workshop on spinors and twistors* (28 June - 3 July 1999) at the Erwin Scrödinger Institute in Vienna I obtained valuable advice on the subject of this note from M. Dunajski, C. N. Kozameh, L. J. Mason, P. Nurowski, and K. P. Tod. When preparing the Letter, I have been supported in part by the Polish Committee for Scientific Research (KBN) under grant no. 2 P03B 060 17.

Kozameh, Newman, and Tod [4] have found a set of two equations defining a class of spaces conformal to Einstein spaces; one of them is $B = 0$, but the other one excludes some of the spaces with a degenerate C. There is an improvement of [4] by Baston and Mason [1], but their equations still do not characterize all of $\mathfrak{M}_c$.

Problems

I consider the following problems to be ordered according to increasing difficulty.

(i) Find a counterexample to GST($\mathfrak{M}_b$).

(ii) If you fail in (i), then prove GST($\mathfrak{M}_b$).

(iii) If you succeed in (i), then find a set of conformally invariant tensor or spinor equations defining $\mathfrak{M}$, without reference to the degeneracy of C, such that $\mathfrak{M}_c \subset \mathfrak{M}$ and GST($\mathfrak{M}$) is true.

Note that if $\mathfrak{M}_c \subsetneq \mathfrak{M}$, then GST($\mathfrak{M}$) is stronger than the classical theorem.

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