62

ASD Null Kähler metrics with symmetry

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A four-dimensional Riemannian manifold which admits a covariantly constant spinor has to be hyper-Kähler: in the Euclidean signature a spinor and its complex conjugate form a basis of a spin space. This argument breaks down in the $(++--)$ signature (also called ultra-hyperbolic, Kleinian, neutral or split signature), where

$$\text{Spin}(2, 2) = SL(2, \mathbb{R}) \times \widetilde{SL}(2, \mathbb{R}),$$

and the representation space of the spin group splits into a direct sum of two real two-dimensional spin spaces S^A and $S^{A'}$. The conjugation of spinors is involutive and maps each spin space onto itself, and there exists an invariant notion of *real spinors*. One can therefore look for $(++--)$ non-Ricci flat metrics with a parallel real spinor (which we choose to be $\iota^{A'} \in \Gamma(S^{A'})$. The Ricci identities imply that the self-dual Weyl spinor is of type N: $C_{A'B'C'D'} = c\iota_{A'}\iota_{B'}\iota_{C'}\iota_{D'}$ for some c such that $\iota^{A'}\nabla_{AA'}c = 0$. I shall consider the anti-self-dual (ASD) case of $c = 0$. The corresponding metrics will be called *ASD null Kähler*. They have a null Ricci spinor, and vanishing scalar curvature. The resulting twistor theory is rich, and leads to a new integrable system in four dimensions [1]. In this paper I shall consider ASD null Kähler metrics which admit a Killing vector preserving the parallel spinor. Let us call them *ASD null Kähler metrics with symmetry.* I shall show that all such metrics are (at least in the real analytic case) locally determined by solutions to a certain integrable equation and its linearisation.

Proposition 1 *Let $H = H(x, y, t)$ and $W = W(x, y, t)$ be smooth real-valued functions on an open set $\mathcal{W} \subset \mathbb{R}^3$ which satisfy*

$$H_{yy} - H_{xt} + H_x H_{xx} = 0, \qquad (1)$$

$$W_{yy} - W_{xt} + (H_x W_x)_x = 0. \qquad (2)$$

Then

$$g = W_x(\mathrm{d}y^2 - 4\mathrm{d}x\mathrm{d}t - 4H_x\mathrm{d}t^2) - W_x^{-1}(\mathrm{d}z - W_x\mathrm{d}y - 2W_y\mathrm{d}t)^2 \qquad (3)$$

is an ASD null Kähler metric on a circle bundle $\mathcal{M} \to \mathcal{W}$. All real analytic ASD null Kähler metrics with symmetry arise from this construction.

Before proving this proposition I shall review some facts about Einstein–Weyl (EW) spaces associated to equation (1). In [2] it has been demonstrated that if an EW space admits a parallel weighted vector, the coordinates can be found in which the metric and the one form are given by

$$h = \mathrm{d}y^2 - 4\mathrm{d}x\mathrm{d}t - 4H_x\mathrm{d}t^2, \qquad \nu = -4H_{xx}\mathrm{d}t, \qquad (4)$$

and the EW equations reduce to (1)[1]. If $H(x,y,t)$ is a smooth real function of real variables then (4) has signature $(++-)$. It has also been shown that there exists a one to one correspondence between EW spaces (4) and two-dimensional complex manifolds (mini-twistor spaces) with a rational curve with normal bundle $\mathcal{O}(2)$ and a global section $\kappa^{-1/4}$, where κ is the canonical bundle. These structures should be invariant under an anti-holomorphic involution fixing a real slice in the twistor space.

Proof of Proposition 1. Let (h,ν) be a three–dimensional EW structure given by (4) and let (V,α) be a pair consisting of a function and a one-form which satisfy the generalized monopole equation

$$*_h(\mathrm{d}V + (1/2)\nu V) = \mathrm{d}\alpha, \tag{5}$$

where $*_h$ is taken with respect to h. It then follows from the Jones and Tod construction that

$$g = Vh - V^{-1}(\mathrm{d}z+\alpha)^2 \tag{6}$$

is a $(++--)$ ASD metric with an isometry $K=\partial_z$. Using the relations

$$*_h\mathrm{d}t = \mathrm{d}t\wedge \mathrm{d}y, \qquad *_h\mathrm{d}y = 2\mathrm{d}t\wedge\mathrm{d}x, \qquad *_h\mathrm{d}x = \mathrm{d}y\wedge\mathrm{d}x + 2H_x\mathrm{d}y\wedge\mathrm{d}t$$

we verify that equation (2) is equivalent to $\mathrm{d}*_h(\mathrm{d}+\nu/2)W_x = 0$. Therefore

$$W_{xx}\mathrm{d}y\wedge\mathrm{d}x + (2(H_xW_x)_x - W_{tx})\mathrm{d}y\wedge\mathrm{d}t + 2W_{xy}\mathrm{d}t\wedge\mathrm{d}x = \mathrm{d}\alpha,$$

and we deduce that W_x is the general solution to the monopole equation (5) on the EW background given by (4). We choose a gauge in which $\alpha = Q\mathrm{d}y + P\mathrm{d}t$. This yields

