Three Stories about Epsilon

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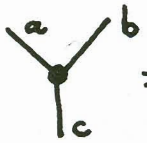
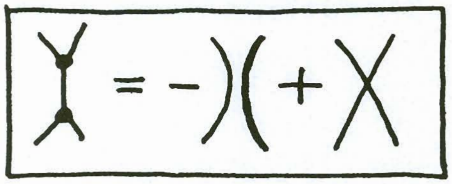
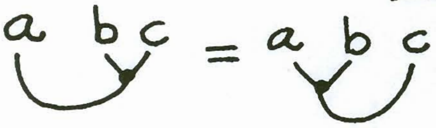
I.  $= \varepsilon_{abc} := \begin{cases} 1 & abc = 123, 231, 312 \\ -1 & abc = 213, 132, 321 \\ 0 & \text{otherwise} \end{cases}$

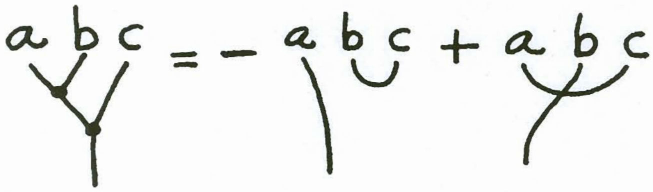
$\Rightarrow$ 

 $= \delta^a_b := \begin{cases} 1 & a = b \\ 0 & a \neq b \end{cases}$.

$a :=$  (a vector)  $:= a_i$

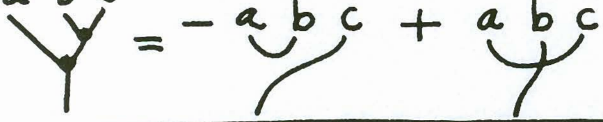
$\boxed{a \cdot b :=}$  , $\boxed{a \times b =}$ 

 $\Leftrightarrow a \cdot (b \times c) = (a \times b) \cdot c$



whence

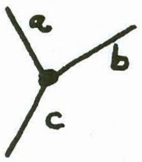
$$\boxed{(a \times b) \times c = -a(b \cdot c) + b(a \cdot c)}$$



$$\boxed{a \times (b \times c) = -(a \cdot b)c + b(a \cdot c)}$$

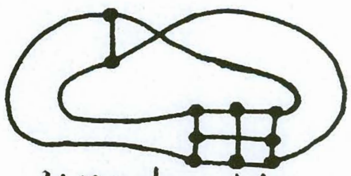
$$\boxed{(a \times b) \times c - a \times (b \times c) = (a \cdot b)c - a(b \cdot c)}$$

II. Coloring a cubic graph.

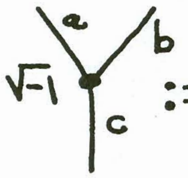


In a proper coloring
a, b, c = <u>3 distinct labels</u>
using $\{1, 2, 3\}$.

e.g. 

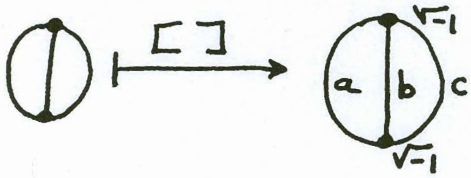
e.g. 

uncolorable
(Petersen graph)



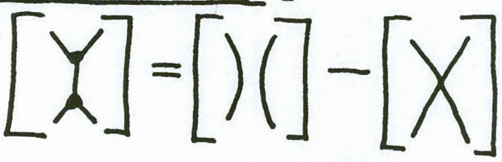
$:= \sqrt{-1}\, \varepsilon_{abc}$

$G \looparrowright \mathbb{R}^2$ a cubic graph immersed in plane.
G <u>planar</u> if no extra crossings $\times$.
$[G] :=$ contraction of the tensor obtained by replacing each 3-vertex by $\sqrt{-1}\,\varepsilon$.

e.g. 

$= (\sqrt{-1})^2 \varepsilon_{acb}\, \varepsilon_{abc}$
$= \varepsilon_{abc}\, \varepsilon_{abc}$
$= 6.$

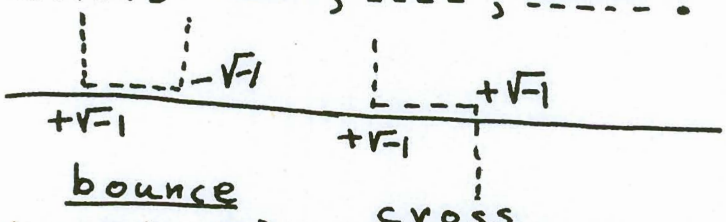
<u>Theorem (Penrose)</u>.

