Twistor Newsletter

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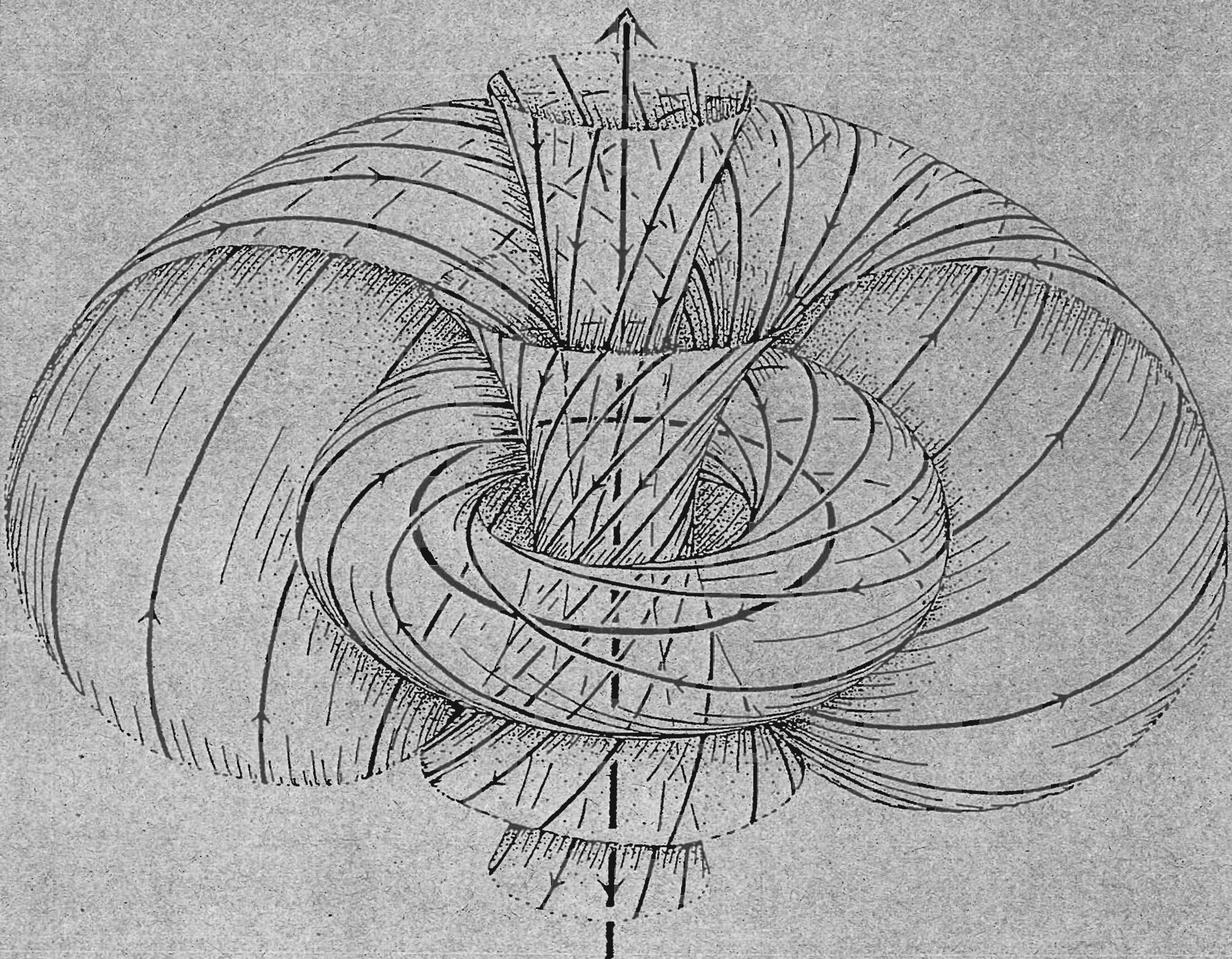

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On Extracting the Googly Information

For a quarter of a century, the most fundamental and problematic issue confronting twistor theory has been the googly problem, i.e. how do we represent the self-dual (SD) part of the gravitational field (and also gauge fields) in terms of twistor (as opposed to dual twistor) geometry. The original "non-linear graviton — i.e. leg break — construction and the Ward construction (found in, respectively, 1975 and 1976) showed how a very natural extension of flat-space twistor ideas could accommodate anti-self-dual (ASD) fields, but no way was seen how to treat the corresponding SD fields. Twistor theory has thus had 25 years of lop-sidedness; worse, the lack of a comprehensive googly construction has held up progress in almost all other areas of twistor physics. In particular, the main twistor approach to quantum field theory, namely twistor diagram theory, lacks a non-perturbative means of treating gauge interactions which could be supplied by a googly-modified Ward construction. Moreover, the twistor particle programme needed, among other things, a satisfactory way of getting at the concept of mass, so that it is not just an operator commuting with everything else. Mass, of course, is the source of gravity, so an appropriate twistor way of handling both SD and ASD aspects of gravity could supply a profound ingredient that is, at present, missing. Most importantly, there is no way that twistor theory can satisfy its original aims of supplanting the normal ideas of space-time by some appropriate non-local "quantum geometry" unless both ASD and SD parts of the gravitational field can be treated together. The hope that twistor theory might shed light on space-time singularities and, perhaps, on the quantum measurement problem, would have to be abandoned without such a development being forthcoming.

It should be remarked that the ambitwistor approach, for all its merits, cannot supply what is needed. A "non-linear graviton", as its name suggests, is to be regarded as a non-linear version of a wavefunction. An ordinary (i.e. linear) twistor wavefunction for a massless particle, of either positive or negative helicity, is a 1-function of a single twistor Z^α, not of a twistor together with a dual twistor. The canonical conjugate variable $W_\alpha = \bar{Z}_\alpha$ is taken over by the operator $-i\hbar\frac{\partial}{\partial Z^\alpha}$ rather than being considered as an independent variable. Moreover, to use a twistor description for the ASD part and a dual twistor description for the SD part makes no physical sense, since gravitons of mixed helicity (e.g. plane-polarized waves) should be as acceptable as gravitons of one or the other helicity.

In the 25 years of search for a "googly graviton", in which a 1-function of homogeneity -6 is to provide some kind of deformation of flat twistor space, the only way of doing this that has emerged was that originally described in RP, TN 3 (1976) pp. 12-17, where the fibres in $\mathbb{T}$ which project to single points in $\mathbb{PT}$ are deformed in a curious way, where on overlaps

$$\underset{i}{Z}^\alpha = (1 + f_{(-6)ij})^{1/6} \underset{j}{Z}^\alpha ,$$

giving an affine bundle of the 6^{th} power of the Euler fibres, considered as 1-dimensional vector spaces. (The "Euler fibres" are the inverse images $\mathbb{P}^{-1}x$ of points x, in the map from twistor space $\mathbb{T}$ to projective twistor space $\mathbb{PT}$. In general, I shall use the notation $\mathbb{P}^{-1}\mathcal{Q}$ to denote that part of the non-projective space that lies above a region $\mathcal{Q}$ of the projective space.) Despite the un-natural — even absurd — appearance of the above expression, it was later shown by MGE (in TN 14) that it had some genuine relationship to a very natural-looking construction involving a 6-dimensional affine bundle over $\mathbb{PT}$.

A significant development occurred in early 1998, when it was found that twistor spaces whose Euler fibres are deformed in accordance with the above can be constructed explicitly. This construction has been described in R.P., TN 44 (1998) pp. 1-9. Let us briefly recall this construction here. Take $\mathcal{M}$ be a either a real analytic Ricci-flat space-time with complexification $\mathbb{C}\mathcal{M}$ (a small "thickening" of $\mathcal{M}$) or else, itself, simply a complex Ricci-flat space-time. To begin, we see how to construct a curved twistor space $\mathcal{T}_x$ relative to an arbitrary point $x \in \mathcal{M}$. The projective relative twistor space $\mathbb{P}\mathcal{T}_x$ is essentially the projective hypersurface twistor space of the (if necessary, complexification $\mathbb{C}\mathcal{C}_x$ — but henceforth I shall drop the "$\mathbb{C}$" — of the) light cone $\mathcal{C}_x$ of x. Thus, each point of $\mathbb{P}\mathcal{T}_x$ corresponds to a complex curve z in $\mathcal{C}_x$ called an α-curve (or twistor line) with tangent vector of the form $o^A\mu^{A'}$, with $\mu^{A'}$ propagated parallel — in the sense of proportional — to itself along z:

$$o^A\mu^{A'}\nabla_{AA'}\mu^{B'} \propto \mu^{B'}, \quad \text{i.e.} \quad \mu^{A'}\mu^{B'}\nabla_{0A'}\mu^{B}_{B'} = 0,$$

where $o^A \tilde{o}^{A}$ are tangent vectors to the generators of $\mathcal{C}$ (and where lower indices 0 or $0'$ refer to contraction with o^A and $\tilde{o}^{A'}$, respectively). For the moment, let us assume $\mu_{0'} \neq 0$, so the α-curve is excluded from being a generator of $\mathcal{C}_x$. As LJM has shown (D.Phil. thesis, Oxford), if we were to take "$\mathcal{T}_x$" to be the ordinary hypersurface twistor space of $\mathcal{C}_x$ (taking scalings for $\mu_{B'}$ to be fixed by $\mu^{A'}\nabla_{0A'}\mu_{B'} = 0$) then this "twistor space" would encode only the ASD information of the (complex) gravitational field. To encode the SD gravitational information as well, we adopt the more complicated scaling, according to the twistor propagation equation:

$$\mu^{B'}\nabla_{0B'}\mu_{A'} = K\mu_{A'}(\mu_{0'})^{-5}\,\text{Þ}_c\,\tilde{\Psi}_{0'0'0'0'},$$

where Þ_c is the conformally invariant "thorn" of Spinors and Space-Time (RP & WR) vol. 1, p. 395 ($\text{Þ}_c = \nabla_{00'} - 4\varepsilon - 5\tilde{\rho}$, in spin-coefficient notation) where $\tilde{\Psi}_{A'B'C'D'}$ is the "massless field" of helicity $+2$ that is equal to the SD Weyl spinor

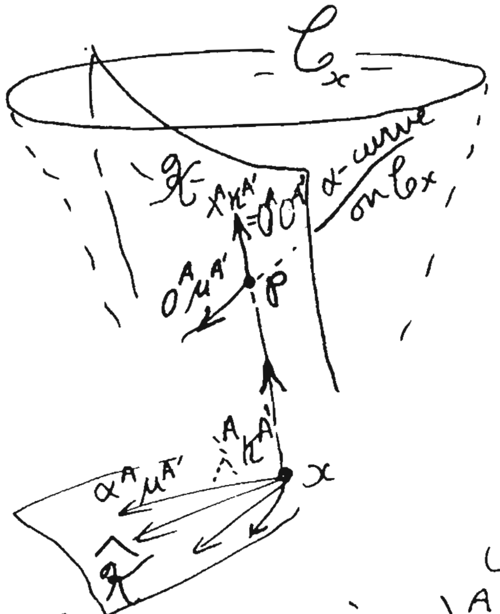
$\tilde{\Psi}_{A'B'C'D'}$ when the physical metric is used, but which scales according to $\tilde{\Psi}_{A'B'C'D'} \rightsquigarrow \Omega^{-1} \tilde{\Psi}_{A'B'C'D'}$

under conformal rescaling $g_{ab} \rightsquigarrow \Omega^2 g_{ab}$ (with $\tilde{\Psi}_{A'B'C'D'} \rightsquigarrow \tilde{\Psi}_{A'B'C'D'}$, $\varepsilon_{AB} \rightsquigarrow \Omega \varepsilon_{AB}$, $\varepsilon_{A'B'} \rightsquigarrow \Omega \varepsilon_{AB}$).
The above scaling equation for $\mu_{A'}$ is then conformally invariant and independent of rescalings of o^A and $\tilde{o}^{A'}$. In fact, the form of the twistor propagation equation is basically fixed by these invariance requirements. The quantity K is a specific numerical constant whose value should be fixed by the later considerations of this article.

We note that the different scalings for $\mu_{A'}$, for a given α-curve, constitute a 1-dimensional family. This family provides the fibre (Euler curve) $\mathbb{P}^{-1}q$ in $\mathcal{T}_x$ that lies above the point q in $\mathbb{P}\mathcal{T}_x$ that corresponds to this α-curve. If we take two points on the same α-curve, then the different scalings of $\mu_{A'}$ at one point are related to those at the other by a factor of the form $(1+f_{-6})^{\frac{1}{6}}$, as above, so the bundle structure of $\mathcal{T}_x$ over $\mathbb{P}\mathcal{T}_x$ indeed encodes the information of a 1-function of homogeneity degree −6 (in fact, an ordinary linear 1-function).

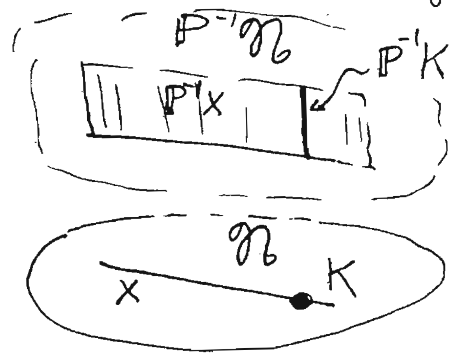
Up to this point, I have excluded the <u>generators</u> of $(\mathbb{C})\mathcal{C}_x$ as counting as "α-lines". If those were to be included in the definition of $\mathbb{P}\mathcal{T}_x$, then we would obtain a "blown-up" projective twistor space for which the point x itself would be represented as a <u>quadric</u> in $\mathbb{P}\mathcal{T}_x$, rather than as a projective line. There is an important shift in emphasis, however, in <u>this</u> article. The significance of the blown-up twistor space is (at least temporarily) being downgraded in importance. To complete the definition of $\mathbb{P}\mathcal{T}_x$, and subsequently of $\mathcal{T}_x$ itself, I propose, in effect to "blow-down" this quadric to obtain an actual projective line X in $\mathbb{P}\mathcal{T}_x$ to represent x. LJM has provided a procedure for doing this using an actual blow-down, but it is perhaps neater to use the following, which obtains $\mathbb{P}\mathcal{T}_x$ (and indeed $\mathcal{T}_x$) directly.

Consider a particular primed spinor $\kappa^{A'}(\neq 0)$ at x and the SD 2-plane element there consisting of tangent vectors of the form $\lambda^A\kappa^{A'}$, for varying λ^A. Construct the "α-like" 2-surface $\mathcal{X}$ swept out by the (null) geodesics through x having these tangent vectors at x, and along each such null geodesic, we parallel-propagate the (co-)tangent vector $\lambda_A\kappa_{A'}$. (If $\mathcal{M}$ were ASD then $\mathcal{X}$ would indeed be an α-surface in $\mathcal{M}$, but in general the SD condition will fail at most points of $\mathcal{X}$ owing to the presence of SD Weyl curvature.) It is important to note that $\mathcal{X}$ is completely smooth (holomorphic) in some open neighbourhood $\mathcal{W}$ of x, this being an instance of the regularity of the exponential map at x. Consider the bundle of non-zero primed spinors $\mu_{A'}$ at points of $\mathcal{W}\cap\mathcal{X}$ with $\mu_{A'}\kappa^{A'}\neq 0$. This, also, is a holomorphic manifold. Moreover, at points away from x, it is biholomorphic to an open neighbour[hood] of $\mathcal{T}_x$ because, locally at each point p of $\mathcal{W}\cap\mathcal{X}$, we can regard p as a point of an α-curve on $\mathcal{C}_x$ meeting $\mathcal{X}$ at p, where we carry $\mu_{A'}$ away from p according to the twistor propagation equation. What is particularly noteworthy is that we can now apply this at the point x itself; here (and only here



do we obtain (locally) a non-uniqueness for the "α-curve" (actually a generator of $\mathcal{C}_x$) meeting $\mathcal{X}$. In fact, the entire α-like plane $\hat{\mathcal{X}}$ through x, with SD 2-plane element defined by $\mu^{A'}$ at x, corresponds to the single choice of $\mu_{A'}$ at x, with tangents of the form $\lambda^A\mu^{A'}$ at x for varying λ^A. Thus, the "blow-down" is automatically performed, giving us $\mathbb{P}\mathcal{T}_x$ directly as a smooth (holomorphic) 3-manifold locally. This also gives us $\mathcal{T}_x$ directly, but we must bear in mind that the scaling in the twistor propagation equation becomes singular (in the presence of SD Weyl curvature) owing to the fact that $\mu_{0'}=0$. Here, the α-curves are simply generators of $\mathcal{C}_x$ and the relevant scaling for each corresponding twistor is determined by its value of $\mu_{A'}$ at x, without propagation.

This construction gives a description of that part of $\mathbb{P}^{-1}\mathcal{N}$ of $\mathcal{T}_x$ lying above a neighbourhood $\mathcal{N}$ of some portion of the line $X\subset\mathbb{P}\mathcal{T}_x$ representing x, this portion merely having to exclude the point K of X which represents the α-like plane $\mathcal{X}$ itself. By choosing two different (non-proportional) values of κ^A, we can cover the whole of a neighbourhood of X with two patches. Most points of $\mathcal{T}_x$ (in this neighbourhood of $\mathbb{P}^{-1}X$) have descriptions in each patch which are directly holomorphically



related to each other. The only issue needing further consideration is whether this holomorphicity extends to the points of $\mathbb{P}^{-1}X$. (That it extends to the points of X in $\mathbb{P}\mathcal{T}_x$ follows from Hartogs-type theorems.) It seems that it does, but further clarification on this point is needed.

To pass from $\mathcal{T}_x$ to a canonical twistor space $\mathcal{T}$, let us assume that M is strongly asymptotically flat in the sense that it has an (analytic) future null infinity $\mathscr{I}^+$ and a regular point i^+ at future timelike infinity — so that no material sources or black holes are present in the remote future. We then define $\mathcal{T} = \mathcal{T}_{i^+} (= \mathcal{T}^+)$, taking advantage of the conformal invariance properties of the construction. (It is usual to use spinors $\iota_A, \bar{\iota}_{A'}$ in place of $o_A, \bar{o}_{A'}$, when $\mathcal{C}_x = \mathscr{I}^+$, so this substitution is to be made in the twistor propagation equation.) It is to be expected that the more general situation when there are material sources or black holes in the remote future, there will be some appropriate way of incorporating this information into the structure of $\mathcal{T}$. This is an issue to be addressed by future considerations.

A comment is pertinent, at this stage, concerning the choice of $\mathscr{I}^+$ for the definition of $\mathcal{T}$, rather than $\mathscr{I}^-$. We know from the work of Friedrich that a broad ("generic") class of vacuum space-times exists satisfying the above condition of strong asymptotic flatness. But we do not yet know, for sure, whether there are any such vacuum space-times apart from the trivial case of Minkowski space for which there is also an analytic $\mathscr{I}^-$. Indeed, there is a school of thought according to which regularity at $\mathscr{I}^-$ and at $\mathscr{I}^+$ would be essentially incompatible, in general. Be that as it may, a choice has to be made, in the present attempt at a twistor description of general (asymptotically flat) vacuum space-times, as to whether to base this on $\mathscr{I}^+$ or $\mathscr{I}^-$. Given this choice, $\mathscr{I}^+$ is definitely to be preferred. In realistic situations involving gravitationally radiating systems we would expect this radiation to be retarded. The role of $\mathscr{I}^-$ would be merely for specifying this condition of retardedness and for this a $\mathscr{I}^-$ with a low order of differentiability would provide an amply adequate framework. On the other hand, $\mathscr{I}^+$ would contain all the interesting information about the nature of the radiation, mass loss, angular momentum, charges, NP constants, etc., these things requiring a much more refined differentiability structure for $\mathscr{I}^+$.

In relation to all this, we must bear in mind that one of the (distant?) aims of twistor theory is to provide an appropriate marriage between general relativity and quantum mechanics according to which the rules of quantum mechanics will have to be modified — in addition to anticipated modifications that space-time structure will have to undergo at (say) the Planck-length scale. On many occasions I have expressed the view that this marriage ought to provide a solution to the measurement problem of quantum mechanics (see, most particularly, R.P. in _Mathematical Physics 2000_, Eds. A. Fokas, A. Grigoryan, T. Kibble, B. Zegarlinski (2000, Imperial College Press) pp. 266-282 and R.P. (1996) Gen. Rel. Grav. **28** pp. 581-600) and that it must possess a fundamental _time asymmetry_ (cf. R.P. in TN 22 (1986) pp. 1-3). The main reason to expect a time-asymmetry in the correct quantum/gravity marriage is the gross time-asymmetry in space-time singularity structure that we find in Nature: future-type singularities appear to be "generic" whereas in the past (the big bang) we find a singularity that is extraordinarily constrained (to a part in <u>at least</u> $10^{10^{123}}$), whence the 2^{nd} law of thermodynamics and the enormously great uniformity of the early universe arise. An explanation of the singularity structure that we find in the universe ought presumably to be a consequence of the correct quantum/gravity.

This suggests that we ought to seek a twistor description of general relativity that is fundamentally time-asymmetric. Accordingly, if some form of time-asymmetry indeed appears to be forced on us, as here, we should not try to resist it, for it may actually be no bad thing. For 25 years, the twistor approach to general relativity has suffered from a manifest <u>chiral</u> asymmetry (whence the googly problem) whereas it is <u>time</u>-asymmetry that would appear to be the more evident physical requirement (and we recall that time-asymmetry is a feature of quantum state reduction as much as of singularity structure (cf. above).

The space $\mathcal{T}$ has a local structure that is given by a 1-form ι up to proportionality and a 3-form Θ up to proportionality subject to

$$\iota\wedge d\iota = 0\,, \quad \iota\wedge\Theta = 0$$

so that ι defines a foliation by 3-manifolds (except where $\iota=0$, which occurs only on $\mathbb{P}^{-1}I$, where the line $I\subset\mathbb{P}\mathcal{T}$ represents i^+) and Θ defines a foliation of these 3-manifolds (and hence of $\mathcal{T}$ itself) by a family of curves — the <u>Euler curves</u>. There is a further structure, which restricts the allowed scalings for ι and Θ, that can be specified as the two quantities

$$\Pi = d\Theta\otimes\iota \quad \text{and} \quad \Sigma = d\Theta\otimes d\Theta\otimes\Theta$$

being given, where we also demand that

$$\Pi = -2\Theta\odot d\iota.$$

(The bilinear operator $\odot$, acting between an n-form α and a 2-form is defined by $\alpha\odot(dp\wedge dq) = (\alpha\wedge dp)\otimes dq - (\alpha\wedge dq)\otimes dp$.)

