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ASD FOUR MANIFOLDS AND FROBENIUS MANIFOLDS

G. Sanguinetti

N.M.J Woodhouse

ABSTRACT. We describe here a simple geometric construction to obtain an anti-self-dual Riemannian four manifold from a three dimensional Frobenius manifold. This establishes a relationship between the isomonodromy deformation description of Frobenius manifolds (leading naturally to Painlevé's VI equation) and the twistor approach to integrable systems, that views them as a dimensional reduction of anti-self-dual Yang-Mills equations.

1 Introduction

Frobenius manifolds were introduced by Boris Dubrovin in the late eighties to encode the geometric information contained in a certain set of PDEs, the so called Witten- Dijkgraaf- Verlinde- Verlinde equations (WDVV). These arise naturally in the context of topological quantum field theories, and lead to a structure of associative algebra on the tangent space of certain moduli spaces of topological field theories.

For a more detailed analysis of the physical motivations that led to the introduction of Frobenius manifolds we refer to [1], while a detailed description of the geometrical features arising from Frobenius manifolds can be found in [3] or again in [1].

Among the most interesting features that arose in the work of Dubrovin there is the fact that WDVV equations are (particular) equations of isomonodromic deformations, and in particular in dimension three they are equivalent to a special form of Painlevé's sixth equation.

On the other hand, one of the greatest developments in the theory of integrable systems came from the application of ideas coming from twistor theory.

In this context, many classical integrable systems (among which the Painlevé's equations) can be seen as a dimensional reduction of anti-self-dual Yang-Mills equations, and hence solved via the Penrose-Ward transform [4].

It follows from this picture that to any solution of Painlevé's sixth equation there should be associated an anti-self-dual Riemannian four manifold. While it is well understood how to find this four manifold starting from the Yang-Mills equations, it is not clear what should be the link with Frobenius manifolds, and what particular anti-self-dual four manifolds should be associated to solutions of WDVV equations.

2 Construction

For our purposes, we can look at a Frobenius manifold X as a complex manifold endowed with a *pencil of flat metrics*, i.e. such that there exist two (contravariant) flat metrics g_1 and g_2 s.t. the linear combination $g_1 + zg_2$ is again a flat metric for any z in $\mathbb{C}$. This is a nontrivial condition (the flatness of a metric is a nonlinear condition on it), and it corresponds, from the integrable systems point of view, to the existence of two compatible Poisson brackets of hydrodynamic type on the loop space of X [2].

In addition, on any Frobenius manifold there exist two canonical vector fields, the identity vector field e and the Euler vector field E, which have the following property (see [1]):

denote by C the Christoffel symbol of the Levi-Civita connection of g_2, then

$$\mathcal{L}_e C = 0 \qquad \mathcal{L}_E C = C$$

(the second is often referred to as a quasihomogeneity condition).

We have the commutation relation

$$[E, e] = -e.$$

We will work in the flat coordinates associated to the metric g_1. In these coordinates the two vector fields take the form [1]

$$e = \frac{\partial}{\partial t^1}, \quad E = \sum_{\alpha=1}^{dim X} d_\alpha t^\alpha \frac{\partial}{\partial t^\alpha}$$

and can be normalized so that $d_1 = 1$.

Let us now specialize to the three dimensional case.

Let us lift the vector fields E and e to T^*X using the natural Lie lift. Define then them on the manifold $T^*X \times \mathbb{C}$ by adding a term $-z\frac{\partial}{\partial z}$ to E. Explicitly, the new vector field $\tilde{E}$ will read

$$\tilde{E} = \sum_{\alpha=1}^{dim X} d_\alpha t^\alpha \frac{\partial}{\partial t^\alpha} - \sum_{\alpha=1}^{dim X} d_\alpha p_\alpha \frac{\partial}{\partial p_\alpha} - z\frac{\partial}{\partial z}.$$

By virtue of the quasihomogeneity condition the commutation relations are unchanged, hence we get a two dimensional integrable distribution on the seven dimensional manifold $T^*X \times \mathbb{C}$.

If we consider the quotient of $T^*X \times \mathbb{C}$ by this distribution, we end up with a five dimensional manifold $\mathcal{F}$, which will still be fibred with fibre $\mathbb{C}$ (this follows from the fact that the vector field $\frac{\partial}{\partial z}$ descends up to scale to the quotient). The base manifold of this fibration will be a four dimensional manifold $\mathcal{M}$ which is the natural candidate to be our complex space time.

We have now to define an ASD conformal structure on the four dimensional manifold $\mathcal{M}$.

We will use the following proposition (see [5])

Proposition 2.1. *Let W, Z, $\tilde{Z}$, $\tilde{W}$ be independent holomorphic vector fields on a four dimensional complex manifold $\mathcal{M}$. Then W, Z, $\tilde{Z}$, $\tilde{W}$ determine an ASD conformal structure if and only if there exist two holomorphic functions u and v on $\mathcal{M} \times \mathbb{P}^1$ such that the distribution on $\mathcal{M} \times \mathbb{P}^1$ spanned by*

$$L = W - z\tilde{Z} + u\frac{\partial}{\partial z}, \qquad M = Z - z\tilde{W} + v\frac{\partial}{\partial z}$$

is integrable.

The vector fields W, Z, $\tilde{Z}$, $\tilde{W}$ are going to define a null tetrad at any point on the manifold $\mathcal{M}$, which is equivalent to give a conformal structure.

Let us consider now the vector fields on X $\frac{\partial}{\partial t_2}$ and $\frac{\partial}{\partial t_3}$. We have the following commutation relations

$$[E, \frac{\partial}{\partial t_2}] = d_2\frac{\partial}{\partial t_2}, \qquad [E, \frac{\partial}{\partial t_3}] = d_3\frac{\partial}{\partial t_3}$$

Consider their horizontal lift with respect to the Levi-Civita connection ∇_z associated to $g_1 + zg_2$. In the flat coordinates of g_1 this connection has the form

$$d + zC$$

so that these lifts are respectively

$$L = \frac{\partial}{\partial t_2} + zC^b_{2a} p_b \frac{\partial}{\partial p_a}$$

$$M = \frac{\partial}{\partial t_3} + zC^b_{3a} p_b \frac{\partial}{\partial p_a}$$

(C^c_{ab} are the components of C). We denote by p_a the coordinates on the fibers in TX. Define W, Z, $\tilde{Z}$, $\tilde{W}$ to be the degree 0 and degree 1 terms in z of these two vector fields (as in the previous proposition), then the quasihomogeneity condition guarantees that each of these vector fields descends up to scale on $\mathcal{M}$ (the commutation with the identity vector field is trivial).

The commutation of L and M is equivalent to the condition that ∇_z has vanishing curvature, and is thus equivalent to the WDVV equations. Furthermore, changing our vectors $\frac{\partial}{\partial t_2}$, $\frac{\partial}{\partial t_3}$ by a transformation in $Sl(2, \mathbb{C})$ clearly won't change the picture.

Since the vector fields L and M satisfy the hypothesis of the previous propositions, we end up with a four dimensional manifold $\mathcal{M}$ with an ASD conformal structure and an action of $Sl(2, \mathbb{C})$ on it.

Furthermore, this generalizes without any further problem to the n-dimensional case (isomonodromic deformations in any dimension have been given a twistor description [4] [6]).

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