(A) 

(B) G planar $\Rightarrow [G]$ = the number of proper colorings of $G \hookrightarrow \mathbb{R}^2$.

<u>Proof</u>. (A) Use section I.

(B) Colors ——, - - - -, ═══ :

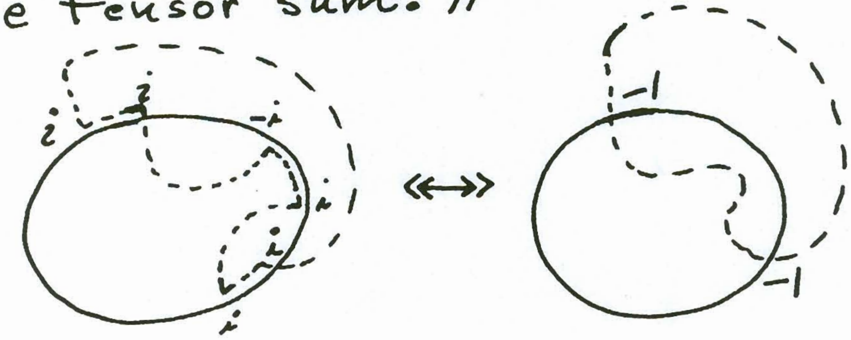


<u>bounce</u>
<u>$+1 = (\sqrt{-1})(-\sqrt{-1})$</u>

<u>cross</u>
<u>$-1 = (\sqrt{-1})(\sqrt{-1})$</u>

Now use the Jordan curve theorem to conclude that each proper coloring of the planar graph contributes $(-1)^{\#(\text{crossings})} = (-1)^{\text{even}} = +1$ to the tensor sum. //

e.g.



III. Witten's Functional Integral For Knot Invariants

$A(x) = A^a_i(x) T_a dx^i$, $i = 1,2,3$
gauge field on $\mathbb{R}^3$. $a = 1,2,\ldots,d$

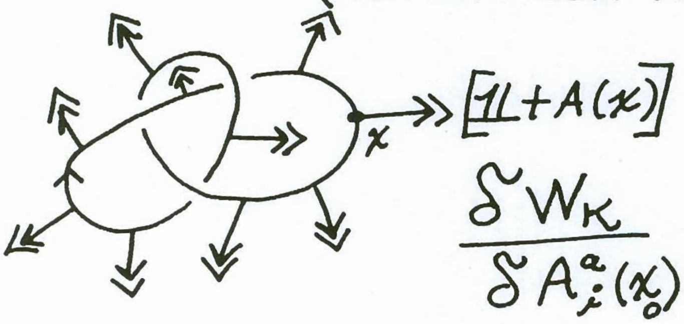
$\{T_1, T_2, \ldots, T_d\}$ matrix Lie algebra basis.

 $K \subset \mathbb{R}^3$ a knot.

$W_K(A) = tr(\mathbb{P} e^{\oint_K A})$ Wilson Loop

$W_K(A) := tr\left(\prod_{x \in K} (\mathbb{1} + A(x))\right)$

(a limit over finite partitions of K)



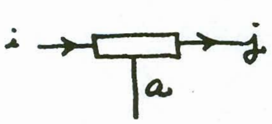
$$\frac{\delta W_K}{\delta A^a_i(x_0)} := \square^a_i W_K$$

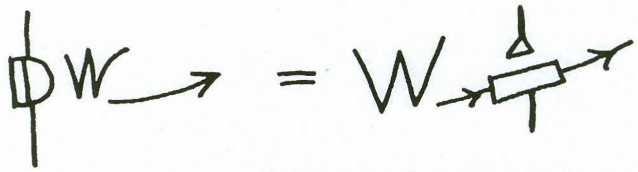
$$\square^a_i W_K = \square^a_i \, tr \prod_{x \in K} (\mathbb{1} + A^a_i(x) \underline{T_a dx^i})$$

$$= \underbrace{T_a dx^i}\, tr \prod_{x \in K} (\mathbb{1} + A(x))$$

insert at x_0

Let  $= T_a$,  $= dx^i$

$\Big($  $= (T_a)_{ij} \Big)$.

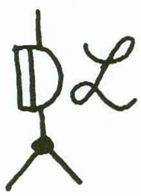
Then $\boxed{\quad}$ 

Insert $T_a dx^i$ at x_0 in Wilson Loop.