The invariance of Π and Σ demands that on an overlap between two patches $\mathcal{U}$ and $\hat{\mathcal{U}}$ of $\mathcal{T}$, on which different scalings for ι and Θ are chosen, we have

$$\hat{\iota} = k\iota \ , \quad \hat{\Theta} = k^2\Theta \ , \quad d\hat{\Theta} = k^{-1}d\Theta .$$

The equality between the two expressions for Π tells us that ι (and $d\iota$) have homogeneity degree 2 with respect to the Euler operator $\Upsilon = \Theta \div (\frac{1}{4}d\Theta)$ (which means $\Upsilon(a)\,d\Theta = 4\,da \wedge \Theta$ for any scalar a): $\pounds_\Upsilon \iota = 2\iota$ (and $\pounds_\Upsilon d\iota = 2d\iota$). We also have (automatically) that Θ (and $d\Theta$) have homogeneity degree 4. We find that, on an overlap between patches $\mathcal{U}$ and $\hat{\mathcal{U}}$ that

$$\hat{\Upsilon} = k^3\Upsilon \quad \text{and} \quad k^3 = 1 + F z^{-6}$$

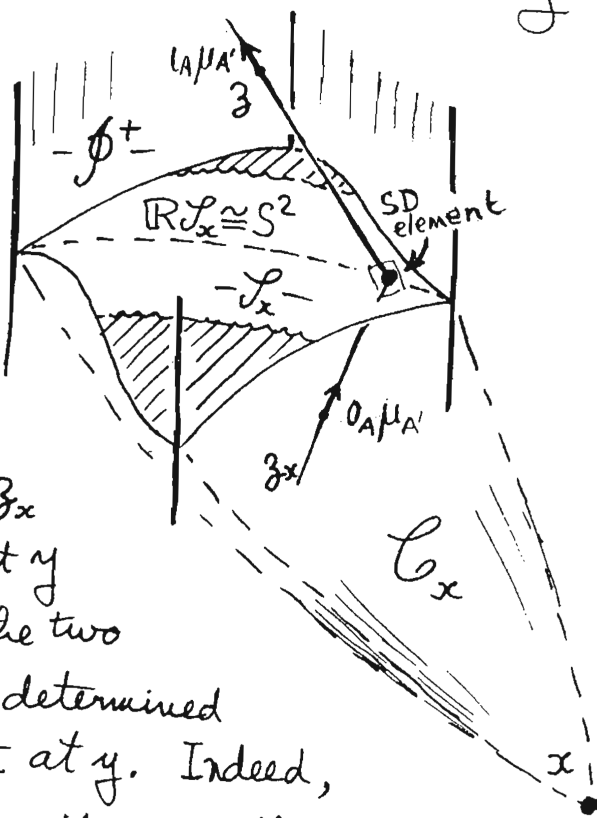
with respect to a "natural" parameter z along Euler curves satisfying $\Upsilon(z) = z$, where F is constant along Euler curves, i.e.

$$k^3 = 1 + f_{-6}$$

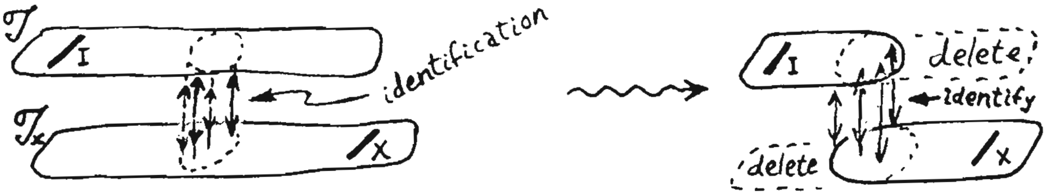
for some f_{-6} defined on $\mathcal{U} \cap \hat{\mathcal{U}}$ which is homogeneous of degree -6 with respect to Υ. This is basically the same type of behaviour as that referred to at the beginning of this article.

All this structure can be seen to be present in the explicit definition of $\mathcal{T}$ given above. In the flat case, we can take $\iota = \pi_{A'}d\pi^{A'} = I_{\alpha\beta}Z^\alpha dZ^\alpha$ and $\Theta = \frac{1}{6}\epsilon_{\alpha\beta\gamma\delta}Z^\alpha dZ^\beta \wedge dZ^\gamma \wedge dZ^\delta$, in standard descriptions. In the leg-break twistor space for ASD vacuums, the canonical projection from the twistor space to the $\pi_{A'}$-space collapses the 2-surfaces, along which the (simple) 2-form $d\iota$ vanishes, down to points of $\mathbb{P}\mathcal{T}$. In this case, the quantities ι and Θ are both invariant, which is stronger than merely Π and Σ being invariant.

The question that I attempt to address in this article is: how do we reconstruct the (complex) space-time $\mathcal{M}$ (or $\mathbb{C}\mathcal{M}$ for real $\mathcal{M}$ — but I am dropping the "$\mathbb{C}$") from the twistor space $\mathcal{T}$? The general way in which this is intended to work was indicated in R.P. TN 44. For any point $x \in \mathcal{M}$, close enough to i^+ (but not on $\mathscr{I}^+$), there will be an intersection $\bar{\mathcal{C}}_x \cap \mathscr{I}^+ = \mathscr{S}_x$ (what ETN and collaborators call a "light-cone cut") that encodes the point x itself in terms of information at $\mathscr{I}^+$. (I shall be concerned with complex cuts, so in the situation of a real space-time $\mathcal{M}$, the notation "$\mathscr{S}_x$" refers to a local complexification of this intersection which would otherwise have been denoted by "$\mathbb{C}\mathscr{S}_x$".) There is a non-trivial S^2 in the complex space $\mathscr{S}_x$ (which would be the anti-celestial sphere, for a real point x, or a deformation of this for a complex x). Let us refer to this sphere as $\mathbb{R}\mathscr{S}_x$ (even in the case of complex x). When x is close enough to i^+ (as is being assumed here), then $\mathbb{R}\mathscr{S}_x$ will be non-singular. But as x moves away from i^+, we expect $\mathbb{R}\mathscr{S}_x$ to acquire caustics and crossing regions.

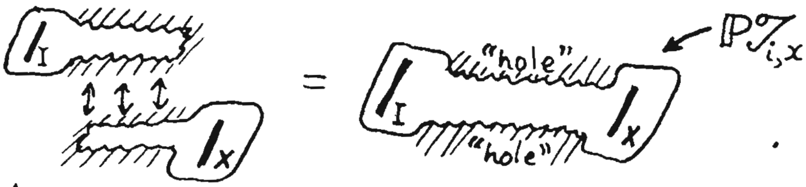


Let y be an arbitrary point of $\mathscr{S}_x$. A general α-curve z in $\mathscr{I}^+$, through y will have a $\mu_{A'}$-value at y (the tangent vector to z at y having the form $\iota^A \mu^{A'}$) and we use this same $\mu_{A'}$-value to continue with a ve curve z_x in $\mathcal{C}_x$ (with tangent vector to z_x at y having the form $o^A \mu^{A'}$) so that the two α-curves are tangent to (and are determined by) the same SD 2-plane element at y. Indeed, we can use this procedure to provide us with a continuation of the scaling for the twistor (scaling provided by $\mu_{A'}$) from z to z_x and thereby provide an identification between elements of $\mathcal{T}$ and elements of $\mathcal{T}_x$. This identification does not apply unambiguously to all elements of $\mathcal{T}$ and of $\mathcal{T}_x$, but only to parts of $\mathcal{T}$ represented by α-curves z which meet $\mathcal{I}_x$ cleanly (by which I mean transversally in a single point), and to parts of $\mathcal{T}_x$ represented by α-curves z_x which meet $\mathcal{I}_x$ cleanly. We may be able to extend the identification between $\mathcal{T}$ and $\mathcal{T}_x$ if we allow this correspondence not to be 1-1, but if we insist on a 1-1 relationship (for x near to i^+) we can obtain a "surgery" of $\mathcal{T}$ in which part of $\mathcal{T}_x$ (near X) is "glued" into a "hole" in $\mathcal{T}$ obtained by deleting the

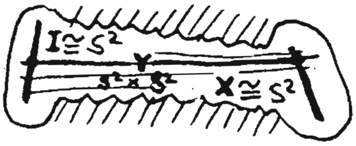


part "near X" in $\mathcal{T}$. I shall call this "glued" twistor space $\mathcal{T}_{i,x}$.

In fact this is not a proper surgery in the usual sense, as significant "holes" will survive, in $\mathbb{P}\mathcal{T}_{i,x}$, as indicated very schematically:



The diagrams are to indicate that the entire I and X have neighbourhoods (with compact closure) in $\mathbb{P}\mathcal{T}_{i,x}$, and the parts of $\mathcal{T}_{i,x}$ above these regions will have neighbourhoods (but not with compact closure) of the entire $\mathbb{P}^{-1}I$ and $\mathbb{P}^{-1}X$, respectively. The "neck" joining the I-region to the X-region in $\mathbb{P}\mathcal{T}_{i,x}$ will contain a non-trivial topological S^2, in fact an S^2's worth of projective lines Y, corresponding to the points of $\mathbb{R}\mathcal{I}_x$. Each point of I and each point of X will lie on such a line Y, and this family of Ys traces out a (non-holomorphic $S^2 \times S^2$.

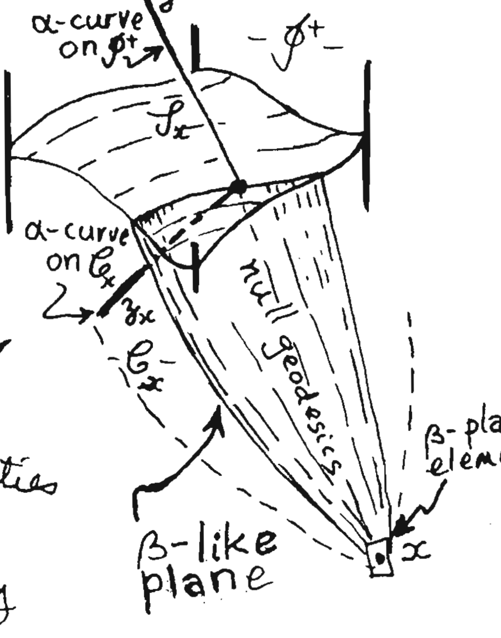


The key problem, with regard to extracting the googly information contained in $\mathcal{T}$, is in deciding which are allowable surgeries $\mathcal{T}_{i,x}$, whence the points x of the complexified) space-time $\mathcal{M}$ to be constructed are thereby determined. A secondary problem is the twistor expression for the <u>metric</u> of $\mathcal{M}$, once its family of points (i.e. family of cuts of $\mathscr{I}^+$) is known. I shall address this secondary question first, since its essential resolution has served to point to a new direction for resolving the first (key) problem.

A suggestion for the metric that I had tentatively suggested earlier turns out not to be correct, as the accompanying article by SF, FH, LJM, and ETN shows, but a new suggestion for the metric, developed by FH and RP from KPT's twistor formulation of ETN's remarkable original formula for the reciprocal of the metric in terms of an integral of W^{-2} over $\mathbb{R}\mathcal{S}$, where W represents the displacement of the cut defining the space-time point x from that defining a neighbouring point x' to x. To translate this into the present twistor framework, we need a 1-form ξ for $\mathcal{T}_x$, defined locally up to proportionality, in a way similar to the way that ι is defined for $\mathcal{T}$. The quantity $\Pi_x = d\Theta \otimes \xi$ is an invariant structure for $\mathcal{T}_x$ (as is Σ) and we require $\Pi_x = -2\Theta \otimes d\xi$ also, so on the overlap between patches $\mathcal{U}$ and $\hat{\mathcal{U}}$ we have

$$\hat{\xi} = k\,\xi$$

just as was the case for ι. To define ξ up to proportionality, we consider the family of β-like planes ruled by the null geodesics through x that are tangent to fixed β-plane elements (ASD plane elements) at x. In the pure googly (SD) case, these β-like planes will be actual β-planes, and their intersections with $\mathscr{I}^+$ will be β-curves on $\mathscr{I}^+$ and α-curves on $\mathcal{C}_x$. In the general case, however, none of these properties will normally hold. These β-like planes will meet $\mathscr{I}^+$ in curves that I shall refer to as <u>ξ-curves</u>. The family



of α-curves on $\mathscr{I}^+$, on ∂n $\mathcal{C}_x$, that meet a fixed ξ-curve will provide a 3-surface in $\mathbb{T}$, or in $\mathbb{T}_x$, whose tangent directions annihilate ξ. (In $\mathbb{PT}$ or in $\mathbb{PT}_x$ this would be a 2-surface, referred to as a "crinkly cone in $\mathbb{PT}$.)

This construction does not fix the scaling for ξ, and a reasonable-looking requirement for restricting this scale freedom (consistent with the "$\gamma + \lambda = \Theta$" of R.P. in TN43, p.3) is to demand

$$\xi \wedge dI + d\xi \wedge I = 2\Theta$$

(which implies $d\xi \wedge dI = 4d\Theta$). We note that there is an entire (holomorphic) $\mathbb{CP}^1$'s worth of ξ-curves on $\mathscr{I}_x$, even though individual ξ-curves generally encounter singularities. The freedom in the ξ-scaling — where I am assuming that the above relationship can be (and is) satisfied — is given by $\xi \mapsto (1-e^{\lambda})\xi$, where

$$d\lambda \wedge \xi \wedge I = 2\Theta.$$

This is an equation governing the propagation of λ, along each projective line Y, which has no global solution over Y (a whole $\mathbb{CP}^1$). (Note that $d\lambda$ has to become infinite at both ends, at I and X.) Thus, the scaling for ξ appears to be fixed by these considerations.

The suggested twistor expression for the metric (discussed more fully in the accompanying article by FH, LJM, and ETN) is given by

$$\frac{1}{g} = \frac{1}{8\pi^2}\oint_{S^3}\frac{\Theta}{\gamma^2} \qquad \text{where } \xi \wedge \xi' \wedge I = \gamma\Theta,$$

the metric $g = g(\delta x, \delta x)$ being evaluated at the point x, corresponding to ξ, where δx represents the infinitesimal displacement from the point x to a neighbouring point $x' = x + \delta x$, this neighbouring point corresponding to the 1-form ξ', which differs from ξ by an infinitesimal amount. (N.B. In this expression, terms in $(\delta x)^2$ play no role, although in my earlier incorrect expression "$d\xi \wedge d\xi' = -\frac{1}{2} g\, d\Theta$" these would have been the crucial terms.) Note that the scalar quantity γ scales as $\hat{\gamma} = k^2 \gamma$ from patch to patch, whence the integrand Θ/γ^2 is unchanged from patch to patch. We know, from the accompanying article by SF, FH, LJM, and ETN that this expression is correct in the ASD and SD cases. It is at least conformally correct in the general case, since it gives the null cones correctly.

Let us now attempt to address the key issue of fixing the cuts $\mathcal{J}_x$ to be actually light-cone cuts. My present ideas on this are still not fully formulated, but they appear to have been driven relentlessly in what I hope is the appropriate direction towards this goal (and, I hope, is not too far from it). The essential idea is to provide a condition that is to hold separately at each point Y of the cut $\mathcal{J}_x$ (or $\mathbb{R}\mathcal{J}_x$, which would be sufficient), this condition to depend upon the scalings of the relavent forms and on the -6 homogeneity functions that relate these scalings from patch to patch. The special role that these functions have that is relevant to our purposes is the following.

Let ξ, η, and ζ be 1-forms, each of homogeneity degree 2, scaling from patch to patch according to

$$\hat{\xi} = k\xi, \quad \hat{\eta} = k\eta, \quad \hat{\zeta} = k\zeta.$$

Consider the 3-form $\xi \wedge \eta \wedge \zeta$. This will have a homogeneity degree, namely 6, and must therefore be proportional to Θ. On an overlap between patches, we have

$$\hat{\xi} \wedge \hat{\eta} \wedge \hat{\zeta} - \xi \wedge \eta \wedge \zeta = (1 + f_{-6})\, \xi \wedge \eta \wedge \zeta - \xi \wedge \eta \wedge \zeta$$

$$= f_{-6}\, \xi \wedge \eta \wedge \zeta.$$

This has homogeneity degree 0, so it is defined on the projective space $\mathbb{P}\mathcal{T}$ (or $\mathbb{P}\mathcal{T}_x$ or $\mathbb{P}\mathcal{T}_{i,x}$). This space is 3-dimensional so $d(f_{-6}\xi\wedge\eta\wedge\zeta)$ must vanish. Hence

$$d(\hat{\xi}\wedge\hat{\eta}\wedge\hat{\zeta}) = d(\xi\wedge\eta\wedge\zeta),$$

the 4-form $d(\xi\wedge\eta\wedge\zeta)$ extends globally across patches. Note that in an application of the fundamental theorem of exterior calculus (FTEC — i.e. the theorem commonly referred to as "the generalized Stokes theorem") we have to pay attention to these -6-functions when we pass from one patch to another. For example, the integral of "$d(\xi\wedge\eta\wedge\zeta)$" over a compact 4-contour need not vanish, for this reason.

We wish to apply these ideas to the particular case when $\zeta = \iota$ and η is a 1-form, like ξ, but which refers to an arbitrary point y on $\mathscr{I}_x$ (or on $\mathbb{R}\mathscr{I}_x$) rather than to x itself. We cannot now expect to use the normalization "$\eta\wedge d\iota + d\eta\wedge\iota = 2\Theta$" since the left-hand side now vanishes (because there are not enough variables in η and ι to span all three projective dimensions). In what follows, a normalization for η plays no role. We wish to obtain a condition on these forms which will amount to the fixing of one complex number per point (i.e. y) of $\mathscr{I}_x$ so, in principle, this could fix the cut of $\mathscr{I}^+$ that determines a point x. The idea is to integrate the 4-form

$$\chi = d(\xi\wedge\eta\wedge\iota),$$

which by the above discussion is global across patches, over some appropriate 4-contour with boundary, in $\mathcal{T}_{i,x}$, and then use the FTEC to convert this to an integral over the boundary, where patching issues bring the f_{-6} functions into play.

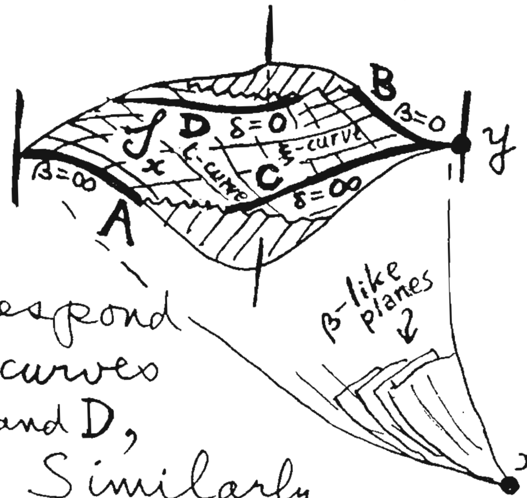
At the time of writing there are still some uncertainties as to the exact regions that would be needed. I shall first describe a prescription that is not (quite?) correct and then make some suggestions as to how this might appropriately be amended. To begin, we find scalar functions A, B, C, D which scale from patch to patch according to

$$\hat{A} = k^{\frac{1}{2}}A, \quad \hat{B} = k^{\frac{1}{2}}B, \quad \hat{C} = k^{\frac{1}{2}}C, \quad \hat{D} = k^{\frac{1}{2}}D,$$

each of A, B, C, D being homogeneous of degree 1, where

$$\iota = A\,dB - B\,dA, \quad \eta = B\,dC - C\,dB, \quad \xi = C\,dD - D\,dC.$$

We do this by noting, first, that there is a complete Riemann sphere's worth of ι-curves (ι-curves being the intersections of $\mathcal{S}_x$ with β-planes on $\mathcal{P}^+$) and of ξ-curves (from the primed spin-spaces at i^+ and at x, respectively, so the complex stereographic coordinates $\beta = B/A$, $\delta = D/C$ can be defined on $\mathcal{T}$ and on $\mathcal{T}_x$. Geometrically, $\delta = \infty$ and $\delta = 0$ are to correspond to two ξ-curves on $\mathcal{S}_x$, these curves to be where α-lines meeting **C** and **D**, respectively, become zero. Similarly $\beta = \infty$ and $\beta = 0$ correspond to two ι-curves on $\mathcal{S}_x$ where α-lines meeting **A** and **B**, respectively, become zero.



We observe that ι is proportional to $d\beta$ and ξ is proportional to $d\delta$, so we can define A and C by

$$\iota = A^2 d\beta, \quad \xi = C^2 d\delta$$

(there being no obstruction to taking the square roots), whence we can now define

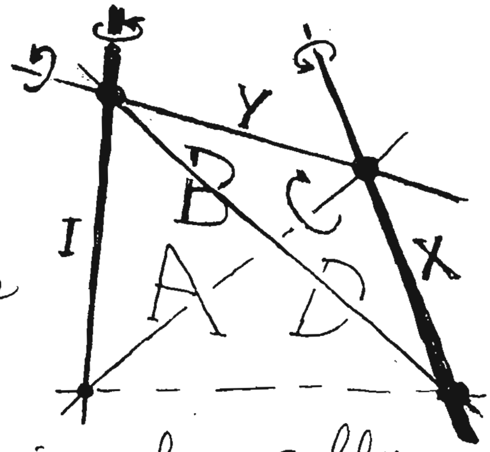
$$\mathbf{B} = A\beta, \quad \mathbf{D} = C\delta.$$

Each constant ratio A:B gives an ι-curve and each constant ratio D:C gives a ξ-curve. In the above, we can also choose the curves C ($\delta = \infty$) and B ($\beta = \infty$) to pass through the point y. Then we can take a new Riemann sphere (stereographic) coordinate γ defined by

$$\gamma = C/B$$

and we have $\eta = B^2 d\gamma$. (This is basically the <u>definition</u> of η that I am adopting here. Previously, I had given merely a vague description).

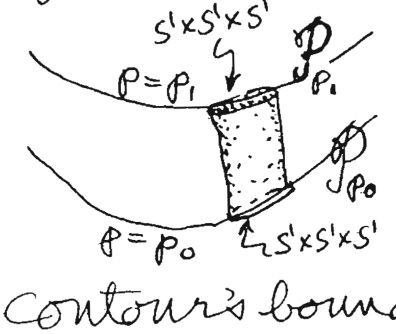
The geometric role of the three Riemann sphere coordinates β, γ, δ, in the case of flat twistor space $\mathbb{PT}$, is given in the figure. Planes through I are labelled by β, through Y by γ, and through X by δ. The coordinate space is $S^2 \times S^2 \times S^2$ which is, of course, topologically distinct from $\mathbb{CP}^3$. The relation to $\mathbb{CP}^3$ involves a blow-up of I and X, a "double blow-up at the points $I \cap Y$ and $X \cap Y$ and a blow-up followed by a blow-down in the opposite direction along Y (so that the line Y itself is reduced to a point but the family of planes through it provides a new line Y′). Note that this "blow-up situation" is quite different from that which motivated many earlier attempts at the googly.