$$Q_x = -W_{xx}, \qquad P_x = -2W_{xy}, \qquad P_y - Q_t = 2(H_xW_x)_x - W_{xt}, \tag{7}$$

so $Q=-W_x + A(y,t)$, $P=-2W_y + B(y,t)$ and $\alpha = -W_x\mathrm{d}y - 2W_y\mathrm{d}t + A\mathrm{d}y + B\mathrm{d}t$. The integrability conditions $P_{xy} = P_{yx}$ are given by (2), and $A_t = B_y$. Therefore there exists $C(y,t)$ such that $A = C_y, B = C_t$. We now replace z by $z - C$ and the metric (6) becomes (3). This proves that (3) is ASD. It is also scalar-flat, because, as a consequence of (2),

$$R = 8(W_{xyy} - W_{xxt} + (H_xW_x)_{xx})W_x = 0. \tag{8}$$

We now choose the null tetrad

$$e^{00'} = -2W_x\mathrm{d}t,\ e^{10'} = \frac{\mathrm{d}z - 2W_y\mathrm{d}t}{2W_x},\ e^{01'} = \mathrm{d}z - 2W_x\mathrm{d}y - 2W_y\mathrm{d}x + ze^{00'},\ e^{11'} = \mathrm{d}x + H_x\mathrm{d}y + ze^{10'},$$

such that $g = 2(e^{00'}e^{11'} - e^{10'}e^{01'})$. The basis of SD two forms $\Sigma^{A'B'}$ is given by

$$\begin{aligned}
\Sigma^{0'0'} &= = \iota_{A'}\iota_{B'}\Sigma^{A'B'} = e^{00'}\wedge e^{10'} = \mathrm{d}z\wedge\mathrm{d}t\\
\Sigma^{0'1'} &= = \iota_{A'}o_{B'}\Sigma^{A'B'} = e^{10'}\wedge e^{01'} - e^{00'}\wedge e^{11'} = \mathrm{d}t\wedge\mathrm{d}(z^2) + 2\mathrm{d}t\wedge\mathrm{d}W + \mathrm{d}y\wedge\mathrm{d}z.\\
\Sigma^{1'1'} &= = o_{A'}o_{B'}\Sigma^{A'B'} = e^{01'}\wedge e^{11'} = 2W_x\mathrm{d}x\wedge\mathrm{d}y + 2(zW_x + W_y)\mathrm{d}x\wedge\mathrm{d}t - \mathrm{d}x\wedge\mathrm{d}z\\
&\quad + (2H_xW_x - 2zW_y)\mathrm{d}t\wedge\mathrm{d}y + z\mathrm{d}z\wedge\mathrm{d}y + (H_x+z^2)\mathrm{d}z\wedge\mathrm{d}t.
\end{aligned}$$

[1] With definition $u = H_x$ the x derivative of equation (1) becomes $(u_t - uu_x)_x = u_{yy}$, which is the dispersionless Kadomtsev–Petviashvili equation originally used in [2]. There are some computational advantages in working with the 'potential' form (1).

These two-forms satisfy:

$$-2\Sigma^{0'0'} \wedge \Sigma^{1'1'} = \Sigma^{0'1'} \wedge \Sigma^{0'1'}, \qquad \mathrm{d}\Sigma^{0'0'} = 0, \qquad \mathrm{d}\Sigma^{0'1'} = 0,$$

$$\mathrm{d}\Sigma^{1'1'} = \mathrm{d}(H_x - 2W) \wedge \mathrm{d}t \wedge \mathrm{d}z + (W_{xt} - W_{yy} - (H_x W_x)_x)\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}t. \tag{9}$$

Therefore the metric (3) admits a constant spinor[2] which is preserved by $K = \partial_z$.
Converse : Let g be a real analytic ASD metric with a covariantly constant spinor $\iota_{A'}$, which is Lie derived along a Killing vector K. From $C_{A'B'C'D'} = 0$ it follows that there exist coordinates $\pi^{A'}$ on the fibers of $S^{A'} \to \mathcal{M}$ such that $\pi^{A'}\nabla_{AA'}\pi^{B'} = 0$. Therefore a parallel section $\iota_{A'}$ of $S_{A'}$ determines a function $l = \pi^{A'}\iota_{A'}$ constant along the twistor distribution. The line bundle given by l on $\mathcal{PT}$ is isomorphic to $\kappa^{-1/4}$, where $\kappa = \Omega^3\mathcal{PT}$ is the canonical bundle. The Killing vector K gives rise to a holomorphic vector field on $\mathcal{PT}$ which preserves the divisor l . Therefore the minitwistor space $\mathcal{Z}$ (the space of trajectories of K in $\mathcal{PT}$) also admits a divisor with values in the $-1/4$ power of the canonical bundle. The minitwistor space $\mathcal{Z}$ satisfies the assumptions of Proposition 5.1 of Ref. [2] and the corresponding EW metric is of the form (4). Therefore $\hat{g} = \Omega^2 g$, where g is given by (3). Both $\hat{g}$ and g are scalar flat (this follows from the spinor Ricci identities and from equation (8) respectively). As a consequence we deduce that $\Omega = \Omega(t)$. Now we can use the coordinate freedom [2] to absorb Ω in the solution to the equation (1).