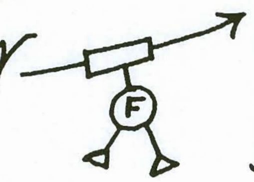
<u>The functional integral</u>:

$$Z_K = \int DA\, e^{\frac{ik}{4\pi}\mathcal{L}(A)}\, W_K(A).$$

$$\mathcal{L}(A) = \int_{S^3} tr\left(A\wedge dA + \tfrac{2}{3}A\wedge A\wedge A\right)$$

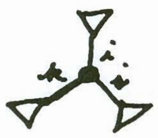
(•)  $=$  curvature tensor for $dA + A\wedge A$.

(•) $\delta\mathcal{W}_{\to} := \mathcal{W}_{\rightsquigarrow} - \mathcal{W}_{\to}$

$\delta\mathcal{W}_{\to} = \mathcal{W}$ 

Lie algebra and curvature insertion for small variation of the loop.

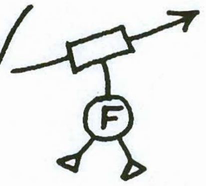
Curvature tensor arises from varying the field <u>or</u> varying the loop. Note the epsilon appearing in relation to the field variation. Note that

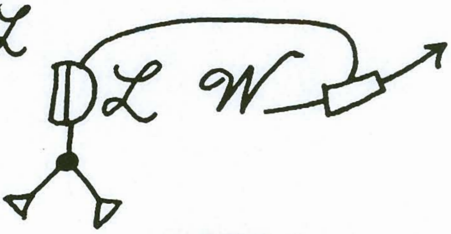
 $= dx^i dx^j dx^k\, \varepsilon_{ijk}$

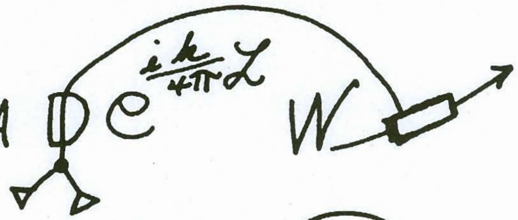
is a volume form.

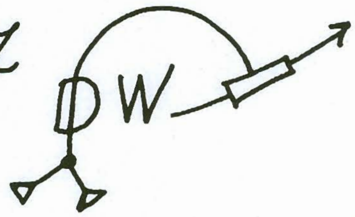
Let $\delta Z_K := Z_{\rightsquigarrow} - Z_{\to}$.

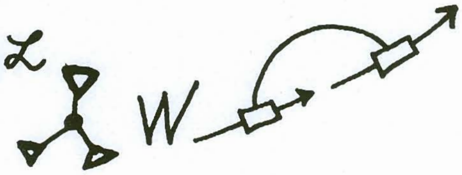
$$\delta Z_K = \int DA\, e^{\frac{ik}{4\pi}\mathcal{L}}\, \delta W_K$$

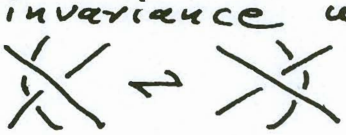
$$\delta Z_{\nearrow} = \int DA\, e^{\frac{ik}{4\pi}\mathcal{L}}\, W$$ 

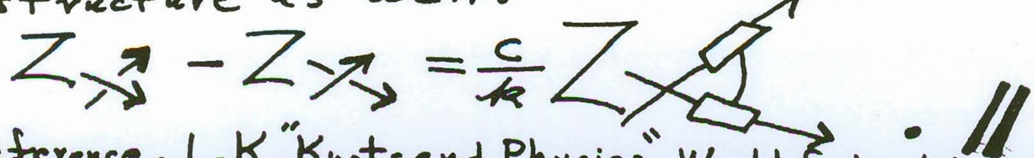
$$= \int DA\, e^{\frac{ik}{4\pi}\mathcal{L}}$$ 

$$= \frac{-i4\pi}{k}\int DA$$ 

$$= \frac{i4\pi}{k}\int DA\, e^{\frac{ik}{4\pi}\mathcal{L}}$$ 

$$\delta Z_{\nearrow} = \frac{i4\pi}{k}\int DA\, e^{\frac{ik}{4\pi}\mathcal{L}}$$ 

Because of the appearance of the volume form , $\delta Z_{\nearrow} = 0$ when the small deformation of the knot does not generate volume. Thus Z_K has formal invariance under  and  but not under .
Thus it is a framed invariant, and the Lie algebra is involved in the framing and in the switching structure as well:



Reference. L.K., "Knots and Physics", World Scientific (1994).