As our first try for a contour arrangement, we try to integrate the quantity χ $(= d(\xi \wedge \eta \wedge \iota))$ over a 4-region bounded by two 3-surfaces $\mathcal{P}_\rho$, given by two values of ρ, where

$$AB^2C^2D = \rho.$$

If this can be done entirely in one patch, the integral of χ over the 4-volume (real 4-contour with boundary) would be equal to the differen[ce] between the integrals of $\xi \wedge \eta \wedge \iota$ over (a closed contour in) $\mathcal{P}_{\rho_1}$ and over (a closed 3-contour in) $\mathcal{P}_{\rho_0}$. If our contour's boundaries were some appropriate pair of $S^1 \times S^1 \times S^1$ regions, then each

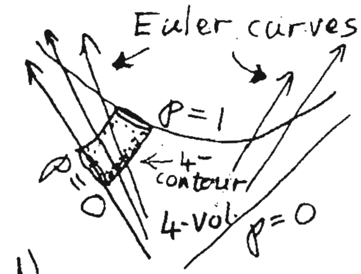


$$\oint_{\mathcal{P}_\rho} \xi \wedge \eta \wedge \iota = \rho \oint_{\mathcal{P}_\rho} \frac{\xi \wedge \eta \wedge \iota}{\rho} = \rho \oint \frac{(AdB - BdA)}{AB} \wedge \frac{(BdC - CdB)}{BC} \wedge \frac{(CdD - \ldots}{CD \ldots}$$

$$= \rho\,(2\pi i)^3,$$

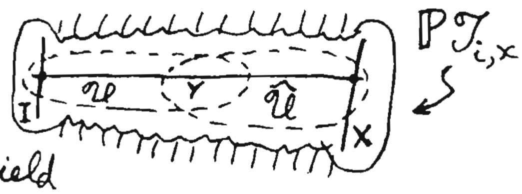
so the integral of χ between $\mathcal{P}_{\rho_0}$ and $\mathcal{P}_{\rho_1}$ would turn out to be $(\rho_1 - \rho_0)(2\pi i \ldots$

The idea is now to choose (say) $p_1 = 1$ and $p_0 = 0$, whence the entire integral should turn out to be $(2\pi i)^3$. We note that in the above displayed expression, the third $\oint$ is actually a projective twistor integral, since the integrand has homogeneity 0. When $p=0$, however, this does not apply, and the contour gets squashed into a region that contains Euler curves, so that projectively it is 2-dimensional, one of the contour directions being actually along the Euler curves, so the entire 3-dimensional boundary integral (at that end) is essentially non-projective. The integrand is now zero, however, because $\xi \wedge \eta \wedge \iota$ is proportional to Θ and vanishes along the Euler direction.



Now suppose that our 4-volume needs to be covered by two patches. A complication now arises because the equation $AB^2C^2D = p$ is not invariant under the googly rescalings, so the 3-surface $\mathcal{P}$ ($=\mathcal{P}_1$) "jumps" to a new 3-surface $\hat{\mathcal{P}}$, given by $\hat{A}\hat{B}^2\hat{C}^2\hat{D} = 1$, i.e. $AB^2C^2D = \frac{1}{1+f_{-6}}$. The situation is like that indicated in the diagram. How do we deal with this?

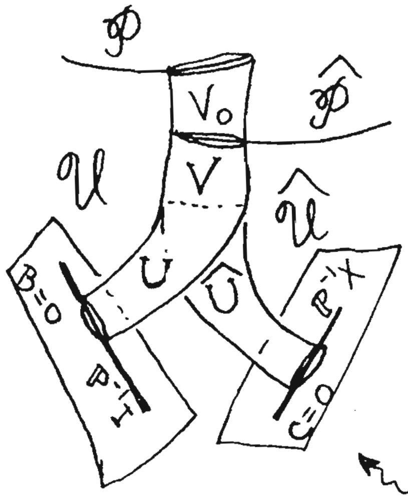
I am going to suppose that we are in a situation that is simple enough that, for a given choice of Y, the patches can be arranged (using "cohomology freedom") so that for a neighbourhood of the line $Y \in \mathbb{P}\mathcal{T}_{i,x}$ we can use just two patches, one of which contains $I \cap Y$ and the other, $X \cap Y$. (This is certainly OK in the weak-field limit, say with an elementary state.) We can arrive



at the $\rho=0$ boundary in various different ways, but let us choose our contour as collapsing down to $\rho=0$ by means of the limit $\varepsilon\to 0$ $(\varepsilon^2=\rho)$ with

$$A=e^{2i\theta},\quad B=\varepsilon e^{-i(\theta+\phi+\psi)},\quad C=e^{i\phi},\quad D=e^{2i\psi}$$

near I (in the $\mathcal{U}$ region) and by means of $\varepsilon\to 0$ with



$$A=e^{2i\theta},\quad B=e^{i\varphi},\quad C=e^{-i(\theta+\varphi+\psi)},\quad D=e^{2i\psi}$$

near X (in the $\hat{\mathcal{U}}$ region). Note that there is a non-Hausdorfness at the $\rho=0$ boundary, in the coordinate space of (A,B,C,D), where $B=0$ and $C=0$ are not Hausdorffly separated. The χ-volumes $U, \hat{U}, V$, and V_0 are as indicated in the diagram, all boundaries lying in $B=0$, $C=0$, $\mathcal{P}$, and $\hat{\mathcal{P}}$. Applying FTEC we obtain

$$U+V+V_0 = \oint_{\mathcal{P}} \xi\wedge\eta\wedge\iota$$

and

$$\hat{U}+V = \oint_{\hat{\mathcal{P}}} \hat{\xi}\wedge\hat{\eta}\wedge\hat{\iota}.$$

But we see that $\oint_{\mathcal{P}}\xi\wedge\eta\wedge\iota$ is just the same expression as $\oint_{\hat{\mathcal{P}}}\hat{\xi}\wedge\hat{\eta}\wedge\hat{\iota}$ when we use the "multiplying top and bottom by ρ" device that we employed earlier in order to evaluate the integrals (for $S^1\times S^1\times S^1$ contours) that yielded $(2\pi i)^3$. Hence

$$V_0 = \hat{U}-U.$$

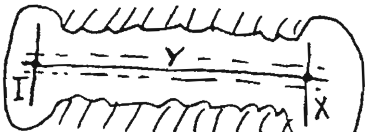
But we can also use FTEC directly to V_0, using $\chi = d(\xi\wedge\eta\wedge\iota)$ (or equivalently $\chi = d(\hat{\xi}\wedge\hat{\eta}\wedge\hat{\iota})$), whence

$$V_0 = \oint_{\mathcal{P}}\xi\wedge\eta\wedge\iota - \oint_{\hat{\mathcal{P}}}\xi\wedge\eta\wedge\iota = \oint_{\hat{\mathcal{P}}}\hat{\xi}\wedge\hat{\eta}\wedge\hat{\iota} - \oint_{\hat{\mathcal{P}}}\xi\wedge\eta\wedge\iota$$

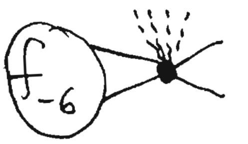
$$= \oint_{\hat{\mathcal{P}}} f_{-6}\,\xi\wedge\eta\wedge\zeta = \oint_{\mathcal{P}} f_{-6}\,\xi\wedge\eta\wedge\iota$$

the final integrands being of degree 0, so the integral is projective

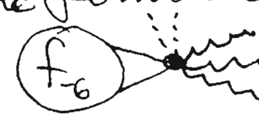
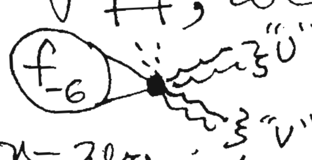
Thus, if we expect a non-zero result from this integral, then we must expect that U and $\hat{U}$ are <u>different</u>. This difference is a reflection of the difference between the

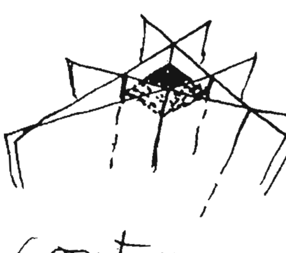


behaviour of the Y-lines (in the neighbourhood of the chosen Y-line) near I and near X. (Note that this difference is independent of how much of V is incorporated into U and $\hat{U}$. Owing to the non-Hausdorffness in coordinate space, we are really concerned with "relative cohomology" here.) For example, the family of Y lines passing through a fixed point of X will look "crinkly" rather than planar near I, in general. The "crinkliness" is a measure of the <u>shear</u> of the cut $\mathscr{I}_x$ at the point Y (at least, it is directly related to the shear in the pure googly case). Thus, we begin to see a connection between the "helicity +2 massless field" that is defined by the f_{-6} cohomology and the measure of shearing of the cut, which is just the kind of thing that is needed — giving the googly information in the curiously twisted Euler fibres an actual geometric content. Accordingly, we could conjecture that the condition for a correct "light-cone cut" is that, for each y, the difference between the U and the $\hat{U}$ x-volumes is indeed equal to the expression $\oint f_{-6}\,\xi\wedge\eta\wedge\iota$, and that perhaps $U+V=\hat{U}+V+V_0=(2\pi i)^3$.

However, when we look at what has been achieved by the above a little more carefully, we find that we are far from finished. For, as things stand, we are not able to get a non-zero answer for $\oint_{\mathbb{P}} f_{-6}\,\xi\wedge\eta\wedge\iota$. The reason for this is that the integral is now projective, and we have lost the handy poles that we obtained from the AB^2C^2D in the denominator when we integrated $\xi\wedge\eta\wedge\iota$ rather than $f_{-6}\,\xi\wedge\eta\wedge\iota$. (Our integral now has the character of trying to get a non-zero answer out of a twistor diagram without the needed "ears" present in .) The difficulty is that we need a non-zero answer both when $(AB^2C^2D)^{-1}$ is present and when it is absent.

The most promising route out of this dilemma appears to be to take advantage of a curious property of boundary contours in non-projective contour integral arising in twistor diagram theory. An inhomogeneous boundary like $A = a$, for some non-zero constant a, can occur simultaneously with a pole at $A = 0$, and this situation plays an important role in many basic twistor diagrams. (See articles by APH and his students in many TN articles; cf. also APH's articles in Twistors in Mathematics and Physics, eds Bailey & Baston (Cambridge 1990) and The Geometric Universe, eds. Huggett, Mason, Tod, Tsou, and Woodhouse (Oxford 1998).) Thus, instead of our f_{-6} function manifesting itself in the form 

, which would evaluate the (+2)-helicity field at the specific point defined by $\Upsilon\Upsilon$, we look at expressions like  or 

. The first of these certainly gives a non-zero integral (cf. RP in TN 1), but the second may possibly be the more appropriate here. The difference is that the first is concerned basically with "triangle areas" →  (×S¹) and the second, with "quadrilateral areas"  (×S¹) ← (these latter being expressible in terms of the difference between two "triangle areas" in the particular case when U, U', V, V' are all linearly dependent; this is not the same contour as for the tetrahedral volume that occurs with 

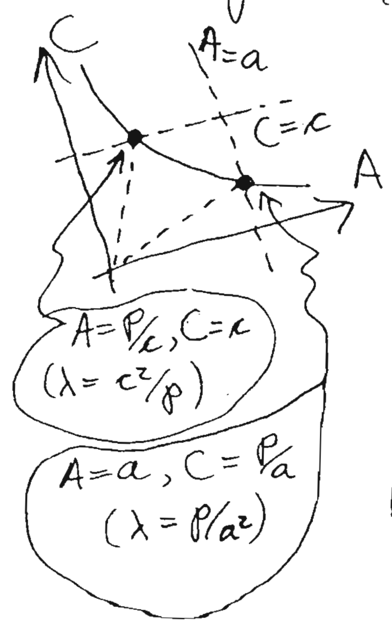
). It seems that the integral that we require has the form 

, but I have not been able to

sort out all this in any detail. It ought to have a space-time interpretation of integrals of the null-datum for the (+2)-helicity field along generators of $\mathscr{I}^+$, as is required for obtaining the asymptotic shear from this null datum (at least in the pure googly case). Something similar may also be (implicitly?) involved in relation to $\mathscr{C}_x$, but this needs further investigation.

The boundary lines $\leadsto A, \dots, \leadsto D$ would have to be interpreted as "inhomogeneous" boundaries in APH's sense, namely boundaries on $A=a$, $B=b$, $C=c$, $D=d$, where a, b, c, d are numerical constants. APH's work would suggest that perhaps $a=b=c=d=e^{\gamma}$ where γ is Euler's constant. The way that the boundaries are arranged in $\begin{smallmatrix}\leadsto A\\ \leadsto C\end{smallmatrix}\}$ and $\begin{smallmatrix}\leadsto B\\ \leadsto D\end{smallmatrix}\}$ would suggest that our $\oint f_{-6}\,\xi_\wedge\eta_\wedge\iota$ could be re-expressed by the incorporation of a factor $(2\pi i)^{-2}\log\left(\frac{A-a}{C-c}\right)\log\left(\frac{B-b}{D-d}\right)$ instead of using boundaries.

The integrals $\int\chi$ needs re-examination in the light of these considerations. To understand the role of these boundaries, it may be helpful to look at a 2-dimensional example. Take the two variables to be A and C, with the "surface" $\mathscr{P}$ defined by $AC=p$, with boundaries on $A=a$ and $C=c$. Our integral is now



$$\oint (A\,dC - C\,dA) = \oint \frac{pA^2 d\left(\frac{C}{A}\right)}{AC} = p\int_{c^2/p}^{p/a^2}\frac{d\lambda}{\lambda}$$

[over $AC=p$ boundary from $C=c$ to $A=a$]

$$= 2p\log\left(\frac{p}{ac}\right)$$

where $\lambda = \frac{C}{A}$. Thus, we have a log term instead of the $2p\pi i$ that we had before. The 4-dimensional case is presumably similar.

In the full situation we must see how to deal with the jump in the boundary location as we pass from patch to patch (since A, ..., D rescale, as before, whereas a, ..., d are simply fixed numbers). I have not yet been able to ascertain the plausibility and the full nature of this proposed scheme. The idea for fixing the "light-cone cuts" would be that we have analogues of the $\mathcal{X}$-volume integrals U and $\hat{U}$ that we had before, and that for each y this difference is equal to the appropriate f_{-6} integral. Moreover, we might expect that there is a condition analogous to the "$U+V = \hat{U}+V+V_0 = (2\pi i)^3$" that I suggested earlier, where the correct curved-space volumes should be equal to the corresponding flat-space ones.

Clearly a great deal needs to be done in order to ascertain the status of this (kind of) proposal. Nevertheless, I have the impression that the correct answer to the determination of the correct surgeries, giving the actual "light-cone cuts" has to follow the general lines that I have indicated above (at least in the pure googly case), although a considerable clarification of the ideas will be needed. (But even if this programme is fully successful, we must expect that the determination of the surgeries and cuts is very implicit, rather than providing an effective constructio of vacuum space-times.) As a final comment, I find it gratifying that there seems to be a strong connection wi twistor diagram theory. (Some other connections have been mention to me by APH.) I believe that there is a genuine possibility of prog with some of the obstacles mentioned at the beginning of this article.

Many thanks, especially to ETN, APH, LJM, and FH. Roger

Twistors and Legendre Transformations

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Abstract

We discuss how twistors arise in a natural manner via Legendre transformations on a contact bundle over space-time and its restriction to null infinity. In section II this is shown to give two different twistor formulations of the good cut equation and the dual good cut equation. In section III, the framework is 'homogenized' and applied to certain questions arising in connection with Penrose's googly programme.

I. Flat Space Twistors and Legendre Transformations

In this section we show a natural relationship—via a Legendre transformation—between twistor space and certain contact spaces (specifically, the projective cotangent bundle over (1) Minkowski space and (2) over Scri). In the latter case it can be generalized to non-flat spaces and to asymptotic twistors. The correspondence between space-time and twistor space can be studied by means of the double fibration of the space of projective null covectors over both Minkowski space and twistor space. When this bundle of null covectors over Minkowski space is restricted to a Cauchy surface it has the structure of a contact manifold. These fibrations will be seen to be related by a Legendre transform.

We begin with the canonical one-form on the cotangent bundle over Minkowski space, M, treated projectively, leading to a contact form on the seven dimensional contact manifold, PT^*M

$$\alpha = p_a dx^a \sim \alpha = \lambda p_a dx^a. \tag{1}$$

For ease of presentation it is perhaps best to work with a representative form ($p_0 = 1$) so that the contact coordinates are (t, x^i, p_i) and

$$\alpha = dt + p_i dx^i. \tag{2}$$

Suppose a null surface (Arnold's "big" wavefront; a solution to the eikonal equation, $g^{ab}\partial_a S \partial_b S = 0$) is given as $S(x^a) = 0$. We can solve this for $t = -T(x^i)$, so that the normal, $p_a = (1, \partial_i T(x^i))$ is a null vector, $p_a p^a = 0$. The projective cotangent bundle on M has the structure of the first jet bundle over a space-like hypersurface coordinatized by x^i with the fiber coordinates (t, p_i) so that on the cross-section $(t, p_i) = (-T(x^i), \partial_i T(x^i))$ we have that $\alpha = 0$. This defines the cross section as a Legendrian submanifold (three dimensional) of the contact manifold (the first jet bundle over the x^i). $T(x^i)$ is an example of what Arnold calls a "generating function" for the Legendrian submanifold.

What we will do is give the *same* contact space (with the same contact form) a different fibration, that is, we will represent it as the first jet bundle over a different base space. As an illustration of this procedure, equation (2) could be rewritten, via a *Legendre transformation*, as

$$\alpha = d(t + p_x x + p_y y + p_z z) - x dp_x - y dp_y - z dp_z = dg - x dp_x - y dp_y - z dp_z \tag{3}$$

with $g(p_x, p_x, p_y) = t + p_x x + p_y y + p_z z$. This same contact form could be considered as arising from the first jet bundle over the base (p_x, p_y, p_z) with fiber coordinates (g, x, y, z). If a Legendrian submanifold is given by a generating function $t = -T(x^i)$, it can alternatively be given by the new generating function

$$g = G(p_x, p_y, p_z) = -T(x, y, z) + p_x x + p_y y + p_z z$$

where $x^i = x^i(p_x, p_y, p_z)$ are obtained by inverting $p_i = \partial_i T(x, y, z)$. In terms of the new generating function, we have that the Legendrian submanifold is given by

$$\begin{aligned} x &= \partial_{p_x} G, \qquad y = \partial_{p_y} G, \qquad z = \partial_{p_z} G. \\ g &= -T(x, y, z) + p_x x + p_y y + p_z z. \end{aligned}$$

This in effect transforms from a fibration over position space to a fibration over momentum space.

We now turn to a fibration over twistor space using a different Legendre transformation. We first rewrite equation (1) making the null character of p_a explicit:

$$\alpha = \mu_A \pi_{B'} dx^{AB'} \tag{4}$$

which can be rewritten as

$$\alpha = \mu_A d(\pi_{B'} x^{AB'}) - \mu_A x^{AB'} d\pi_{B'}. \tag{5}$$

Defining $\omega^A = ix^{AB'}\pi_{B'}$, the contact form becomes

$$\alpha = -i\{\mu_A d\omega^A - i\mu_A x^{AB'} d\pi_{B'}\} = -i(\mu_A d\omega^A + \lambda^{A'} d\pi_{B'}\} = -iW_\alpha dZ^\alpha \tag{6}$$

and we see that *all* the differentials in equation (6) are differentials of the twistor $Z^\alpha = (\omega^A, \pi_{B'})$, i.e., the contact coordinates, for the base, are $(\omega^A,\ \pi_{B'})$ and the fiber coordinates, treated projectively, are the dual twistors, $W_\alpha = (\mu_A,\ \lambda^{A'} = -i\mu_A x^{AA'})$. From this point of view one again sees that twistors and dual twistors are canonically conjugate to each other.

A Legendrian submanifold can be constructed from a generating function that is an arbitrary homogeneous function *of degree zero* on twistor space

$$G(\omega^A, \pi_{B'}) = \textit{constant}$$

The Legendrian submanifold thus is given by $W_\alpha = \partial_{Z^\alpha} G$ or

$$\begin{aligned} \mu_A &= \partial_{\omega^A} G, \\ \lambda^{A'} &= \partial_{\pi_{A'}} G. \end{aligned} \tag{7}$$

The vanishing of $G(Z^\alpha)$ defines a null 3-surface in Minkowski space by the 3-parameter set of lines in the twistor space that are tangent to the two-surface $G(Z^\alpha) = 0$. [See R. Penrose, TN #42 for 4 other things one can do with such a two-surface]. Note that, with an arbitrary $G(Z^\alpha)$, by inverting the Legendre transformation one automatically obtains an arbitrary solution of the eikonal equation. One can see this explicitly by first writing $\lambda^{A'} = -i\mu_A x^{AA'}$, from equation (7), as

$$\partial_{\pi_{A'}} G = -ix^{AA'} \partial_{\omega^A} G \tag{8}$$

which tell us two things: (1) it can be thought of as an equation defining $\pi_{B'} = \pi_{B'}(x^{AA'})$ implicitly and (2) by multiplying by $\pi_{A'}$ we see from the condition $Z^\alpha W_\alpha = 0$ and Euler's theorem, that G must be homogeneous of degree zero (requiring that $Z^\alpha W_\alpha = 0$ even where $G \neq 0$). Now by taking the gradient of $G(x^{AA'}\pi_{A'}, \pi_{B'})$ treating $\pi_{B'}$ as a function of $x^{AA'}$, via (8), and using (8) explicitly we have that

$$\partial_{AA'} G = i\mu_A \pi_{A'}$$

is a null vector and hence that $G(x^{AA'}\pi_{A'}(x^{BB'}), \pi_{B'}(x^{BB'}))$ satisfies the eikonal equation.

II. Asymptotic Twistors and Legendre Transformations

We now consider the projective cotangent bundle over Scri, $\mathcal{I}^+$. We first let Scri, ($\mathcal{I} = \mathbb{R}\mathrm{x}\mathbb{S}^2$), have the standard Bondi type coordinates, $(u, \zeta, \bar{\zeta})$ with the projectivized "momenta" $(1, p, \bar{p})$ and then complexify, letting $\bar{\zeta} \rightsquigarrow \tilde{\zeta}$ and $\bar{p} \rightsquigarrow \tilde{p}$, independent of, respectively, ζ and p.

The contact form on $PT^*\mathcal{I}^+$ is

$$\alpha = du - pd\zeta - \tilde{p}d\tilde{\zeta} \tag{9}$$

which can be rewritten as

$$\begin{aligned} \alpha &= d(u - \tilde{p}\tilde{\zeta}) - pd\zeta + \tilde{\zeta}d\tilde{p} & (10)\\ &= dg - pd\zeta + \tilde{\zeta}d\tilde{p} & (11) \end{aligned}$$

with $g = u - \tilde{p}\tilde{\zeta}$. This can be thought of as defining the first jet bundle over the space coordinatised by $(\zeta, \tilde{p})$ with fibers parametrized by

$$(g,\ p = \partial_\zeta g,\ \tilde{\zeta} = -\partial_{\tilde{p}} g)\,. \tag{12}$$

A cross-section of $\mathcal{I}^+$ given by $u = Z(\zeta, \tilde{\zeta})$ can be lifted to the contact bundle via $p = \partial_\zeta Z$, $\tilde{p} = \partial_{\tilde{\zeta}} Z$ and yields a two-dimensional Legendre submanifold. The same Legendrian submanifold can be described after the Legendre transformation by

$$
\begin{aligned}
g &= G(\zeta, \tilde{p}) = Z(\zeta, \tilde{\zeta}) - \tilde{p}\tilde{\zeta}, && (13)\\
p &= \partial_\zeta G, \quad \tilde{\zeta} = -\partial_{\tilde{p}} G && (14)
\end{aligned}
$$

where $\tilde{\zeta} = \tilde{P}(\zeta, \tilde{p})$ via the inversion of $\tilde{p} = \partial_{\tilde{\zeta}} Z(\zeta, \tilde{\zeta})$.