□

Remarks:

- This Proposition is analogous to a result of LeBrun [5] who constructs all scalar-flat Kähler metrics with symmetry in Euclidean signature from solutions to the $SU(\infty)$ Toda equation and its linearisation.

- If $H = const$ then (2) reduces to the wave equation in $2+1$ dimension, and consequently the metric (3) is the $(++--)$ Gibbons–Hawking solution [3].

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[2] This is a consequence of the following result: *Let $\Sigma^{A'B'}$ be a basis of real SD two-forms on an ASD scalar-flat manifold such that*

$$\mathrm{d}(\iota_{A'}\iota_{B'}\Sigma^{A'B'}) = \mathrm{d}(o_{A'}\iota_{B'}\Sigma^{A'B'}) = 0. \tag{10}$$

Then there exists a covariantly constant real section of $S^{A'}$.
Proof. Let $(o_{A'}, \iota_{A'})$ be a normalised spin basis. The covariant derivatives of the basis can be expressed as

$$\nabla_a \iota_{B'} = U_a \iota_{B'} + V_a o_{B'}, \qquad \nabla_a o_{B'} = W_a \iota_{B'} - U_a o_{B'}.$$

The first condition in (10) can be rewritten as $\nabla_A{}^{A'}(\iota_{A'}\iota_{B'}) = 0$, which implies $V_a = 2U_{AB'}\iota^{B'}\iota_{A'}$. The second condition in (10) yields $U_{AA'} = \alpha_A \iota_{A'}, W_{AA'} = \beta_A \iota_{A'}$ for some α_A, β_A. Therefore

$$\nabla_{AA'}\iota_{B'} = \alpha_A \iota_{A'}\iota_{B'}. \tag{11}$$

Contracting the RHS of the above equation with $\nabla^A{}_{C'}$, and symmetrising over $(A'B')$ gives 0 because g is ASD and scalar-flat. As a consequence $\nabla^A{}_{(C'}[\iota_{A')}\iota_{B'}\alpha_A] = 0$ which gives $\nabla^A{}_{A'}\alpha_A = 0$. Consider the real spinor $\hat{\iota}_{A'} := \iota_{A'}\exp f$, where $\iota^{A'}\nabla_{AA'}f = 0$ in order to preserve $\mathrm{d}\Sigma^{0'0'} = 0$. Integrability conditions for $\nabla_{AA'}f = \alpha_A \iota_{A'}$ are satisfied, as α_A solves the neutrino equation. Therefore we can find f for each α_A, and equation (11) implies that $\hat{\iota}_{A'}$ is covariantly constant.

□

- Note that $d\Sigma^{1'1'} \neq 0$ unless $W = H_x/2 + f(t)$, in which case

$$d\Sigma^{1'1'} = d(H_{xt} - H_x H_{xx} - H_{yy}) \wedge dy \wedge dt = 0,$$

and we are working in a covariantly constant real spin frame. The metric

$$g = \frac{H_{xx}}{2}(dy^2 - 4dxdt - 4H_x dt^2) - \frac{2}{H_{xx}}(dz - \frac{H_{xx}dy}{2} - H_{xy}dt)^2 \qquad (12)$$

is therefore pseudo hyper-Kähler. In [2] it was shown that all pseudo hyper-Kähler metrics with a symmetry satisfying $dK_+ \wedge dK_+ = 0$ are locally given by (12). Here dK_+ is an SD part of dK.

In the split signature we can arrange for one of the complex structures to be real and for the other two to be purely imaginary:

$$-I^2 = S^2 = T^2 = 1, \qquad IST = 1,$$

and S and T determine a pair of transverse null foliations. Now $g(TX, TY) = g(SX, SY) = -g(X, Y)$ for any pair of real vectors X, Y. The endomorphism I endows $\mathcal{M}$ with the structure of a two-dimensional complex Kähler manifold, as does every other complex structure $aI + bS + cT$ parametrised by the points of the hyperboloid $a^2 - b^2 - c^2 = 1$.

- If $W_x \neq H_{xx}/2$ then (3) is not Ricci–flat. This can be verified by a direct calculation. It also follows from more geometric reasoning: The Killing vector $K = \partial_z$ acts on SD two-forms by a Lie derivative. One can choose a basis $\Sigma^{A'B'}$ such that one element of this basis is fixed, and the Killing vector rotates the other two. The components of the SD derivative of K are coefficients of these rotations. Therefore $(dK)_+ = const$ if g is pseudo hyper-Kähler. In our case $dK_+ = (H_{xx}/W_x)dz \wedge dt$. Therefore H_{xx}/W_x must be constant for (3) to be Ricci-flat. An example of a non-vacuum metric is given by $W = H_y/2$.

I am grateful to David Calderbank, Lionel Mason and Paul Tod for useful discussions.

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