We now *define* the projective twistor space to be the space coordinatized by the functions

$$(ig, i\tilde{p}, \zeta) \tag{15}$$

whose differentials appear in the contact form, or non-projectively

$$Z^\alpha = \pi_{0'}(ig, i\tilde{p}, 1, \zeta) = (\omega^A, \pi_{B'}). \tag{16}$$

To see that this definition agrees with the standard one when $Z(\zeta, \tilde{\zeta})$ represents a flat-space lightcone cut, we consider the intersection of a twistor 2-surface with $\mathcal{I}^+$. It takes the form, with $\pi_{B'} = \pi_{0'}(1, \zeta) =$ constant,

$$
\begin{aligned}
u &= Z(\zeta, \tilde{\zeta}) = x^{00'} + x^{01'}\tilde{\zeta} + x^{10'}\zeta + x^{11'}\zeta\tilde{\zeta}\\
&= (x^{00'} + x^{10'}\zeta) + (x^{01'} + x^{11'}\zeta)\tilde{\zeta}\\
u &= -i\{(\omega^0/\pi_{0'}) + (\omega^1/\pi_{0'})\tilde{\zeta}\} && (17)
\end{aligned}
$$

i.e., the intersection of twistor surfaces with Scri are the straight lines in the $\zeta =$ constant plane with $\tilde{\zeta}$ a parameter along the line, and the intercept and gradient given by $-i(\omega^0, \omega^1)/\pi_{0'}$. If we perform the above mentioned Legendre transformation we have that

$$g = G = -i\omega^0/\pi_{0'}, \quad \tilde{p} = -i\omega^1/\pi_{0'}$$

as was claimed.

We have just shown how flat twistor space arises naturally from a Legendre transformation on the contact bundle over $\mathcal{I}^+$; we now show how this can be generalized to other forms of twistor theory. There are two distinctly different points of view one could take: (1) we can define a type of flat local twistor space over each point of the sphere, $\tilde{\zeta} \epsilon S^2$, or (2) define a global curved twistor space associated with certain second order ODEs given on $\mathcal{I}^+$.

(1) Returning to either equation (15) or equation (16), we see that for each value of $\tilde{\zeta}$ we have a flat twistor space, i.e. a local twistor space for each $\tilde{\zeta}$. There are a wide range of applications for this observation. One of them is to reexpress the Null Surface Version of GR in this language. This is being investigated. For the present, we describe two special applications: (a) to the Leg-Break construction of anti-self-dual space-times and (b) to a googly construction of self-dual space-times.

(a) The Leg-Break construction is based on the solution of the "good cut" equation

$$\partial_{\tilde{\zeta}}\partial_{\tilde{\zeta}} Z = \tilde{\sigma}(Z, \zeta, \tilde{\zeta})$$

for the function $Z(x^a, \zeta, \tilde{\zeta})$ where, with regularity conditions, the x^a are four constants of integration. The Leg-Break metric can be constructed from the $Z(x^a, \zeta, \tilde{\zeta})$. The above Legendre transformation transforms the good cut equation to

$$G_{\tilde{p}\tilde{p}} = \frac{1}{\tilde{\sigma}(G - \tilde{p}G_{\tilde{p}}, \zeta, -G_{\tilde{p}})} \tag{18}$$

a second order equation for $g = G(x^a, \zeta, \tilde{p})$. The point to be emphasized is that *the equation is completely given in terms of the local twistor variables and the free data* $\tilde{\sigma}(G - \tilde{p}G_{\tilde{p}}, \zeta, -G_{\tilde{p}})$, *also expressed in terms of the twistor variables.* (The local twistor variables can here be thought of as the flat twistor space associated to the given Bondi coordinate system as if it were built from shear free cuts in Minkowski space.) There is a more symmetric version of this. If the solution $g = G(\zeta, \tilde{p})$ is given implicitly by the function $F(g, \zeta, \tilde{p}) = 0$ then $F(g, \zeta, \tilde{p})$ satisfies the differential equation

$$F_{\tilde{p}\tilde{p}} - 2\frac{F_{\tilde{p}}}{F_g}F_{\tilde{p}g} + \left(\frac{F_{\tilde{p}}}{F_g}\right)^2 F_{gg} - \frac{F_g}{\tilde{\sigma}(g + \tilde{p}\frac{F_{\tilde{p}}}{F_g}, \zeta, \frac{F_{\tilde{p}}}{F_g})} = 0 \tag{19}$$

which, unfortunately, is not a very pretty equation.

(b) A self-dual space-time can be constructed as the space of solutions to the dual good cut equation $\eth_{\zeta}\eth_{\zeta}Z = \sigma(Z,\zeta,\tilde{\zeta})$ and these solutions can, in turn, be transformed by the same Legendre transformation into an equation for $g = G(x^a,\zeta,\tilde{p})$ in the (googly) twistor space

$$G_{\tilde{p}\tilde{p}} = \frac{-(G_{\tilde{p}\zeta})^2}{\sigma(G - \tilde{p}G_{\tilde{p}},\zeta,-G_{\tilde{p}}) - G_{\zeta\zeta}} \tag{20}$$

again an equation completely stated in terms of the local twistors which are now the canonical asymptotic twistors. This also can be easily converted into an equation for the function $H(g,\zeta,\tilde{p}) = 0$ that implicitly defines $g = G(\zeta,\tilde{p})$.

This approach to the Googly relates to that of Roger Penrose as follows; the twistor space is the flat asymptotic twistor space as used in the googly construction, and the equation above can be viewed as a differential equation for the envelope of the level surfaces of the 'googly maps' (see the next section for these). More work is still required to fully understand equations (18) and (20).

(2) An alternate view is to consider the cut function as a solution to a second order ODE which in most applications is of the form, $\eth_{\tilde{\zeta}}\eth_{\tilde{\zeta}}Z = \tilde{\sigma}(Z,\zeta,\tilde{\zeta})$. The cut, or, of more relevance, the curve obtained by fixing ζ, then depends on two parameters, the constants of integration, (α,β) (e.g., the initial position and slope at some $\tilde{\zeta} = \tilde{\zeta}_0$) and has the form $u = Z(\alpha,\beta;\zeta,\tilde{\zeta})$. We refer to the cut function *with a fixed* ζ as a twistor curve parametrized by $\tilde{\zeta}$. The space of twistor curves is parametrized by the (α,β,ζ). Below we will take the twistor variables as a specific choice of the parametrization.

We now have a two parameter family of Legendrian submanifold given, for each (α,β), by

$$\begin{aligned} u &= Z(\alpha,\beta;\zeta,\tilde{\zeta}), \\ p &= \eth_{\zeta}Z, \\ \tilde{p} &= \eth_{\tilde{\zeta}}Z. \end{aligned}$$

After the Legendre transformation, equation (13), we have the same family of submanifolds and twistor curves, ($\zeta = const$), but now described by

$$g = G(\alpha,\beta;\zeta,\tilde{p}) = Z(\alpha,\beta;\zeta,\tilde{\zeta}) - \tilde{p}\tilde{\zeta}, \tag{21}$$

$$\tilde{\zeta} = -\eth_{\tilde{p}}G(\alpha,\beta;\zeta,\tilde{p}) \tag{22}$$

$$p = \eth_{\zeta}G(\alpha,\beta;\zeta,\tilde{p}). \tag{23}$$

If we fix the $\tilde{\zeta} = \tilde{\zeta}_1$ in equation (22), the Eqs.(21) and (22) can be solved (in some region) for (α,β), i.e., $(\alpha,\beta) = (\alpha(g_1,\tilde{p}_1,\zeta;\tilde{\zeta}_1),\ \beta(g_1,\tilde{p}_1,\zeta;\tilde{\zeta}_1))$ which are then substituted back into Eqs.(21) and (22) yielding

$$g = G^*(g_1,\tilde{p}_1;\tilde{\zeta}_1;\zeta,\tilde{p}), \tag{24}$$

$$\tilde{\zeta} = -\eth_{\tilde{p}}G^*(g_1,\tilde{p}_1;\tilde{\zeta}_1;\zeta,\tilde{p}) \tag{25}$$

$$p = \eth_{\zeta}G^*(g_1,\tilde{p}_1;\tilde{\zeta}_1;\zeta,\tilde{p}). \tag{26}$$

The (projective) twistor coordinates $(g_1,\tilde{p}_1,\zeta)$ are the local coordinates for the (anti-self-dual) twistor space determined by $\tilde{\zeta} = \tilde{\zeta}_1$. Different coordinate patches (at least one extra is necessary) can be obtained by choosing different values for the fiducial $\tilde{\zeta}$, e.g., $\tilde{\zeta} = \tilde{\zeta}_2$; the overlap transformation is then given implicitly by

$$g_2 = G^*(g_1,\tilde{p}_1;\tilde{\zeta}_1;\zeta,\tilde{p}_2), \tag{27}$$

$$\tilde{\zeta}_2 = -\eth_{\tilde{p}}G^*(g_1,\tilde{p}_1;\tilde{\zeta}_1;\zeta,\tilde{p}_2). \tag{28}$$

The twistor coordinates on the two patches, $(g_1,\tilde{p}_1,\zeta)$ and $(g_2,\tilde{p}_2,\zeta)$, with the overlap transformation *define* the asymptotic twistor space and this space, in turn, parametrizes the twistor curves which rule the "good cuts" or the solution to the "good cut equation".

We thus see that the anti-self dual space-times can be described via either the local twistor point of view, i.e., from equation (18) or from the twistor space point of view, namely from equations (24),(25),(26).

III. The Homogeneous Scri Formalism and 'Googly maps'

In this section we express the correspondence between asymptotic twistor space and Scri in homogeneous coordinates (due originally to Sparling, see Eastwood & Tod 1982 and Tod 1981). We first apply the formalism to give a Lorentz invariant formulation of equations (20). We then calculate the metric of an $\mathcal{H}$-space in terms of the cut function, reformulating the calculations in Ko, Ludvigsen, Newman and Tod in terms of homogeneous coordinates. This can be compared with an expression proposed for the metric by RP in the context of his googly programme. The most basic form of this expression does not work, although the full features of RP's framework are not used so the results are inconclusive. The formula is not in any case central to the programme. A formula that does work (a transcription of the Newman integral formula adapted from one given in Tod 1979) is presented at the end. Although these calculations are inconclusive, the purpose of presenting them is to show how the homogeneous coordinate formulation allows one to do quite detailed calculations in relation to the googly with relative ease in complete generality.

We work with homogeneous coordinates on scri (with apologies for the conflict with earlier notation; the u of this section should be identified with $\pi_{0'}\bar{\pi}_0 u$ from the previous section)

$$(u, \pi_{A'}, \bar{\pi}_A) \sim (\lambda\bar{\lambda}u, \lambda\pi_{A'}, \bar{\lambda}\bar{\pi}_A).$$

These can be related to the coordinates used earlier by setting $\pi_{0'} = 1$ and $\pi_{1'} = \zeta$. This can be tied into the GHP formalism by choosing u so that (thorn)$'u = 1$. The main advantage of this formalism is that it allows one to maintain explicit Lorentz invariance.

We first set up the homogeneous versions of the correspondence between twistor space and null infinity and the dual good cut equation (see also Tod 1981 and eastwood & Tod 1982).

We will confine ourselves to space-times with self-dual Weyl tensor and consider the (flat) asymptotic twistors (i.e. the 'googly' situation). A twistor $(\omega^A, \pi_{A'})$ determines a line in scri given by the equation $iu = \omega^A\bar{\pi}_A$. A 'dual good cut' is determined by a homogeneity $(1,1)$ function $Z(x^a, \pi_{A'}, \bar{\pi}_A)$ by $u = Z$. In flat space $Z(x^a, \pi_{A'}, \bar{\pi}_A) = x^{AA'}\pi_{A'}\bar{\pi}_A$. See equation (17). For agreement with the previous section we have that Z and σ in this section are identified with $\bar{\pi}_0\pi_{0'}Z$ and $\bar{\pi}_0(\pi_{0'})^3\sigma$ from the last. In a self-dual space-time we have that Z satisfies the good cut equation

$$\partial^{A'}\partial^{B'}Z = \pi^{A'}\pi^{B'}\sigma\,, \qquad \partial^{A'} = \partial/\partial\pi_{A'}\,.$$

where the form of the right hand side is determined by the homogeneity of Z and $\sigma = \sigma(Z, \pi_{A'}, \bar{\pi}_A)$ is the Bondi shear. Note that although this is formally 3 equations, it is really just one in the sense that both sides are proportional to $\pi^{A'}\pi^{B'}$.

The twistor generating function for dual good cuts

Solutions to the dual good cut equation were shown to give rise to 2-surfaces in twistor space subject to equation (20) in terms of affine coordinates on twistor space. We now reformulate that equation in terms of homogeneous coordinates. This has the advantage that the equation now has explicit Lorentz invariance, but the drawback that it becomes three equations. This 2-surface will be the envelope of the level surfaces of the googly map corresponding to the dual good cut given below, but will not otherwise relate to the subsequent development.

The dual good cut determines a Legendrian 2-surface in the contact space $PT^*\mathcal{I} = PT^*\mathbb{PT}$ and this can in turn be projected down to give a 2-surface in twistor space. We can represent a 1-parameter family of such 2-surfaces by $G := G(\omega^A, \pi_{A'}) =$ constant. (One can simply just focus on $G = 0$.)

The corresponding two surface in $PT^*\mathcal{I} = PT^*\mathbb{PT}$ can be expressed first in terms of coordinates $(Z^\alpha, W_\beta) = (\omega^A, \pi_{A'}, \mu_A, \lambda^{A'})$ with $Z^\alpha W_\alpha = 0$ by $\mu_A = \partial G/\partial\omega^A$ and $\lambda^{A'} = \partial G/\partial\pi_{A'}$. We can identify $\partial Z/\partial\pi_{A'} = \partial G/\partial\pi_{A'}$ although one should be careful to note that $\partial/\partial\pi_{A'}$ has different meanings on each side of the equation, on the left holding $\bar{\pi}_A = \mu_A$ constant and on the right holding ω^A constant. To impose the dual good cut equation we need to take the $\partial/\partial\pi_{A'}$ derivative holding $\mu_A = \bar{\pi}_A$ constant. When expressed in twistor coordinates, this derivative operator becomes

$$\frac{\partial}{\partial\pi_{A'}} - \frac{2}{G_{CD}G^{CD}}G^{AB}\frac{\partial^2 G}{\partial\omega^B\partial\pi_{A'}}\frac{\partial}{\partial\omega^A} \quad \text{where} \quad G_{AB} = \frac{\partial^2 G}{\partial\omega^A\partial\omega^B}\,.$$

Thus the dual good cut equation becomes

$$\frac{\partial^2 G}{\partial\pi_{A'}\partial\pi_{B'}} - \frac{2}{G_{CD}G^{CD}}G^{AB}\frac{\partial^2 G}{\partial\omega^B\partial\pi_{A'}}\frac{\partial^2 G}{\partial\omega^A\partial\pi_{B'}} - \pi^{A'}\pi^{B'}\sigma = 0 \tag{29}$$

where $\sigma := \sigma\left(\pi_{C'}\frac{\partial G}{\partial\pi_{C'}}, \pi_{A'}, \frac{\partial G}{\partial\omega^A}\right)$. Note that, as above, this is actually one equation being proportional to $\pi^{A'}\pi^{B'}$.

Googly maps

The dual good cut determines a map from twistor space to the unprimed spin space, $Z^\alpha \mapsto \tilde\pi^A$ which is given projectively by mapping the twistor to the value of $\tilde\pi_A$ at which its line hits the good cut. Thus $\tilde\pi^A$ is determined implicitly by $\omega^A\tilde\pi_A = iZ(x,\pi_{A'},\tilde\pi_A)$. The scale of this map can be fixed by imposing

$$\tilde\pi^A = \omega^A - i\tilde\partial^A Z\,, \tag{30}$$

where $\tilde\partial^A = \partial/\partial\tilde\pi_A$. This gives $\tilde\pi^A$ as an implicit function of $(x,\omega^A,\pi_{A'})$. This is the homogeneous 'googly map' (as obtained by K.P.Tod in TN9). Note that although this choice of scaling is canonical, it is not clear that it is unique or indeed the 'correct' one in the context of the googly construction especially in view of RP's proposal to deform the scalings on the twistor space which would certainly require some alteration to the above formula. Nevertheless, one can see that a change in the prescribed scalings is unlikely to improve the situation described below, and the purpose of this calculation is to show firstly that this is not the likely to lead to the correct formulation, and secondly how relatively straightforward this formalism is for testing such conjectures.

Penrose recently conjectured a formula for the space-time metric in terms of these structures. He introduces a 1-form $\xi = \tilde\pi_A d\tilde\pi^A$ thought of as a 1-form on twistor space depending on the space-time point x. To obtain the metric, consider a tangent vector v at x, then introduce a small displacement operator

$$\delta = v^a\frac{\partial}{\partial x^a}\bigg|_{\omega^A,\pi_{A'}} \tag{31}$$

in the direction of v^a while keeping the twistor $(\omega^A,\pi_{A'})$ constant. Note that thus far partial x^a-derivative implicitly meant keeping $(\tilde\pi_A,\pi_{A'})$ constant.

Now define $\tilde\pi'_A \equiv \delta(\tilde\pi_A)$. Then RP's earlier proposal was that the metric be obtained from the formula

$$d(\tilde\pi'_A d\tilde\pi'^A)\wedge d\xi = g(v,v)\Phi\,,\qquad 24\Phi = \varepsilon_{\alpha\beta\gamma\delta}dZ^\alpha dZ^\beta dZ^\gamma dZ^\delta$$

This formula can be checked directly using the homogeneous scri formalism as follows. Firstly we need to know what the metric actually is. We have that, holding π and $\tilde\pi$ fixed, Z is constant on null hypersurfaces. Hence

$$g^{ab}Z_{,a}Z_{,b} = 0\,,\qquad Z_{,a} = \partial Z/\partial x^a$$

We can differentiate this with respect to π and $\tilde\pi$. To introduce some notation, note that if we set

$$Z^A = \tilde\partial^A Z,\quad Z^{A'} = \partial^{A'}Z\,,\quad Z^{AA'} = \tilde\partial^A\partial^{A'}Z\,,$$

then by homogeneity

$$\pi_{A'}Z^{AA'} = Z^A\,,\quad \tilde\pi_A Z^{AA'} = Z^{A'}\,,\qquad \partial^A\partial^B Z = \pi^A\pi^B\tilde\Lambda$$

where the last equation defines $\tilde\Lambda$ as a function of x and π and $\tilde\pi$. Recall also that the **dual** good cut equation implies

$$\partial^{A'}Z^{B'} = \pi^{A'}\pi^{B'}\sigma.$$

First, differentiating twice with respect to π and using the good cut equation we find that

$$g^{ab}\partial^{A'}Z_{,a}\partial^{B'}Z_{,b} = 0,\qquad \text{since}\quad g^{ab}Z_{,a}\sigma_{,b} = \dot\sigma g^{ab}Z_{,a}Z_{,b} = 0\,.$$

Differentiating twice with respect to $\tilde\pi$ we get the following nine equations that, at a fixed value of π and $\tilde\pi$ determine the conformal structure

$$g^{ab}Z^{C(C'}{}_{,a}Z^{D')D}{}_{,b} = -\tilde\pi^C\tilde\pi^D\tilde\Lambda^{C',D'}\,,\quad \tilde\Lambda^{C',D'} = g^{ab}\partial^{(C'}\tilde\Lambda_{,a}Z_{,b}^{D')}\,.$$

Thus a scale for g^{ab} exists such that

$$g^{ab}Z^{CC'}{}_{,a}Z^{DD'}{}_{,b} = \varepsilon^{CD}\varepsilon^{C'D'} - \tilde\pi^C\tilde\pi^D\tilde\Lambda^{C',D'}$$

and this scale turns out to be the one that gives a Ricci flat space-time. One can therefore see that

$$\theta^{AA'} = Z_{,c}^{AA'}dx^c + \frac{1}{2}\tilde\pi^A\tilde\Lambda^{A'}{}_{B'}Z_{,c}^{B'}dx^c$$

is an orthonormal tetrad so that the metric can be given as

$$g_{cd} = \varepsilon_{AB}\varepsilon_{A'B'}Z_{,c}^{AA'}Z_{,d}^{BB'} + \tilde\Lambda_{A'B'}Z_{,c}^{A'}Z_{,d}^{B'}\,.$$

Now we compute RP's expression to see whether we get the above expression for the metric. Keeping the definition in (31) mind, it is not difficult to show that

$$d\tilde{\pi}^A = (i\Lambda\tilde{\pi}^A\tilde{\pi}_B - \varepsilon^A{}_B)(d\omega^B - iZ^{BB'}d\pi_{B'}),$$

and $d\xi := \varepsilon_{AB}d\tilde{\pi}^A \wedge d\tilde{\pi}^B = \nu_A \wedge \nu^A$, where $\nu^A = d\omega^A - iZ^{AA'}d\pi_{A'}$. Applying the displacement δ gives

$$d\xi \wedge d(\tilde{\pi}'_A d\tilde{\pi}'^A) = \nu_A \wedge \nu^A \wedge \delta\nu_A \wedge \delta\nu^A$$

where $\delta\nu^A = i(Z^{AA'}_{,a} - iZ_{,a}\tilde{\pi}^A\Lambda^{A'})d\pi_{A'}$. This yields the metric expression

$$g_{cd} = \epsilon_{AB}\epsilon_{A'B'}Z^{AA'}_{,c}Z^{BB'}_{,d} - 2iZ^{AA'}_{,(c}Z_{,d)}\tilde{\pi}_A\Lambda_{A'}.$$

Unfortunately one does not pick up the part $v^c v^d \tilde{\Lambda}_{A'B'}Z^{A'}{}_{,c}Z^{B'}{}_{,d}$ of the metric that distinguishes the metric from the flat one. It might be argued that when the scalings are introduced, the above formula changes in such a way as to pick up the missing term. This seems unlikely in view of the fact that the scalings effectively involve $\partial^3\sigma/\partial u^3$ and this is of a different character to the term that is missing. So these arguments are inconclusive and more work is required to pin down these issues (finding the correct specification of the scalings for googly maps is still an open problem). Although this particular calculation is not encouraging, this formalism has lead to the twistorial reformulation of the Newman integral formula for the metric.

Let ξ be the 1-form corresponding to a point x, $\xi' = \mathcal{L}_\delta\xi$ and $\iota = \pi_{A'}d\pi^{A'}$. Then it is a simple consequence of googly geometry that the form $\xi \wedge \xi' \wedge \iota$ has to be proportional to $\theta = \frac{1}{6}\varepsilon_{\alpha\beta\gamma\delta}Z^\alpha dZ^\beta dZ^\gamma dZ^\delta$, the standard volume 3-form on the projective twistor space

$$\xi \wedge \xi' \wedge \iota = \gamma\theta.$$

The metric is then given by the following integral:

$$\frac{16\pi^2}{g(\delta x^a, \delta x^a)} = \oint \frac{\theta}{\gamma^2}.$$

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Real Lorentzian metrics from complex, half flat solutions

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In the 1960's Roger Penrose showed that all real Lorentzian solutions of Einstein's vacuum field equations could, in the approximation where fields are linearized about the Minkowski solution, be constructed from complex half flat solutions, [1]. The extent to which this result can be extended to the full non-linear theory is unknown. Real Lorentzian solutions have been constructed from complex half flat ones but they are all algebraically special and quite simple: see for example, [2] and references therein. The aim of this note is to outline some methods of constructing real Lorentzian metrics from complex half-flat ones. Natural ways to construct, locally, real 2- and 3-differential forms (which can be extended to n- forms, $2 \preceq n \preceq 8$) by using half flat solutions will be exhibited. These real forms, on a complex four manifold M, will be used to construct real metrics on real four manifolds, N, cf also [3]. Hence real metrics on N will be constructed, locally, from complex half flat metrics on M.

By using the type of formalism developed by Plebanski, complex half flat metrics on M can be represented, in terms of a basis of complex co-frames $\chi^{\mathbf{A}A'}$, by a line element

$$ds^2 = \epsilon_{\mathbf{AB}}\epsilon_{A'B'}\chi^{\mathbf{A}A'} \otimes \chi^{\mathbf{B}B'} \tag{1}$$

and the Cartan structure equations

$$d\chi^{\mathbf{A}A'} - \chi^{\mathbf{B}A'} \wedge \omega^{\mathbf{A}}_{\mathbf{B}} - \chi^{\mathbf{A}B'} \wedge \varpi^{A'}_{B'} = 0, \tag{2}$$

$$\Omega^{\mathbf{A}}_{\mathbf{B}} \equiv d\Gamma^{\mathbf{A}}_{\mathbf{B}} + \Gamma^{\mathbf{A}}_{\mathbf{C}} \wedge \Gamma^{\mathbf{C}}_{\mathbf{B}} = \frac{1}{2}\Psi^{\mathbf{A}}_{\mathbf{BCD}}\chi^{\mathbf{C}}_{A'} \wedge \chi^{\mathbf{D}A'} \tag{3}$$

$$d\varpi^{A'}_{B'} + \varpi^{A'}_{C'} \wedge \varpi^{C'}_{B'} = 0. \tag{4}$$

Here, the anti-self dual and self-dual components of the Levi-Civita spin connection are given, respectively, by $\omega^{\mathbf{A}}_{\mathbf{B}}$, with Weyl spinor $\Psi^{\mathbf{A}}_{\mathbf{BCD}}$, and the flat $\varpi^{A'}_{B'}$. Upper case Latin bold, un-primed, and Latin normal, primed, indices represent transformation properties under the structure groups SL(2,C)$_L$ and SL(2,C)$_R$ respectively. Upper case Latin indices sum and range over 0 and 1.

It follows from the above equations that

$$\Omega^{\mathbf{A}}_{\mathbf{B}} \wedge \chi^{\mathbf{B}B'} = 0. \tag{5}$$

It is useful to recall at this point that equations like Eqs. (2) and (5) constitute a simple representation of the Einstein vacuum condition.

Real forms on M, related to half-flat geometries, can be constructed by using the co-frame $\chi^{\mathbf{A}A'}$ and its complex conjugate, $\bar{\chi}^{A\mathbf{A}'}$. First consider the real 2-forms $\sigma^{\mathbf{a}}$ (i.e. the Hermitian matrix-valued 2-form $\sigma^{\mathbf{AA}'}$) defined by

$$\sigma^{\mathbf{AA}'} = i\chi^{\mathbf{A}A'} \wedge \bar{\chi}^{A\mathbf{A}'} \kappa_{AA'}. \tag{6}$$

(Lower case Latin indices represent transformation properties under SO(1,3)$_L$ = $\{SL(2,C)_L \times c.c.\}/\mathbb{Z}_2$ and range and sum over 1 to 4.)

Here the Hermitian matrix-valued function $\kappa_{AA'}$ is covariantly constant with respect to the real flat so(1,3)$_R$-valued connection represented by $\varpi^a_b \leftrightarrow \delta^A_B \varpi^{A'}_{B'} + \delta^{A'}_{B'} \bar{\varpi}^A_B$, that is

$$d\kappa_{AA'} - \kappa_{AB'} \wedge \varpi^{B'}_{A'} - \kappa_{BA'} \wedge \bar{\varpi}^B_A = 0, \tag{7}$$

and so the $\kappa_{AA'}$ label a four-parameter family of 2-forms $\sigma^{\mathbf{AA}'}$. These 2-forms are compatible with the real so(1,3)$_L$-valued connection $\omega^{\mathbf{a}}_{\mathbf{b}} \leftrightarrow \delta^{\mathbf{A}}_{\mathbf{B}} \bar{\omega}^{\mathbf{A}'}_{\mathbf{B}'} + \delta^{\mathbf{A}'}_{\mathbf{B}'} \omega^{\mathbf{A}}_{\mathbf{B}}$, since it follows from the above equations that the covariant exterior derivative of $\sigma^{\mathbf{a}}$, with respect to the latter connection, is zero i.e.

$$D\sigma^{\mathbf{a}} \equiv d\sigma^{\mathbf{a}} + \sigma^{\mathbf{b}} \wedge \omega^{\mathbf{a}}_{\mathbf{b}} = 0. \tag{8}$$

Furthermore, it also follows from the above that

$$\Omega^{\mathbf{A}}_{\mathbf{B}} \wedge \sigma^{\mathbf{BA}'} = 0 \tag{9}$$

and similarly for the complex conjugate equation. Note that from Eq.(8), it follows that, for any n-3 forms $\pi_{\mathbf{a}}$, the n-form $D\pi_{\mathbf{a}} \wedge \sigma^{\mathbf{a}}$ is closed. In a similar way, p-forms, with $3 \preceq p \preceq 8$ which satisfy equations like (8) and (9), can be constructed. All these equations can be pulled back to sub-manifolds of M to define geometries on them, in particular on four dimensional real submanifolds, N. Here only the 2-form equations above and the similar equations for 3-forms will be considered. The latter can be obtained and written in the following way. Let $\rho^{\mathbf{a}} \leftrightarrow \rho^{\mathbf{AA'}}$ be real 3-forms on M, defined by

$$\rho^{\mathbf{AA'}} = i\chi^{\mathbf{A}A'} \wedge \bar{\chi}^{\mathbf{AA'}} \wedge r_{AA'} \tag{10}$$

where $r_{AA'}$ is taken to be a Hermitian matrix valued 1-form which has zero exterior covariant derivative with respect to the flat so(1,3)$_R$ -valued connection ϖ^a_b, that is

$$dr_{AA'} + r_{BA'} \wedge \overline{\varpi}^B_A + r_{AB'} \wedge \varpi^{B'}_{A'} = 0. \tag{11}$$

By choice of gauge, $r_{AA'}$ can be chosen to be closed. It follows, once again from the above equations that the real 3-forms are compatible with the real so(1,3)$_L$-valued connection $\omega^{\mathbf{a}}_{\mathbf{b}}$, and

$$D\rho^{\mathbf{a}} \equiv d\rho^{\mathbf{a}} - \rho^{\mathbf{b}} \wedge \omega^{\mathbf{a}}_{\mathbf{b}} = 0 \ ; \ \Omega^{\mathbf{A}}_{\mathbf{B}} \wedge \rho^{\mathbf{BA'}} = 0. \tag{12}$$

Now constructions leading to 4-geometries, involving the above 2-forms, $\sigma^{\mathbf{a}}$, and then the above 3- forms,$\rho^{\mathbf{a}}$, will be considered.

First the 2- forms $\sigma^{\mathbf{a}}$ will be used to construct real 1-forms, $\varsigma^{\mathbf{a}}$, on M. When these 1-forms can be pulled back to define a real co-frame, $\theta^{\mathbf{a}}$,on a real four manifold N, locally embedded in M, they determine a Lorentz metric on N

$$ds^2 = \eta_{\mathbf{ab}}\theta^{\mathbf{a}} \otimes \theta^{\mathbf{b}}, \tag{13}$$

with Levi-Civita connection the pull-back of $\omega^{\mathbf{a}}_{\mathbf{b}}$, and anti-self dual curvature two-form the pull-back of $\Omega^{\mathbf{A}}_{\mathbf{B}}$. The properties of such metrics will be determined by the original half-flat metric, the embedding map and $\kappa_{AA'}$. Secondly, the real 3-forms $\rho^{\mathbf{a}}$ will be used to construct real 1-forms, $\theta^{\mathbf{a}}$, and a Lorentzian metric on N in a different way. (Alternative procedures, using similar ideas are possible. In particular some apply when the initial complex metric is defined on a real four manifold.)

In the case of the 2-forms, real one-forms, $\varsigma^{\mathbf{a}} = X \rfloor \sigma^{\mathbf{a}}$, can be constructed by contraction with a real vector field, X, on M. When the forms and vector fields satisfy appropriate conditions these one-forms will satisfy the equations on M

$$d\varsigma^{\mathbf{a}} - \varsigma^{\mathbf{b}} \wedge \omega^{\mathbf{a}}_{\mathbf{b}} = 0, \tag{14}$$

$$\Omega^{\mathbf{A}}_{\mathbf{B}} \wedge \varsigma^{\mathbf{BA'}} = 0. \tag{15}$$

A set of (non-gauge covariant) conditions on X which lead to Eqs. (14) is given by

$$\pounds_X \sigma^{a} = f^{a}_{b}\sigma^{b}, \quad X\rfloor\omega^{a}_{b} = -f^{a}_{b}, \tag{16}$$

for some real functions f^{a}_{b}. If, furthermore, X also satisfies

$$\pounds_X \omega^{a}_{b} = -Df^{a}_{b}, \tag{17}$$

Eq.(15) holds too. When the 1-forms, ς^{a}, can be pulled back to a co-frame θ^{a} on N, the pull backs of Eqs.(14) and (15) imply that the metric, given by Eq.(13), satisfies the Einstein vacuum field equations. However, since $X\rfloor\varsigma^{a} = 0$, the construction may in fact lead to solutions of the vacuum constraint equations on a three manifold which is a sub-manifold of N.

In the case of the 3-forms, ρ^{a}, it is natural to construct metrics on N by using the duality of vector densities and 3-forms in four dimensions. When ρ^{a} can be pulled back to a basis of 3-forms on N, τ^{a} say, then there exist real 1-forms θ^{a} on N (and hence a Lorentz metric as in Eq.(13) above) such that $\tau^{a} = \frac{1}{6}\epsilon^{a}_{bcd}\theta^{b}\wedge\theta^{c}\wedge\theta^{d}$. It follows from the pull-backs to N of Eq.(12) that

$$d\theta^{a} - \theta^{b}\wedge\omega^{a}_{b} = \Theta^{a}, \tag{19}$$

$$\Theta^{a} = \frac{1}{2}\Theta^{a}_{bc}\theta^{b}\wedge\theta^{c}, \quad \Theta^{a}_{bc} = -\Theta^{a}_{cb}, \quad \Theta^{a}_{ba} = 0, \tag{20}$$

$$d\tau^{a} - \tau^{b}\wedge\omega^{a}_{b} = 0. \tag{21}$$

Eqs. (19-21) relate real metrics on N to the pull back of the connection ω^{a}_{b} (also written ω^{a}_{b}), and encode, at least in part, the half-flat geometry on M. Explicit examples of all the above geometries and further investigations of the above, including the role of conformal freedom in the last construction and the relation to linearized theory, will be presented elsewhere.

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Combinatorics and field theory

For any given sequence of integers there exists a quantum field theory whose Feynman rules produce that sequence. An example is illustrated for the Stirling numbers. The method employed here offers a new direction in combinatorics and graph theory.

In quantum field theory graphs are used to represent the terms in a diagrammatic perturbation expansion, whereby one associates with each graph a numerical amplitude. It is known [1] that there is an elegant formal construction for the set of all graphs in terms of the exponential of a derivative operator. For example, $Z(\epsilon, g) = \exp(\frac{1}{2!}\epsilon d^2/dx^2)\exp(\frac{1}{4!}gx^4)|_{x=0}$ is the generating function for the set of all 4-vertex vacuum diagrams, connected and disconnected, in which the line amplitude is ϵ and the vertex amplitude is g. Here, for simplicity, we consider field theories in zero-dimensional space so that Feynman integrals become trivial and are merely the product of the line amplitudes.

In analogy with statistical mechanics we refer to the sum of all vacuum diagrams as the *partition function.* The power series expansion of a generic partition function contains both connected and disconnected graphs. If we are interested only in connected graphs, then we consider instead the *free energy* $F = -\ln Z$; the coefficients of the power series expansion of F represent the sum of the symmetry numbers of just the connected graphs [2], where the symmetry number of a graph is defined as the reciprocal of the number of ways in which the graph can be turned into itself by permuting the lines or vertices. These are the basic rules for graphs in field theories. Using these rules, we would like to find field theories whose diagrammatic expansions correspond to graphs that are meaningful in combinatorics. A simple but intriguing example is the field theory of partitions.

Field theory of partitions

The partition of an integer n is the set of all distinct ways to represent n as a sum of positive integers smaller than or equal to n. The number of elements in the partition of n is designated P_n. Defining $P_0 = 1$, the first few P_n are $1,\ 1,\ 2,\ 3,\ 5,\ 7,\ 11, \cdots$. In the case of partitioning of labelled objects, the number of partitions is given by the Bell numbers $\{B_n\} = 1,\ 1,\ 2,\ 5,\ 15,\ 52,\ 203, \cdots$. The labelled partitions can also be grouped into classes characterised by the Stirling numbers $S(n,k)$, which count the number of partitions of n labelled objects into k groups [3]. Specifically, we have $S(1,1) = 1$, $S(2,1) = S(2,2) = 1$, $S(3,1) = 1$, $S(3,2) = 3$, $S(3,3) = 1$, and so on. Clearly, if we sum $S(n,k)$ over k, we recover the Bell numbers: $B_n = \sum_k S(n,k)$.

There is an elementary field theory whose Feynman rules produce precisely the labelled partitions into groups [4]. This is given by the potential energy $V(x) = g(e^x - 1)$ and the *line insertion operator* $D = \epsilon d/dx$ for a grouping variable g and a line amplitude ϵ:

$$Z(\epsilon, g) \;=\; \exp\left(\epsilon \frac{d}{dx}\right) \exp\left[g(e^x - 1)\right]\Big|_{x=0} \;=\; \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!}\left(\sum_{k=1}^{n} S(n,k) g^k\right) .$$

If we set $g = 1$, then this reduces to the field theory of labelled partitions (the Bell numbers). Note that the line insertion operator in the above example is given by a first order derivative operator d/dx. Graphically, such propagators correspond to lines having only one end. This is a strange kind of line; ordinarily, a line has two ends. However, in the field theory, we can have generalised lines having multiple ends.

Generalised partitions

Consider a general field theory that includes all n-point vertices as well as all generalised lines having m ends. This leads to the expression

$$Z(L,V) = \exp\left(\epsilon \sum_{m=1}^{\infty} \frac{L_m}{m!}\frac{d^m}{dx^m}\right) \exp\left(g \sum_{n=1}^{\infty} \frac{V_n}{n!}x^n\right)\Bigg|_{x=0},$$

which represents the set of all vacuum diagrams, connected and disconnected, constructed from n-point vertices whose amplitudes are V_n and generalised lines having m legs whose amplitudes are L_m. If we expand this expression as a formal series in powers of ϵ, then the coefficient of ϵ^n is the sum of the symmetry numbers of all graphs having n lines, and if we expand this expression as a series in powers of g, then the coefficient of g^n is the sum of the symmetry numbers of all graphs having n vertices. We note that in general these formal power series are divergent because the number of graphs grows like a factorial.

In particular, if we set $g = 1$ and $L_m = V_n = 1$, then we obtain the following beautiful result on generalised partitions:

$$Z(\epsilon) = \exp\left(e^{\epsilon\frac{d}{dx}} - 1\right) \exp\left(e^x - 1\right)\Bigg|_{x=0} = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} B_n^2 .$$

The graphs contributing to this expression, up to order ϵ^4, are shown in Fig. 1 below.

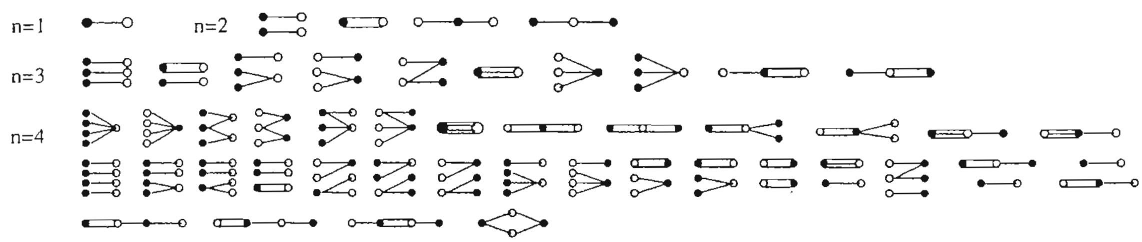

FIG. 1. Graphs in a theory whose Feynman rules allow for n-point vertices ($n = 1, 2, 3, \cdots$) and m-legged lines ($m = 1, 2, 3, \cdots$). If the vertex amplitudes are all unity and the m-legged line amplitude is ϵ^m, then the generating function $Z(\epsilon)$ for the graphs has a Taylor expansion for which the coefficient of ϵ^n is $B_n^2/n!$. Thus, the number of labelled graphs of order n is the square of the nth Bell number.

It is interesting that the graphs of generalised lines in the above figure have formal resemblance to the graphs in twistor diagrams. These generalised lines are, however, quite natural in the context of quantum field theory. For example, the strong-coupling expansion of the Lagrangian for quantum chromodynamics are known to involve these multilegged propagators. Note that the number of these graphs form a sequence $1, 1, 4, 10, 33, \cdots$. We do not know how to generate this sequence, however, if we label the vertices in Fig. 1, then the number of graphs for generalised labelled partitions are given by the square of the Bell numbers.

Topology numbers

We have illustrated how the Feynman rules in quantum field theory can naturally be associated with ideas in combinatorics. The structures we introduced above are, however, quite primitive in the sense that we worked only in zero-dimensional space, and we have not introduced Fermion lines, which give rise to directed graphs. Despite such simplicity in the underlying field theory, the corresponding graphical expansions are already quite intricate. This suggests that, by introduction of additional structures (or perhaps even without such extensions) we might be able to obtain partition functions that would generate unknown, crucially important integer sequences such as topology numbers or partially ordered sets (posets).

Although we do not know how this might be achieved, in the following we would like to sketch the line of thinking involved in such problems. Here, let us consider the labelled topologies. The n-th topology number can be represented by unlabelled transitive graphs of n nodes. By transitivity, we mean if α and β are related and β and γ are related, then α and γ also have to be related. Some of the graphs are shown in Fig. 2.

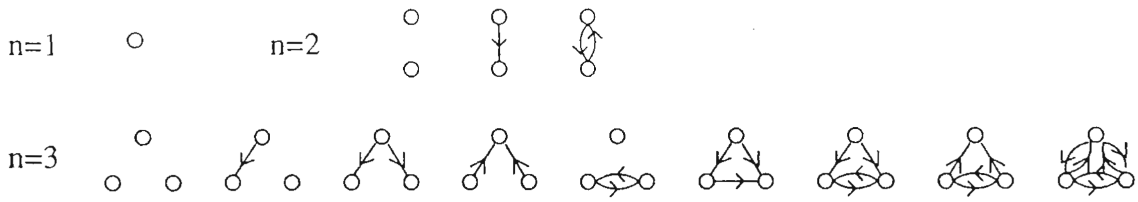

FIG. 2. Connected and disconnected graphs representing topology numbers: 1, 1, 3, 9, 33, 139, 718, 4535, $\cdots$. Only eight terms of the sequence are known.

Since disconnected graphs can be obtained by exponentiating the connected ones in the sense noted above, let us consider only the connected ones (free energy). The labelled connected topologies are then given by the sum of the symmetry numbers of the connected transitive graphs. Because we are interested in the symmetry numbers, all the transformations of the graph that preserve the symmetry numbers are allowed. Thus, we may simplify the above graphs in the following manner. The first step is to replace the double lines in Fig. 2 by single 'Bosonic' lines. The connected graphs obtained after this replacement are shown in Fig. 3.

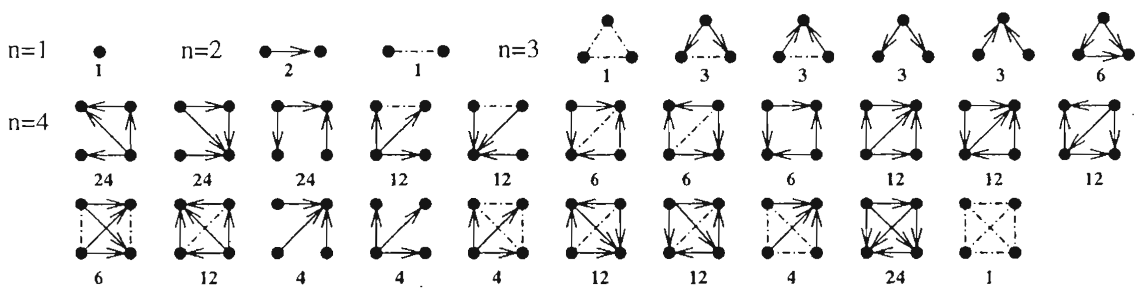

FIG. 3. Graphs representing connected topologies, after replacing the double arrows in Fig. 2 by single Bosonic lines. The numbers are the associated symmetry numbers times $n!$.

The second step is to 'squash' all the Bosonic lines, namely, any graph having n vertices joined by Bosonic lines can be squashed into a single n-vertex. The symmetry number is preserved by associating the factor of $1/n!$ to each n-vertex. The final step is then to lift the transitivity in the sense that, if α is related to β and β is related to γ, then the redundant relation joining α and γ by a 'Fermion' line (arrow) can be removed. It can easily be seen that this simplification does not alter the symmetry numbers. After these transformations, the simplified graphs appear to be identical to the connected posets. An example for $n = 4$ in Fig. 3 is shown in Fig. 4.

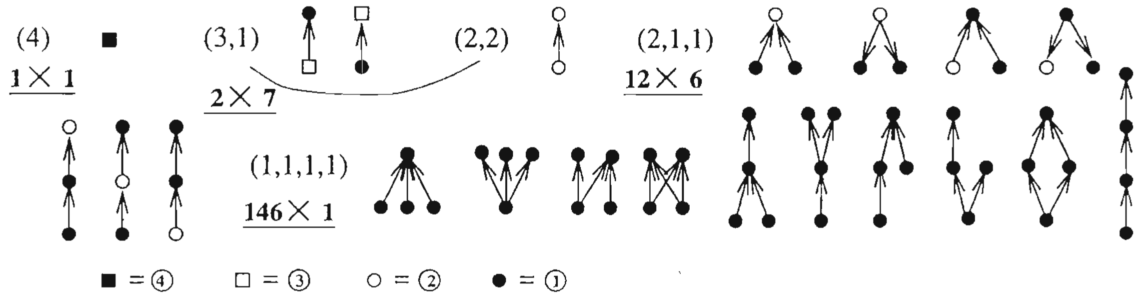

FIG. 4. The 21 graphs for $n = 4$ in Fig. 3 reduce to these graphs, which are just the connected posets. In particular, the squashing of Bosonic lines gives five partitions of 4. The corresponding Stirling numbers are 1, 7, 6, 1 respectively for one, two, three, and four groups to partition 4. The number of connected labelled posets, on the other hand, are given by 1, 2, 12, 146, and so on. Therefore, we deduce that the number of connected labelled topologies for $n = 4$ is given by $1 \times 1 + 2 \times 7 + 12 \times 6 + 146 \times 1 = 233$, without counting the symmetry numbers indicated in Fig. 3.

The number of graphs corresponding to Fig. 4, if we label the vertices, can be determined by counting the associated symmetry numbers. However, we can deduce this number without such a procedure, because the graphs in Fig. 4 are in fact precisely the connected posets, and the Stirling numbers tells us how many ways we can partition the number in the group, as stated in the figure caption. Therefore, if we let $\{d_n\} = 1,\ 2,\ 12,\ 146,\ 3060,\ 101642, \cdots$ denote connected labelled posets and $\{t_n\} = 1,\ 3,\ 19,\ 233,\ 4851,\ 158175, \cdots$ denote the connected labelled topologies, we deduce the following formula:

$$t_n = \sum_{k=1}^{n} S(n,k) d_k \ .$$

By exponentiating the foregoing arguments in the sense noted earlier, we can show that the same identity via Stirling numbers also holds for disconnected graphs, a formula known in combinatorics [3].

In the above discussion on topology numbers our line of thinking in simplifying the problem is motivated by field theoretic ideas. The challenging problem is to find a field theory whose Feynman rules produce graphs corresponding to those in Fig. 4. This may be achieved by introducing Fermions, Wick ordering, and so on. We note that only the first 14 terms in the sequences $\{d_n\}$, $\{t_n\}$ are known.

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Classical fields as statistical states

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We present a rough outline for an idea that characterises the observed, macroscopic realisation of the electromagnetic field in terms of a probability distribution on the underlying quantum electrodynamic state space.

In this note we sketch out some tentative thoughts on the relation between microscopic and macroscopic states in quantum theory. The idea is that classical fields (e.g., electromagnetism, or weak-field gravity) should be thought of as *statistical* states. Our starting point is to take the view that the physical world is inherently quantum mechanical. The problem is thus not how to quantise a given classical theory, but rather how to classicalise the quantum theory appropriately at large scales. Starting with a general multi-particle quantum system characterised by a large state space and a myriad of associated observables, our task is to specify those states of the system that correspond, in some reasonable sense, to the classical macroscopic configurations observed in practice.

This is an interesting question, because it ties in with some of the major open issues in quantum theory that may be of relevance to practical considerations. Even in the case of electromagnetism, it has to be appreciated that there is no general agreement as to what constitutes the precise relation between *microscopic* and *macroscopic* realisations of the electromagnetic field, despite the fact that classical and quantum electrodynamics are both well developed theories. Sometimes it is suggested that the coherent states of electromagnetism correspond to classical electrodynamic fields, but the arguments supporting this idea are not entirely convincing. Coherent states, to be sure, are in one-to-one correspondence with classical solutions of Maxwell's equations—in particular, to nonsingular, normalisable solutions. These states also have the property that they saturate the uncertainty lower bounds for measurements of the field operator. However, there is no explanation for why this should be a 'natural' configuration for the electromagnetic state space. Since coherent states are pure *quantum* states, we are left to wonder if it is possible that these states could *remain* pure on a macroscopic scale. This is questionable.

To put the matter another way, we expect a pure state to have low entropy by any reasonable definition, whereas for the quasi-stable nature of a classical field configuration, a high-entropy state would be the more plausible candidate. Then we could invoke some form of the second law of thermodynamics to explain the natural occurrence of such configurations.

Now let us try to build up a model along these lines in more precise terms. Suppose we consider a complex Hilbert space $\mathcal{H}$, for which we denote the associated multi-particle bosonic Fock space $\mathcal{F}$. We use Greek indices for elements of $\mathcal{H}$, and Roman indices for elements of $\mathcal{F}$. Thus, if $\xi^\alpha \in \mathcal{H}$ and $\eta_\alpha \in \mathcal{H}^*$ (the dual space), then for their inner product we write $\eta_\alpha\xi^\alpha$. Likewise, if $\Psi^a \in \mathcal{F}$ and $\Phi_a \in \mathcal{F}^*$, then we can form the inner product $\Phi_a\Psi^a$. An element Ψ^a of $\mathcal{F}$ is given, more explicitly, by a normalisable set of symmetric tensors in the space $(\mathbf{C}, \mathcal{H}, \mathcal{H}\otimes\mathcal{H}, \mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H}, \cdots)$ given by $\Psi^a = \{\psi, \psi^\alpha, \psi^{(\alpha\beta)}, \psi^{(\alpha\beta\gamma)}, \cdots\}$ with the inner product

$$\Phi_a\Psi^a \;=\; \phi\psi + \phi_\alpha\psi^\alpha + \phi_{\alpha\beta}\psi^{\alpha\beta} + \phi_{\alpha\beta\gamma}\psi^{\alpha\beta\gamma} + \cdots\,. \qquad (1)$$

We have in mind, in particular, the case where $\mathcal{H}$ is the Hilbert space of positive-frequency square-integrable solutions of Maxwell's equations, equipped with the usual gauge independent Hermitian inner product. Then $\mathcal{F}$ is the multi-particle photon space of quantum electrodynamics, and the general element of $\mathcal{F}$ determines a superposition of states consisting of various numbers of photons, where the photon number is the rank of the corresponding tensor. In addition we require a specification of the electromagnetic field operators. The creation operator $C_{\alpha b}^{\ a}$ is a map from $\mathcal{F}$ to $\mathcal{F} \otimes \mathcal{H}^*$ and the annihilation operator $A^{\alpha a}_{\ b}$ is a map from $\mathcal{F}$ to $\mathcal{F} \otimes \mathcal{H}$. They satisfy the commutation relations $C_{\beta b}^{\ a} A^{\alpha b}_{\ c} - A^{\alpha a}_{\ b} C_{\beta c}^{\ b} = \delta^{\alpha}_{\beta} \delta^{a}_{c}$. The specific actions of C and A on $\mathcal{F}$ (see Geroch 1971) are not required here.

The pure states of quantum electrodynamic systems are not represented by elements of $\mathcal{F}$, but rather by points in the associated *projective* Fock space Γ. Let x denote a typical point in Γ, and $\Psi^a(x)$ a point in the fibre above x. Here, we think of $\mathcal{F}$ as a fibre space over Γ. Then the expectation of the annihilation operator $A^{\alpha a}_{\ b}$, conditional to a pure quantum state x, can be represented by a map $A^\alpha(x)$ from Γ to $\mathcal{H}$, given by

$$A^\alpha(x) \;=\; \frac{\bar{\Psi}_a(x) A^{\alpha a}_{\ b} \Psi^b(x)}{\bar{\Psi}_c(x)\Psi^c(x)} \; . \tag{2}$$

Note that $A^\alpha(x)$ is independent of the scale of $\Psi^a(x)$. Thus each quantum electrodynamic state is associated with a unique classical field, given by the map $x \in \Gamma \to A^\alpha(x) \in \mathcal{H}$.

Now suppose we are given a classical solution $\xi^\alpha \in \mathcal{H}$ of Maxwell's equations, and we wish to construct a state on Γ to which ξ^α should correspond in some natural *physical* sense. The general state on the multi-particle photon state space Γ is given by a *probability distribution* $\rho(x)$ over Γ. Thus, if dx represents the natural volume element on Γ associated with the Fubini-Study metric, we have

$$\int_\Gamma \rho(x) dx \;=\; 1 \; . \tag{3}$$

The associated density matrix ρ^a_b, which contains sufficient information to value the expectations of linear observables, is given by

$$\rho^a_b \;=\; \int_\Gamma \rho(x) \frac{\bar{\Psi}_b(x)\Psi^a(x)}{\bar{\Psi}_c(x)\Psi^c(x)} dx \; . \tag{4}$$

For example, the expectation of the annihilation operator can be written in the form:

$$\rho^a_b A^{\alpha b}_{\ a} \;=\; \int_\Gamma \rho(x) A^\alpha(x) dx \; . \tag{5}$$

We come to our key hypothesis. Our suggestion is that, for a given classical field configuration ξ^a, the associated physical quantum electrodynamic state is given by the probability density function $\rho(x)$ that maximises the entropy function

$$S_\rho \;=\; -\int_\Gamma \rho(x) \ln \rho(x) dx \tag{6}$$

over Γ, subject to the constraint

$$\xi^a = \int_\Gamma \rho(x) A^\alpha(x) dx \; . \tag{7}$$

A standard line of argument (cf. [2]) then shows that the choice of $\rho(x)$ that maximises S_ρ is given by a *grand canonical ensemble* on Γ of the form:

$$\rho(x) \; = \; \exp[-\mu_\alpha A^\alpha(x) - \bar{\mu}^\alpha C_\alpha(x)]/Z(\mu) \; . \tag{8}$$

Here, the partition function $Z(\mu)$ is given by the integral

$$Z(\mu) \; = \; \int_\Gamma \exp\left[-\mu_\alpha A^\alpha(x) - \bar{\mu}^\alpha C_\alpha(x)\right] dx \; . \tag{9}$$

The 'chemical potential' $\mu_\alpha \in \mathcal{H}^*$ arises as a Lagrange multiplier in the variation analysis, and is determined by the relation

$$\frac{\partial \ln Z(\mu)}{\partial \mu_\alpha} \; = \; \xi^\alpha \tag{10}$$

for the given value of ξ^α. Thus the dual field μ_α is *thermodynamically conjugate* to the expectation of annihilation operator ξ^α.

The manifold Γ is foliated by the level surfaces of $A^\alpha(x)$, and the grand canonical distribution $\rho(x)$ is constant on each such surfaces. If we introduce the manifold $\mathcal{C} \in \Gamma$ consisting of coherent states, then Γ can be viewed as a fibre space over $\mathcal{C}$, because each level surface of the expectation of the annihilation operator intersects $\mathcal{C}$ at one point, namely, the coherent state corresponding to the given value of $A^\alpha(x)$. Coherent states are those points x for which the eigenvalue relation $A^{\alpha a}_{b}\Psi^b(x) = \xi^\alpha\Psi^a(x)$ is satisfied for some ξ^α.

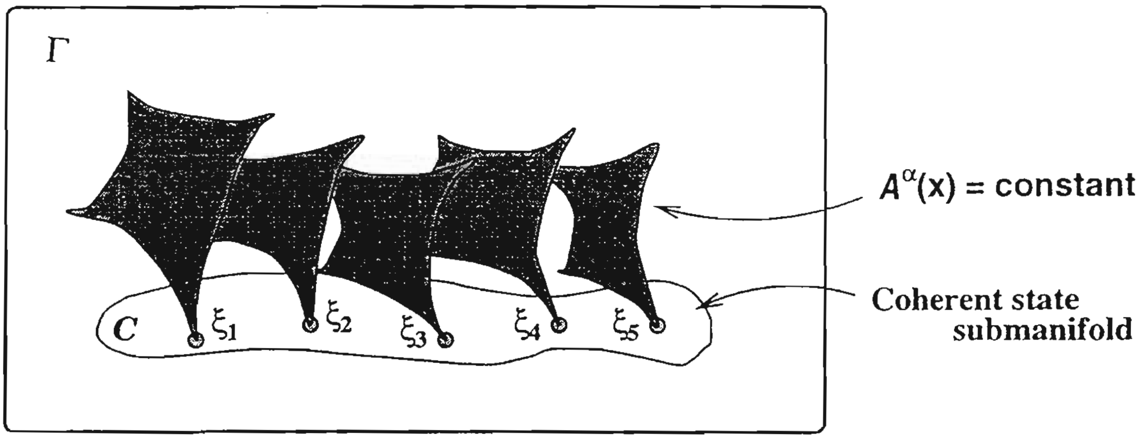

FIG. 1. Foliation of projective Fock space Γ by level surfaces of the expectation of the annihilation operator $A^\alpha(x)$. Each such surface intersects the manifold $\mathcal{C}$ of coherent states at a point.

There are many interesting analogies that follow from the ideas suggested above, and it is tempting to take them seriously. For example, the phenomenon of classicalisation can be viewed as a consequence of the second law of thermodynamics (cf. [3]). Suppose we have a large region of essentially classical configurations (the laboratory) and a small region (the experimental region) where we create, say, a region of pure quantum state. Initially the experimental region is insulated from the rest of the laboratory, but after the shield is removed the experimental region 'decoheres' by adjusting itself to the chemical potential μ_α of the laboratory. This follows from the requirement of matching chemical potentials (in this case, electromagnetic fields) for systems in equilibrium. More precisely, the chemical potential of the laboratory has to shift slightly to accommodate the experimental region, and as this happens the pure state of the experimental region decoheres, or classicalises, just in such a way as to match up with the chemical potential of the laboratory, so that the combined system is now characterised by a new classical state, say ν_α, which differs from the original μ_α by an essentially negligible perturbation.

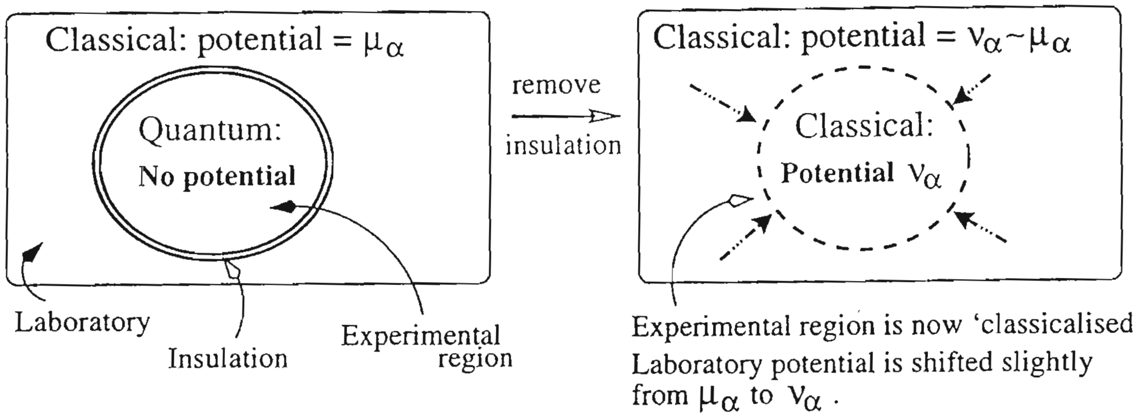

FIG. 2. Classicalisation of a quantum field. An experimental region confining a pure quantum field is insulated from the environment of the surrounding laboratory, which is permeated with a classical field of chemical potential μ_α. After the insulation is removed the pure quantum state decoheres as it comes into equilibrium with the environment, and acquires a potential ν_α equal to that of the laboratory, which has shifted very slightly in response to its interaction with the quantum field.

In the idea outlined above, we implicitly assume that there exists a dynamical mechanism leading to a kind of Boltzmann's H-theorem for quantum electromagnetism. However, such a mechanism may not exist as such in quantum physics, in which case the resulting 'equilibrium' (classical) distribution would have to be altered. For example, we might be led to the density matrix

$$\rho^a_b = \exp(-\mu_\alpha A^{\alpha a}_{\ \ b} - \bar{\mu}^\alpha C^{\ \,a}_{\alpha b})/Q(\mu) , \qquad (11)$$

where the expectation $\rho^a_b A^{\alpha b}_{\ \ a} = \xi^\alpha$ determines μ_α and $Q(\mu)$ is chosen so that $\rho^a_a = 1$. Nevertheless, the idea of representing classical states as statistical distributions is legitimate, and it also ties in naturally with the foundations of statistical mechanics, where the aim is to account for the observed characteristics of macroscopic physics in terms of an underlying microscopic dynamics. We hope to exploit the idea further.

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2. D. C. Brody & L. P. Hughston, J. Math. Phys. **39**, 6502 (1998); **40**, 12 (1999).

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Solving Polynomial Equations with Spinors

Abstract/Disclaimer: Almost surely, everything in this article is well-known (for example, to R.P.) but it seems a story worth telling. The main observation is that spinor methods provide solutions by radicals to cubic and quartic polynomials. For polynomials of degree $\leqslant 7$, they provide a means for reducing to canonical form.

The quadratic equation

$$ax^2 + 2bx + c = 0$$

is easily solved by 'completing the square':

$$ax^2 + 2bx + c = a(x + b/a)^2 + (ac - b^2)/a.$$

Equivalently, the generic homogeneous quadratic polynomial

$$ax^2 + 2bxy + cy^2 \qquad (*)$$

in the two variables x, y may be thrown into the canonical form

$$X^2 + Y^2 = (X + iY)(X - iY)$$

by a suitable linear change of variables:

$$\left.\begin{array}{l} X = \alpha x + \beta y \\ Y = \gamma x + \delta y \end{array}\right\} \quad \text{e.g.} \quad \left.\begin{array}{ll} \alpha = \sqrt{a} & \beta = b/\sqrt{a} \\ \gamma = 0 & \delta = \sqrt{(ac - b^2)/a} \end{array}\right\} \begin{array}{l} \text{if } a \neq 0 \\ \text{and } ac \neq b^2. \end{array}$$

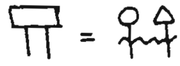
In ⑲ terminology, $(*)$ is a binary quadric and much effort was expended (by Cayley, Sylvester, Gordan, ... a cast of thousands...) in finding invariants, covariants, and canonical forms of binary quadrics, cubics, quartics, ..., quantics, ... The literature is vast. I can recommend two excellent texts [E, GY]. The marvelous though somewhat arcane ⑲ notation can be translated [P] into marvelous ⑳ spinor notation or even more marvelous ⑲/⑳ chemico-algebraic/twistor-bug notation [C]/[PR1].

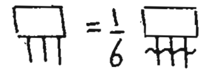
In particular,

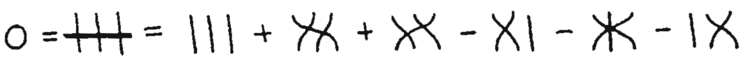
$$ax + by = \phi_A \quad \text{where} \quad \phi_0 = a, \ \phi_1 = b$$

$$ax^2 + 2bxy + cy^2 = \psi_{AB} \text{ where } \psi_{00} = a, \psi_{01} = \psi_{10} = b, \ \psi_{11} = c$$

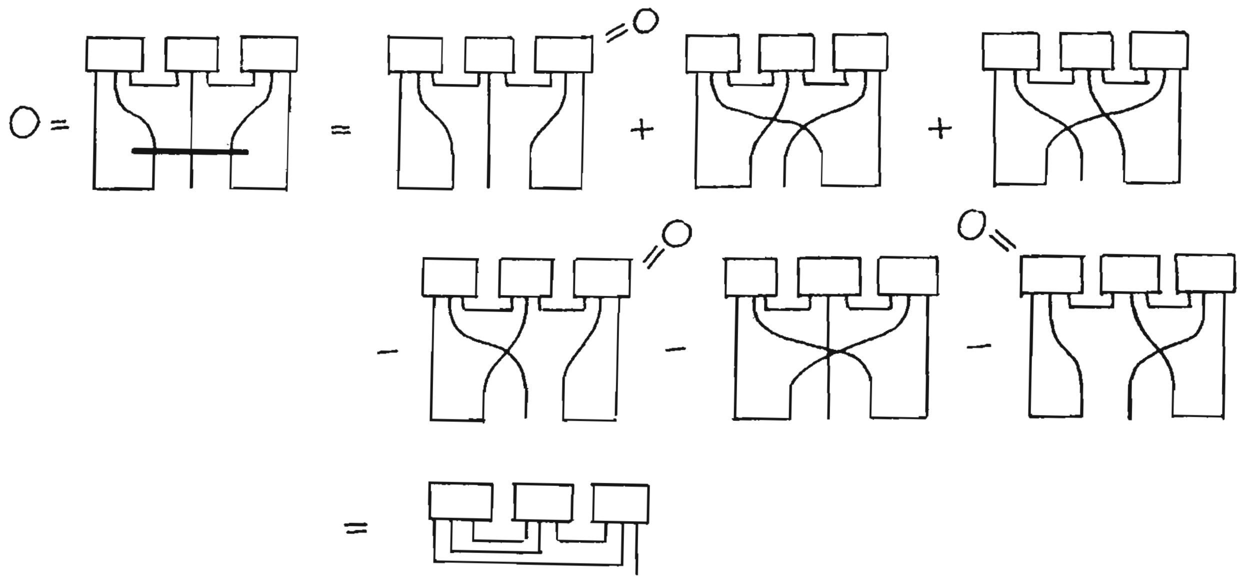
and then

ϕ_A =  ψ_{AB} =  where  $= \frac{X+iY}{\sqrt{2}}$,  $= \frac{X-iY}{\sqrt{2}}$.

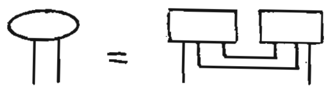
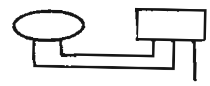
Let ε^{AB} =  and consider the binary cubic  .
There is the spinor identity (cf. TN 1, p.1)

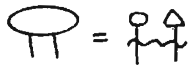


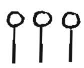
from which all other spinor identities follow. (The corresponding statement in higher dimensions is a second main theorem of invariant theory, essentially due to Weyl [W]—see also [G].) In particular,



In other words, the binary quadric (a covariant called the Hessian)

 satisfies  = 0.

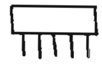
In classical terminology,  and  are said to be apolar.
Now we've already solved the quadratic equation  . If we suppose these two zeroes to be distinct, then it follows, simply by counting dimensions, that

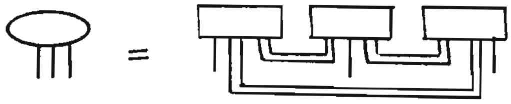
 is a linear combination of  and  .

The coefficients of this combination must be non-zero (assuming the Hessian is non-vanishing). They may be absorbed to yield the canonical form

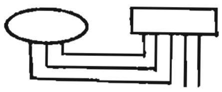
$$X^3+Y^3 \quad \left[= (X+Y)(X+\omega Y)(X+\omega^2 Y) \text{ where } \omega = e^{2\pi i/3}\right]$$

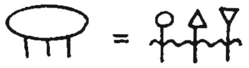
of the generic cubic. Special cases (if  vanishes or has a double zero) can be similarly thrown into special canonical forms. In particular, we have <u>solved the cubic</u> with radicals.

This classical canonical reduction argument applies to all binary quantics of odd degree. In the case of a quintic , for example, the cubic covariant



is apolar to , i.e.

 = 0

and, having solved the cubic , it follows that

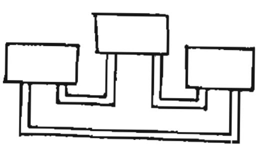
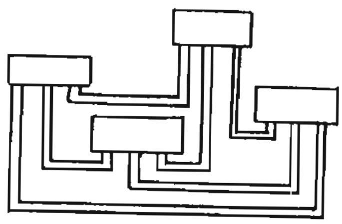
 ∈ span 

(if the roots are distinct). We have therefore written (using only radicals) the generic quintic in the form

$$X^5+Y^5+(\lambda X+\mu Y)^5.$$

Of course, this canonical form does not itself admit a solution by radicals. (This is quite different from a canonical form determined by the principal null directions of the spinor (i.e. zeroes of the polynomial) — for quintics such a reduction cannot be effected by radicals.) Septics may be similarly canonized by radicals, the canonizing apolar being a quartic.

The binary quantics of even degree are more difficult. There is an obstruction to writing a $2n$-tic as a sum of n perfect $2n^{\text{th}}$ powers. It is an invariant called the <u>catalecticant</u>:

 ,  ,  , ...

For quartics, however, we can use the geometric link between spinors and tensors to effect a canonization. This is converse to the more usual use of spinors in classifying tensors as typified by the Weyl tensor in terms of the principal null directions of its associated Weyl spinors (see [PR2, Chapter 8] for a comprehensive discussion (including an optional Maxwell field) and a similar treatment of the Ricci tensor). Introduce the symmetric spinor x^{AB} as coördinates x^{00}, $x^{01}=x^{10}$, x^{11} on $\mathbb{C}^3$. Then

$$\varepsilon_{AC}\varepsilon_{BD}x^{AB}x^{CD} = 2[x^{00}x^{11}-(x^{01})^2] = \left[\frac{x^{00}+x^{11}}{\sqrt{2}}\right]^2 + \left[\frac{ix^{00}-ix^{11}}{\sqrt{2}}\right]^2 + \left[\sqrt{2}\,i x^{01}\right]^2$$

is evidently preserved by $SL(2,\mathbb{C})$, hence the double cover $SL(2,\mathbb{C}) \xrightarrow{2:1} SO(3,\mathbb{C})$.
As matrices,

$$\underbrace{\begin{pmatrix}\hat{x}^0\\ \hat{x}^1\end{pmatrix} = \begin{pmatrix} a & b\\ c & d\end{pmatrix}}_{\in SL(2,\mathbb{C})}\begin{pmatrix}x^0\\ x^1\end{pmatrix} \Rightarrow \begin{pmatrix}\hat{x}^{00}\\ \hat{x}^{01}\\ \hat{x}^{11}\end{pmatrix} = \underbrace{\begin{pmatrix} a^2 & 2ab & b^2\\ ac & ad+bc & bd\\ c^2 & 2cd & d^2\end{pmatrix}}_{\in SO(3,\mathbb{C})}\begin{pmatrix}x^{00}\\ x^{01}\\ x^{11}\end{pmatrix} \qquad (**)$$

Notice that lifting from $SO(3,\mathbb{C})$ to $SL(2,\mathbb{C})$ may be effected by $\sqrt{\ }$. A binary quartic ϕ_{ABCD} may be regarded, equivalently, as a trace-free symmetric form $\phi_{ABCD}x^{AB}x^{CD}$ on $\mathbb{C}^3$ (cf. [PR2, §8.3]). The generic symmetric form on $\mathbb{C}^3$ may be complex orthogonally diagonalized — the argument follows the real version save for the possibility of null eigenvectors which must be treated separately. Furthermore, the characteristic polynomial is a cubic which we already know how to solve (it is not a covariant but is, rather, an ordinary cubic polynomial whose coefficients are invariants). Therefore, we can lift to an explicitly computable $SL(2,\mathbb{C})$ change of coördinates so that, in the new coördinates:

$$\phi_{ABCD}x^{AB}x^{CD} = \lambda\left[\frac{x^{00}+x^{11}}{\sqrt{2}}\right]^2 + \mu\left[\frac{ix^{00}-ix^{11}}{\sqrt{2}}\right]^2 + \nu\left[\sqrt{2}\,ix^{01}\right]^2 \quad \text{where } \lambda+\mu+\nu=0$$

$$= \frac{\lambda-\mu}{2}(x^{00})^2 + (\lambda+\mu)[x^{00}x^{11}+2(x^{01})^2] + \frac{\lambda-\mu}{2}(x^{11})^2$$

Thus, $\quad \phi_{ABCD} = \frac{\lambda-\mu}{2}x^4 + 3(\lambda+\mu)x^2y^2 + \frac{\lambda-\mu}{2}y^4.$

Provided that $\lambda \neq \mu$, a further coördinate rescaling gives

$$X^4 + 6m X^2 Y^2 + Y^4,$$

the canonical form obtained in [E] (by a different method which assumes that the quartic can be solved). As a quadratic in X^2 and Y^2, it is easily solved. As usual, there are some degenerate cases which must be treated separately but, in any case, we have now <u>solved the quartic</u> by radicals. Of course, these procedures follow the (16) formulae of del Ferro, Fontano, and Ferrari or, equivalently, the solutions derived from Galois theory [S].

<u>E.g.</u> We can use these procedures to find the zeroes of

$$\boxed{14x^4 + 68x^3 + 115x^2 + 76x + 14}$$

The corresponding trace-free symmetric form $\phi_{ABCD}x^{AB}x^{CD}$ is

$$14(x^{00})^2 + 68x^{00}x^{01} + 115/3\,(x^{00}x^{11} + 2(x^{01})^2) + 76x^{01}x^{11} + 14(x^{11})^2 \quad (***)$$

$$= \left(\frac{x^{00}+x^{11}}{\sqrt{2}}, \frac{ix^{00}-ix^{11}}{\sqrt{2}}, \sqrt{2}ix^{01}\right) \underbrace{\begin{pmatrix} 199/6 & 0 & -36i \\ 0 & 31/6 & 2 \\ -36i & 2 & -115/3 \end{pmatrix}}_{M} \begin{pmatrix} \frac{x^{00}+x^{11}}{\sqrt{2}} \\ \frac{ix^{00}-ix^{11}}{\sqrt{2}} \\ \sqrt{2}ix^{01} \end{pmatrix}.$$

The symmetric matrix M has characteristic polynomial

$$\lambda^3 - 73/12\,\lambda + 595/108$$

whose Hessian is

$$-\frac{73}{18}x^2 + \frac{595}{54}x - \frac{5329}{648} = -\frac{(438x-595-108\sqrt{3}i)(438x-595+108\sqrt{3}i)}{47304}$$

and so the characteristic polynomial may be written as

$$p\,(438x-595-108\sqrt{3}i)^3 + q\,(438x-595+108\sqrt{3}i)^3$$

for suitable p and q which are easily computed:

$$p = \frac{324 + 595\sqrt{3}\,i}{54449931456} \qquad q = \frac{324 - 595\sqrt{3}\,i}{54449931456}.$$

This gives the roots of the characteristic polynomial as $5/3, 7/6, -17/6$ and enables us complex orthogonally to diagonalize M:

$$R^t M R = \begin{pmatrix} 5/3 & 0 & 0 \\ 0 & 7/6 & 0 \\ 0 & 0 & -17/6 \end{pmatrix} \quad \text{where } R = \begin{pmatrix} 8i & 9 & 4i \\ -4 & 4i & -1 \\ 7 & -8i & 4 \end{pmatrix}.$$

Thus,

$$(\ast\ast\ast) = \frac{5}{3}\left[\frac{\hat{x}^{00} + \hat{x}^{11}}{\sqrt{2}}\right]^2 + \frac{7}{6}\left[\frac{i\hat{x}^{00} - i\hat{x}^{11}}{\sqrt{2}}\right]^2 - \frac{17}{6}\left[\sqrt{2}\,i\,\hat{x}^{01}\right]^2$$

where

$$\begin{pmatrix} \dfrac{\hat{x}^{00} + \hat{x}^{11}}{\sqrt{2}} \\ \dfrac{i\hat{x}^{00} - i\hat{x}^{11}}{\sqrt{2}} \\ \sqrt{2}\,i\,\hat{x}^{01} \end{pmatrix} = R^{-1} \begin{pmatrix} \dfrac{x^{00} + x^{11}}{\sqrt{2}} \\ \dfrac{ix^{00} - ix^{11}}{\sqrt{2}} \\ \sqrt{2}\,i\,x^{01} \end{pmatrix} \quad \text{i.e.} \quad \begin{pmatrix} \hat{x}^{00} \\ \hat{x}^{01} \\ \hat{x}^{11} \end{pmatrix} = \begin{pmatrix} -i/2 & -i & -i/2 \\ 3/2 & 4 & 5/2 \\ 9i/2 & 15i & 25i/2 \end{pmatrix} \begin{pmatrix} x^{00} \\ x^{01} \\ x^{11} \end{pmatrix}$$

see (**)

Incorporating the final rescaling gives

$$\left.\begin{aligned} X &= (1-i)(x+y)/2\sqrt{2} \\ Y &= (1+i)(3x+5y)/2\sqrt{2} \end{aligned}\right\} \text{ and, sure enough, } \quad \begin{pmatrix} (1-i)/2 & (1-i)/2 \\ 3(1+i)/2 & 5(1+i)/2 \end{pmatrix}.$$

$$14x^4 + 68x^3y + 115x^2y^2 + 76xy^3 + 14y^4 = X^4 + 34X^2Y^2 + Y^4,$$

as predicted. The zeroes of the original quartic are, thus: $(-2 \pm \sqrt{2})/2,\ (-10 \pm \sqrt{2})/7.$

Thanks to R.P. and N.M.J.W. (~16 years ago) and Vladimir Ezhov.

C: W.K. Clifford "Theory of Graphs" MacMillan 1881 (See Puzzle in TN 34 (1992) and [P]).

E: E.B. Elliott "Algebra of Quantics" Clarendon Press 1895.

G: A.R. Gover "The algebra of orthogonal group invariants" preprint.

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PR1: } R. Penrose and W. Rindler "Spinors and Space-time" C.U.P.
PR2: } vol. 1 1984 vol. 2 1986.

S: I. Stewart "Galois Theory" Chapman and Hall 1973.

W: H. Weyl "Classical Groups" P.U.P. 1939.

Mike Eastwood
Adelaide

A conjectured form of the Goldberg-Sachs theorem[1]

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The rather well-known Goldberg-Sachs theorem [3] is one of the most beautiful results in the mathematics of general relativity theory. It played a major role in the work on algebraically special solutions of Einstein's equations [5].

For the purposes of this Letter, it is convenient to formulate it as follows. Let $\mathfrak{M}$ be a set of Lorentzian, not conformally flat, manifolds (M, g) of dimension 4. For $(M, g) \in \mathfrak{M}$, let $K \subset TM$ be a null line bundle; its sections are null vector fields. Following the notation and terminology of [7], consider the following two properties of K:

(GSR) K is geodetic and shear-free;

(PND) K is a bundle of *repeated* principal null directions of the Weyl tensor C.

A Goldberg-Sachs theorem GST($\mathfrak{M}$) is a statement of the form: *if* $(M,g) \in \mathfrak{M}$, *then the conditions (GSR) and (PND) are equivalent.* Goldberg and Sachs proved the theorem for $\mathfrak{M}$ = the set of Einstein spaces, i.e. solutions of $R_{\mu\nu} = \frac{1}{4} g_{\mu\nu} R$. Shortly afterwards, Kundt and Thompson [6] and Robinson and Schild [8] pointed out that both conditions (GSR) and (PND) are conformally invariant, but the property of being an Einstein space is not. They proved a generalized Goldberg-Sachs theorem that, in a refined form, is given in §7.3 of [7] and in §7.5 of [5]. This generalized theorem involves only conformal notions, but requires a separate formulation for each degree of degeneracy of the Weyl tensor.

A conformally invariant set of space-times is

$$\mathfrak{M}_c = \{(M, g) \text{ is conformal to an Einstein space}\}$$

and GST($\mathfrak{M}_c$) is true as a consequence of the classical Goldberg-Sachs theorem. It is not easy to find a description of $\mathfrak{M}_c$ by means of tensorial or spinorial equations; see [4] for an account of the early work by Brinkman. To appreciate the difficulty of the subject, recall that Schouten (p. 314 in [9]) attributes to Brinkman the following statement: In dimension 4, if two manifolds are Ricci flat and conformal to each other, but not to a flat space, then they are isometric. Ehlers and Kundt (p. 99 in [2]) give a counterexample to this: there are *pp* waves that are conformal, but not isometric, to each other.

It is known that $\mathfrak{M}_c$ is contained in the set $\mathfrak{M}_b$ of spaces satisfying the conformally invariant *Bach equation* $B = 0$. According to the arguments due to Geroch and Horowitz, presented in [4], there are space-times satisfying the Bach equation that are not conformal to an Einstein space.

[1] During the *Workshop on spinors and twistors* (28 June - 3 July 1999) at the Erwin Scrödinger Institute in Vienna I obtained valuable advice on the subject of this note from M. Dunajski, C. N. Kozameh, L. J. Mason, P. Nurowski, and K. P. Tod. When preparing the Letter, I have been supported in part by the Polish Committee for Scientific Research (KBN) under grant no. 2 P03B 060 17.

Kozameh, Newman, and Tod [4] have found a set of two equations defining a class of spaces conformal to Einstein spaces; one of them is $B = 0$, but the other one excludes some of the spaces with a degenerate C. There is an improvement of [4] by Baston and Mason [1], but their equations still do not characterize all of $\mathfrak{M}_c$.

Problems

I consider the following problems to be ordered according to increasing difficulty.

(i) Find a counterexample to GST($\mathfrak{M}_b$).

(ii) If you fail in (i), then prove GST($\mathfrak{M}_b$).

(iii) If you succeed in (i), then find a set of conformally invariant tensor or spinor equations defining $\mathfrak{M}$, without reference to the degeneracy of C, such that $\mathfrak{M}_c \subset \mathfrak{M}$ and GST($\mathfrak{M}$) is true.

Note that if $\mathfrak{M}_c \subsetneq \mathfrak{M}$, then GST($\mathfrak{M}$) is stronger than the classical theorem.

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[3] J. N. Goldberg and R. K. Sachs, A theorem on Petrov types *Acta Phys. Polon.* **22** Suppl. (1962) 13–23.

[4] C. N. Kozameh, E. T. Newman and K. P. Tod, Conformal Einstein spaces *Gen. Rel. Grav.* **17** (1985) 343–52.

[5] D. Kramer, H. Stephani, M. MacCallum and E. Herlt, *Exact solutions of Einstein's field equations* VEB Deutscher Verlag der Wissenschaften, Berlin 1980.

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[7] R. Penrose and W. Rindler, *Spinors and space-time* vol. 2, C. U. P., Cambridge 1986.

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Three Stories about Epsilon

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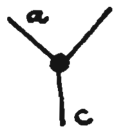
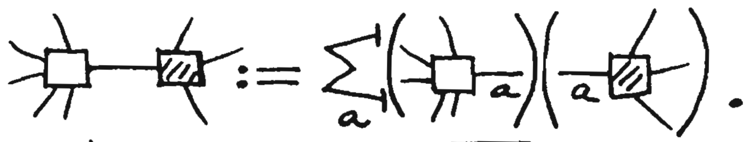
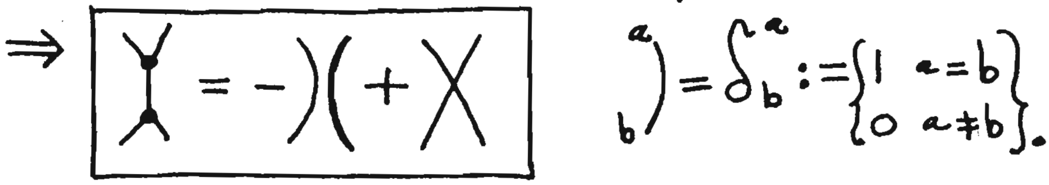
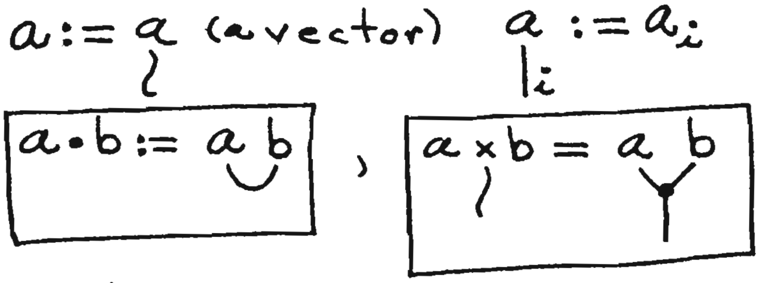
I.

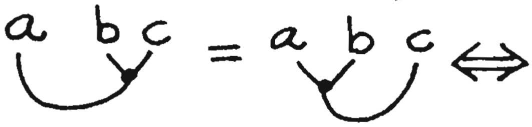


$$= \epsilon_{abc} := \begin{cases} 1 & abc = 123,\ 231,\ 312 \\ -1 & abc = 213,\ 132,\ 321 \\ 0 & \text{otherwise} \end{cases}$$

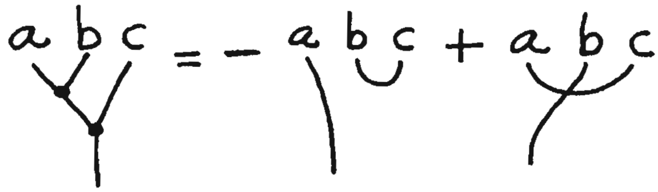








$\Leftrightarrow$ $a \cdot (b \times c) = (a \times b) \cdot c$



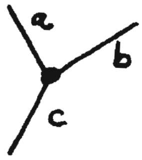
whence

$$(a \times b) \times c = -a(b \cdot c) + b(a \cdot c)$$

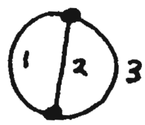
$$a \times (b \times c) = -(a \cdot b)c + b(a \cdot c)$$

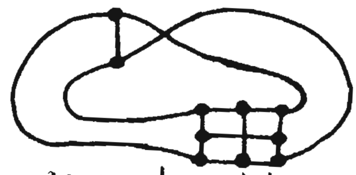
$$(a \times b) \times c - a \times (b \times c) = (a \cdot b)c - a(b \cdot c)$$

II. Coloring a cubic graph.

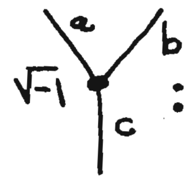


In a proper coloring a, b, c = 3 distinct labels using $\{1, 2, 3\}$.

e.g. 

e.g. 

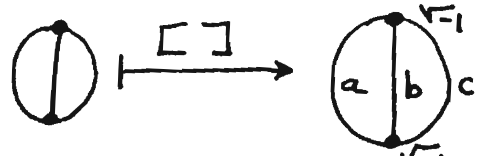
uncolorable (Petersen graph)



$:= \sqrt{-1}\, \varepsilon_{abc}$

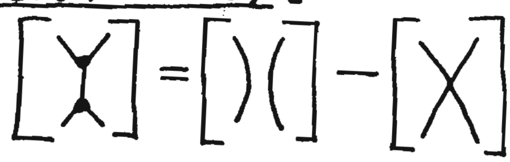
$G \looparrowright \mathbb{R}^2$ a cubic graph immersed in plane.
G planar if no extra crossings $\times$.

$[G]$:= contraction of the tensor obtained by replacing each 3-vertex by $\sqrt{-1}\,\varepsilon$.

e.g. 

$$= (\sqrt{-1})^2 \varepsilon_{acb}\, \varepsilon_{abc} = \varepsilon_{abc}\, \varepsilon_{abc} = 6.$$

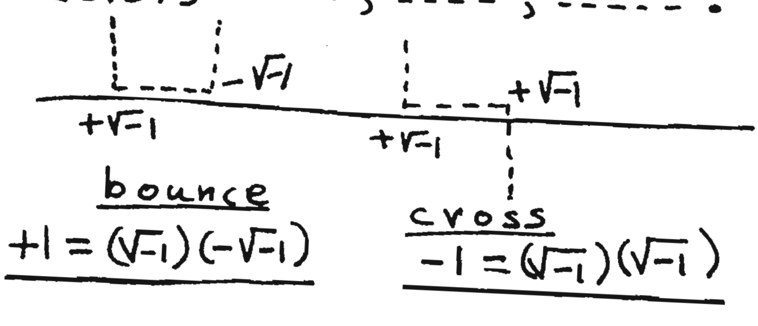
Theorem (Penrose).

(A) 

(B) G planar $\Rightarrow [G]$ = the number of proper colorings of $G \hookrightarrow \mathbb{R}^2$.

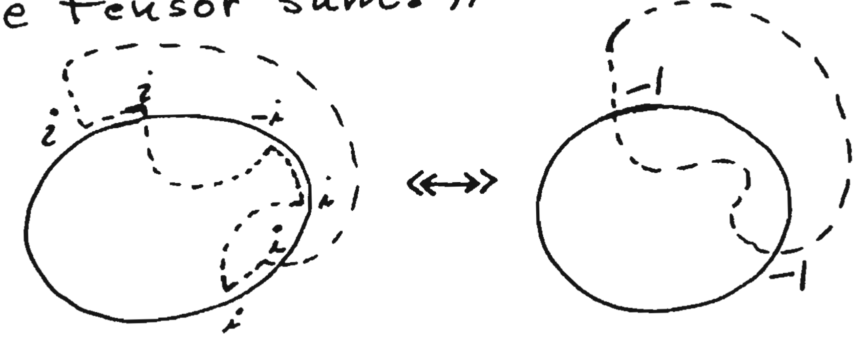
Proof. (A) Use section I.

(B) Colors —— , - - - - , ═ ═ ═ :



Now use the Jordan curve theorem to conclude that each proper coloring of the planar graph contributes $(-1)^{\#(\text{crossings})} = (-1)^{\text{even}} = +1$ to the tensor sum. //

e.g.



III. Witten's Functional Integral For Knot Invariants

$A(x) = A^a_i(x) T_a dx^i$, $i = 1, 2, 3$
$a = 1, 2, \ldots, d$

gauge field on $\mathbb{R}^3$.

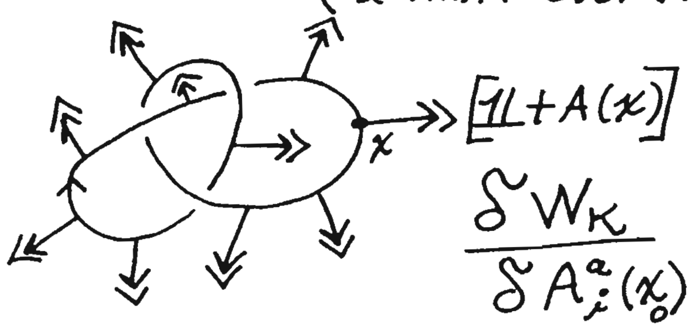
$\{T_1, T_2, \ldots, T_d\}$ matrix Lie algebra basis.

 $K \subset \mathbb{R}^3$ a knot.

$W_K(A) = tr(\mathbb{P}e^{\oint_K A})$ Wilson Loop

$W_K(A) := tr\left(\prod_{x \in K} (\mathbb{1} + A(x))\right)$

(a limit over finite partitions of K)



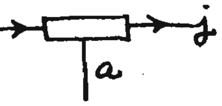
$$\frac{\delta W_K}{\delta A^a_i(x_0)} := \oint^a_i W_K$$

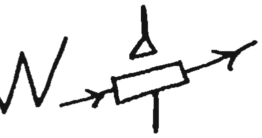
$$\oint^a_i W_K = \oint^a_i tr \prod_{x \in K} (\mathbb{1} + A^a_i(x) \underline{T_a dx^i})$$

$$= T_a dx^i \, tr \prod_{x \in K} (\mathbb{1} + A(x))$$

insert at x_0

Let  $= T_a$,  $= dx^i$

( $= (T_a)_{ij}$).

Then  = W 

Insert $T_a dx^i$ at x_0 in Wilson Loop.

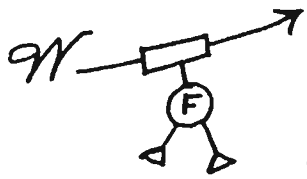
<u>The functional integral</u>:

$$Z_K = \int DA\; e^{\frac{ik}{4\pi}\mathcal{L}(A)}\, W_K(A).$$

$$\mathcal{L}(A) = \int_{S^3} tr\left(A \wedge dA + \tfrac{2}{3} A\wedge A\wedge A\right)$$

(•)  =  curvature tensor for $dA + A\wedge A$.

(•) $\delta W_{\to} := W_{\rightsquigarrow} - W_{\to}$

$\delta W_{\to} =$ 

Lie algebra and curvature insertion for small variation of the loop.

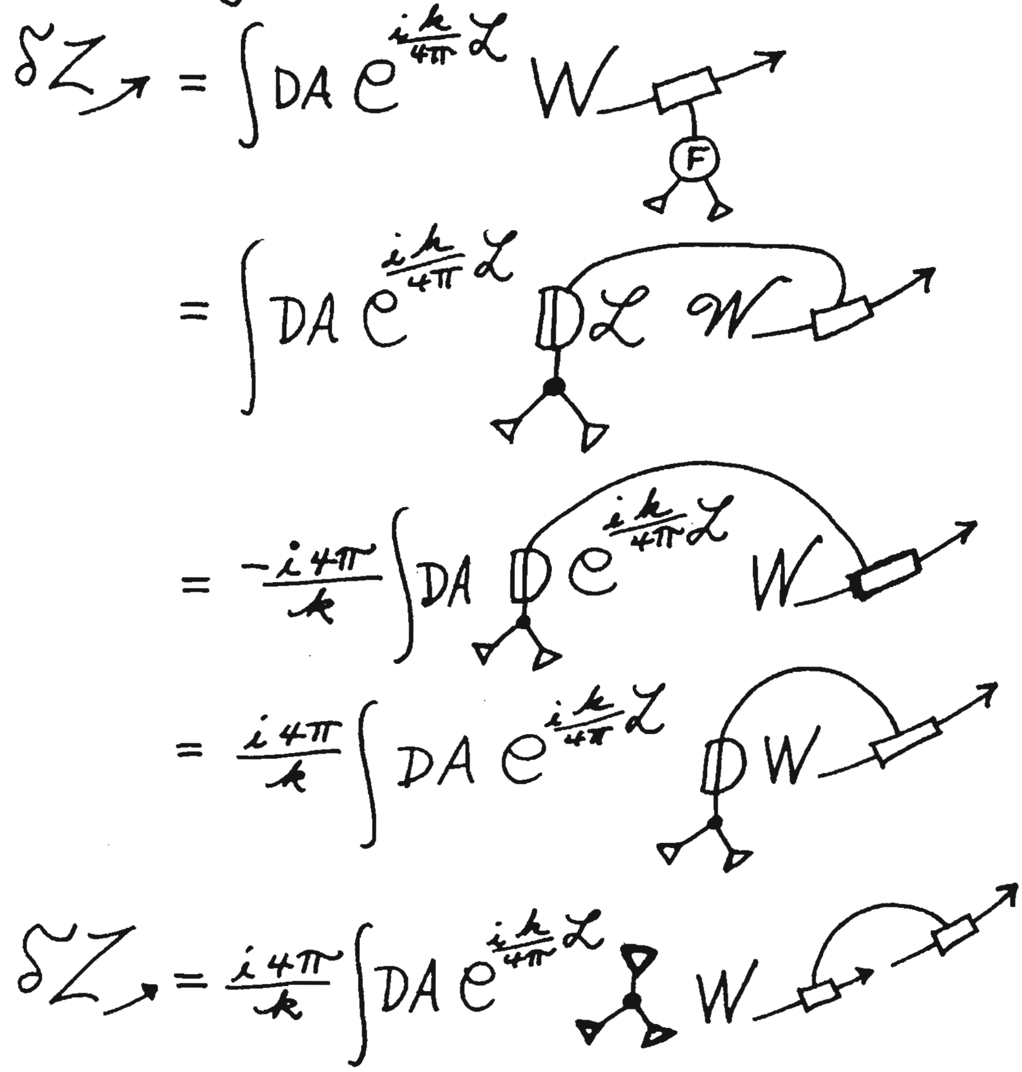
Curvature tensor arises from varying the field <u>or</u> varying the loop. Note the epsilon appearing in relation to the field variation. Note that

 $= dx^i dx^j dx^k\, \varepsilon_{ijk}$

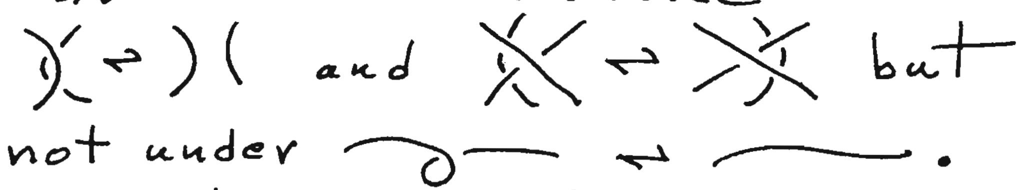
is a volume form.

Let $\delta Z_K := Z_{\rightsquigarrow} - Z_{\to}$.

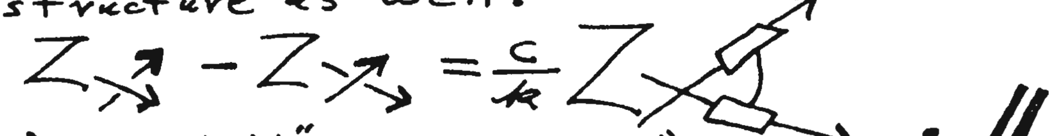
$$\delta Z_K = \int DA\, e^{\frac{ik}{4\pi}\mathcal{L}}\, \delta W_K$$



Because of the appearance of the volume form , $\delta Z_{\nearrow} = 0$ when the small deformation of the knot does not generate volume. Thus Z_K has formal invariance under



Thus it is a framed invariant, and the Lie algebra is involved in the framing and in the switching structure as well:



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ASD FOUR MANIFOLDS AND FROBENIUS MANIFOLDS

G. Sanguinetti
N.M.J Woodhouse

ABSTRACT. We describe here a simple geometric construction to obtain an anti-self-dual Riemannian four manifold from a three dimensional Frobenius manifold. This establishes a relationship between the isomonodromy deformation description of Frobenius manifolds (leading naturally to Painlevé's VI equation) and the twistor approach to integrable systems, that views them as a dimensional reduction of anti-self-dual Yang-Mills equations.

1 Introduction

Frobenius manifolds were introduced by Boris Dubrovin in the late eighties to encode the geometric information contained in a certain set of PDEs, the so called Witten- Dijkgraaf- Verlinde- Verlinde equations (WDVV). These arise naturally in the context of topological quantum field theories, and lead to a structure of associative algebra on the tangent space of certain moduli spaces of topological field theories.

For a more detailed analysis of the physical motivations that led to the introduction of Frobenius manifolds we refer to [1], while a detailed description of the geometrical features arising from Frobenius manifolds can be found in [3] or again in [1].

Among the most interesting features that arose in the work of Dubrovin there is the fact that WDVV equations are (particular) equations of isomonodromic deformations, and in particular in dimension three they are equivalent to a special form of Painlevé's sixth equation.

On the other hand, one of the greatest developments in the theory of integrable systems came from the application of ideas coming from twistor theory.

In this context, many classical integrable systems (among which the Painlevé's equations) can be seen as a dimensional reduction of anti-self-dual Yang-Mills equations, and hence solved via the Penrose-Ward transform [4].

It follows from this picture that to any solution of Painlevé's sixth equation there should be associated an anti-self-dual Riemannian four manifold. While it is well understood how to find this four manifold starting from the Yang-Mills equations, it is not clear what should be the link with Frobenius manifolds, and what particular anti-self-dual four manifolds should be associated to solutions of WDVV equations.

2 Construction

For our purposes, we can look at a Frobenius manifold X as a complex manifold endowed with a *pencil of flat metrics*, i.e. such that there exist two (contravariant) flat metrics g_1 and g_2 s.t. the linear combination $g_1 + zg_2$ is again a flat metric for any z in $\mathbb{C}$. This is a nontrivial condition (the flatness of a metric is a nonlinear condition on it), and it corresponds, from the integrable systems point of view, to the existence of two compatible Poisson brackets of hydrodynamic type on the loop space of X [2].

In addition, on any Frobenius manifold there exist two canonical vector fields, the identity vector field e and the Euler vector field E, which have the following property (see [1]):

denote by C the Christoffel symbol of the Levi-Civita connection of g_2, then

$$\mathcal{L}_e C = 0 \qquad \mathcal{L}_E C = C$$

(the second is often referred to as a quasihomogeneity condition).

We have the commutation relation

$$[E, e] = -e.$$

We will work in the flat coordinates associated to the metric g_1. In these coordinates the two vector fields take the form [1]

$$e = \frac{\partial}{\partial t^1}, \quad E = \sum_{\alpha=1}^{dim X} d_\alpha t^\alpha \frac{\partial}{\partial t^\alpha}$$

and can be normalized so that $d_1 = 1$.

Let us now specialize to the three dimensional case.

Let us lift the vector fields E and e to T^*X using the natural Lie lift. Define then them on the manifold $T^*X \times \mathbb{C}$ by adding a term $-z\frac{\partial}{\partial z}$ to E. Explicitly, the new vector field $\tilde{E}$ will read

$$\tilde{E} = \sum_{\alpha=1}^{dim X} d_\alpha t^\alpha \frac{\partial}{\partial t^\alpha} - \sum_{\alpha=1}^{dim X} d_\alpha p_\alpha \frac{\partial}{\partial p_\alpha} - z\frac{\partial}{\partial z}.$$

By virtue of the quasihomogeneity condition the commutation relations are unchanged, hence we get a two dimensional integrable distribution on the seven dimensional manifold $T^*X \times \mathbb{C}$.

If we consider the quotient of $T^*X \times \mathbb{C}$ by this distribution, we end up with a five dimensional manifold $\mathcal{F}$, which will still be fibred with fibre $\mathbb{C}$ (this follows from the fact that the vector field $\frac{\partial}{\partial z}$ descends up to scale to the quotient). The base manifold of this fibration will be a four dimensional manifold $\mathcal{M}$ which is the natural candidate to be our complex space time.

We have now to define an ASD conformal structure on the four dimensional manifold $\mathcal{M}$.

We will use the following proposition (see [5])

Proposition 2.1. *Let W, Z, $\tilde{Z}$, $\tilde{W}$ be independent holomorphic vector fields on a four dimensional complex manifold $\mathcal{M}$. Then W, Z, $\tilde{Z}$, $\tilde{W}$ determine an ASD conformal structure if and only if there exist two holomorphic functions u and v on $\mathcal{M} \times \mathbb{P}^1$ such that the distribution on $\mathcal{M} \times \mathbb{P}^1$ spanned by*

$$L = W - z\tilde{Z} + u\frac{\partial}{\partial z}, \qquad M = Z - z\tilde{W} + v\frac{\partial}{\partial z}$$

is integrable.

The vector fields W, Z, $\tilde{Z}$, $\tilde{W}$ are going to define a null tetrad at any point on the manifold $\mathcal{M}$, which is equivalent to give a conformal structure.

Let us consider now the vector fields on X $\frac{\partial}{\partial t_2}$ and $\frac{\partial}{\partial t_3}$. We have the following commutation relations

$$[E, \frac{\partial}{\partial t_2}] = d_2\frac{\partial}{\partial t_2}, \qquad [E, \frac{\partial}{\partial t_3}] = d_3\frac{\partial}{\partial t_3}$$

Consider their horizontal lift with respect to the Levi-Civita connection ∇_z associated to $g_1 + zg_2$. In the flat coordinates of g_1 this connection has the form

$$d + zC$$

so that these lifts are respectively

$$L = \frac{\partial}{\partial t_2} + zC^b_{2a} p_b \frac{\partial}{\partial p_a}$$

$$M = \frac{\partial}{\partial t_3} + zC^b_{3a} p_b \frac{\partial}{\partial p_a}$$

(C^c_{ab} are the components of C). We denote by p_a the coordinates on the fibers in TX. Define W, Z, $\tilde{Z}$, $\tilde{W}$ to be the degree 0 and degree 1 terms in z of these two vector fields (as in the previous proposition), then the quasihomogeneity condition guarantees that each of these vector fields descends up to scale on $\mathcal{M}$ (the commutation with the identity vector field is trivial).

The commutation of L and M is equivalent to the condition that ∇_z has vanishing curvature, and is thus equivalent to the WDVV equations. Furthermore, changing our vectors $\frac{\partial}{\partial t_2}$, $\frac{\partial}{\partial t_3}$ by a transformation in $Sl(2,\mathbb{C})$ clearly won't change the picture.

Since the vector fields L and M satisfy the hypothesis of the previous propositions, we end up with a four dimensional manifold $\mathcal{M}$ with an ASD conformal structure and an action of $Sl(2,\mathbb{C})$ on it.

Furthermore, this generalizes without any further problem to the n-dimensional case (isomonodromic deformations in any dimension have been given a twistor description [4] [6]).

Many thanks to P. Boalch, M. Dunajski, L. Mason and P. Tod for useful conversations and suggestions.

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ASD Null Kähler metrics with symmetry

Maciej Dunajski

A four-dimensional Riemannian manifold which admits a covariantly constant spinor has to be hyper-Kähler: in the Euclidean signature a spinor and its complex conjugate form a basis of a spin space. This argument breaks down in the $(++--)$ signature (also called ultra-hyperbolic, Kleinian, neutral or split signature), where

$$\text{Spin}(2,\,2) = SL(2,\mathbb{R}) \times \widetilde{SL}(2,\mathbb{R}),$$

and the representation space of the spin group splits into a direct sum of two real two-dimensional spin spaces S^A and $S^{A'}$. The conjugation of spinors is involutive and maps each spin space onto itself, and there exists an invariant notion of *real spinors*. One can therefore look for $(++--)$ non-Ricci flat metrics with a parallel real spinor (which we choose to be $\iota^{A'} \in \Gamma(S^{A'})$. The Ricci identities imply that the self-dual Weyl spinor is of type N: $C_{A'B'C'D'} = c\iota_{A'}\iota_{B'}\iota_{C'}\iota_{D'}$ for some c such that $\iota^{A'}\nabla_{AA'}c = 0$. I shall consider the anti-self-dual (ASD) case of $c = 0$. The corresponding metrics will be called *ASD null Kähler*. They have a null Ricci spinor, and vanishing scalar curvature. The resulting twistor theory is rich, and leads to a new integrable system in four dimensions [1]. In this paper I shall consider ASD null Kähler metrics which admit a Killing vector preserving the parallel spinor. Let us call them *ASD null Kähler metrics with symmetry*. I shall show that all such metrics are (at least in the real analytic case) locally determined by solutions to a certain integrable equation and its linearisation.

Proposition 1 *Let $H = H(x,y,t)$ and $W = W(x,y,t)$ be smooth real-valued functions on an open set $\mathcal{W} \subset \mathbb{R}^3$ which satisfy*

$$H_{yy} - H_{xt} + H_x H_{xx} = 0, \tag{1}$$

$$W_{yy} - W_{xt} + (H_x W_x)_x = 0. \tag{2}$$

Then

$$g = W_x(\mathrm{d}y^2 - 4\mathrm{d}x\mathrm{d}t - 4H_x\mathrm{d}t^2) - W_x^{-1}(\mathrm{d}z - W_x\mathrm{d}y - 2W_y\mathrm{d}t)^2 \tag{3}$$

is an ASD null Kähler metric on a circle bundle $\mathcal{M} \to \mathcal{W}$. All real analytic ASD null Kähler metrics with symmetry arise from this construction.

Before proving this proposition I shall review some facts about Einstein–Weyl (EW) spaces associated to equation (1). In [2] it has been demonstrated that if an EW space admits a parallel weighted vector, the coordinates can be found in which the metric and the one form are given by

$$h = \mathrm{d}y^2 - 4\mathrm{d}x\mathrm{d}t - 4H_x\mathrm{d}t^2, \qquad \nu = -4H_{xx}\mathrm{d}t, \tag{4}$$

and the EW equations reduce to (1)[1]. If $H(x,y,t)$ is a smooth real function of real variables then (4) has signature $(++-)$. It has also been shown that there exists a one to one correspondence between EW spaces (4) and two-dimensional complex manifolds (mini-twistor spaces) with a rational curve with normal bundle $\mathcal{O}(2)$ and a global section $\kappa^{-1/4}$, where κ is the canonical bundle. These structures should be invariant under an anti-holomorphic involution fixing a real slice in the twistor space.

Proof of Proposition 1. Let (h,ν) be a three–dimensional EW structure given by (4) and let (V,α) be a pair consisting of a function and a one-form which satisfy the generalized monopole equation

$$*_h(\mathrm{d}V + (1/2)\nu V) = \mathrm{d}\alpha, \tag{5}$$

where $*_h$ is taken with respect to h. It then follows from the Jones and Tod construction that

$$g = Vh - V^{-1}(\mathrm{d}z + \alpha)^2 \tag{6}$$

is a $(++--)$ ASD metric with an isometry $K = \partial_z$. Using the relations

$$*_h\mathrm{d}t = \mathrm{d}t\wedge \mathrm{d}y, \qquad *_h\mathrm{d}y = 2\mathrm{d}t\wedge\mathrm{d}x, \qquad *_h\mathrm{d}x = \mathrm{d}y\wedge\mathrm{d}x + 2H_x\mathrm{d}y\wedge\mathrm{d}t$$

we verify that equation (2) is equivalent to $\mathrm{d} *_h (\mathrm{d} + \nu/2)W_x = 0$. Therefore

$$W_{xx}\mathrm{d}y\wedge\mathrm{d}x + (2(H_xW_x)_x - W_{tx})\mathrm{d}y\wedge\mathrm{d}t + 2W_{xy}\mathrm{d}t\wedge\mathrm{d}x = \mathrm{d}\alpha,$$

and we deduce that W_x is the general solution to the monopole equation (5) on the EW background given by (4). We choose a gauge in which $\alpha = Q\mathrm{d}y + P\mathrm{d}t$. This yields

$$Q_x = -W_{xx}, \qquad P_x = -2W_{xy}, \qquad P_y - Q_t = 2(H_xW_x)_x - W_{xt}, \tag{7}$$

so $Q = -W_x + A(y,t)$, $P = -2W_y + B(y,t)$ and $\alpha = -W_x\mathrm{d}y - 2W_y\mathrm{d}t + A\mathrm{d}y + B\mathrm{d}t$. The integrability conditions $P_{xy} = P_{yx}$ are given by (2), and $A_t = B_y$. Therefore there exists $C(y,t)$ such that $A = C_y, B = C_t$. We now replace z by $z - C$ and the metric (6) becomes (3). This proves that (3) is ASD. It is also scalar-flat, because, as a consequence of (2),

$$R = 8(W_{xyy} - W_{xxt} + (H_xW_x)_{xx})W_x = 0. \tag{8}$$

We now choose the null tetrad

$$e^{00'} = -2W_x\mathrm{d}t,\; e^{10'} = \frac{\mathrm{d}z - 2W_y\mathrm{d}t}{2W_x},\; e^{01'} = \mathrm{d}z - 2W_x\mathrm{d}y - 2W_y\mathrm{d}x + ze^{00'},\; e^{11'} = \mathrm{d}x + H_x\mathrm{d}y + ze^{10'},$$

such that $g = 2(e^{00'}e^{11'} - e^{10'}e^{01'})$. The basis of SD two forms $\Sigma^{A'B'}$ is given by

$$\begin{aligned}
\Sigma^{0'0'} &= \;= \iota_{A'}\iota_{B'}\Sigma^{A'B'} = e^{00'}\wedge e^{10'} = \mathrm{d}z\wedge\mathrm{d}t\\
\Sigma^{0'1'} &= \;= \iota_{A'}o_{B'}\Sigma^{A'B'} = e^{10'}\wedge e^{01'} - e^{00'}\wedge e^{11'} = \mathrm{d}t\wedge\mathrm{d}(z^2) + 2\mathrm{d}t\wedge\mathrm{d}W + \mathrm{d}y\wedge\mathrm{d}z.\\
\Sigma^{1'1'} &= \;= o_{A'}o_{B'}\Sigma^{A'B'} = e^{01'}\wedge e^{11'} = 2W_x\mathrm{d}x\wedge\mathrm{d}y + 2(zW_x + W_y)\mathrm{d}x\wedge\mathrm{d}t - \mathrm{d}x\wedge\mathrm{d}z\\
&\quad + (2H_xW_x - 2zW_y)\mathrm{d}t\wedge\mathrm{d}y + z\mathrm{d}z\wedge\mathrm{d}y + (H_x + z^2)\mathrm{d}z\wedge\mathrm{d}t.
\end{aligned}$$

[1] With definition $u = H_x$ the x derivative of equation (1) becomes $(u_t - uu_x)_x = u_{yy}$, which is the dispersionless Kadomtsev–Petviashvili equation originally used in [2]. There are some computational advantages in working with the 'potential' form (1).

These two-forms satisfy:

$$-2\Sigma^{0'0'}\wedge\Sigma^{1'1'}=\Sigma^{0'1'}\wedge\Sigma^{0'1'},\qquad \mathrm{d}\Sigma^{0'0'}=0,\qquad \mathrm{d}\Sigma^{0'1'}=0,$$

$$\mathrm{d}\Sigma^{1'1'}=\mathrm{d}(H_x-2W)\wedge\mathrm{d}t\wedge\mathrm{d}z+(W_{xt}-W_{yy}-(H_xW_x)_x)\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}t. \qquad (9)$$

Therefore the metric (3) admits a constant spinor[2] which is preserved by $K=\partial_z$.
Converse : Let g be a real analytic ASD metric with a covariantly constant spinor $\iota_{A'}$, which is Lie derived along a Killing vector K. From $C_{A'B'C'D'}=0$ it follows that there exist coordinates $\pi^{A'}$ on the fibers of $S^{A'}\to\mathcal{M}$ such that $\pi^{A'}\nabla_{AA'}\pi^{B'}=0$. Therefore a parallel section $\iota_{A'}$ of $S_{A'}$ determines a function $l=\pi^{A'}\iota_{A'}$ constant along the twistor distribution. The line bundle given by l on $\mathcal{PT}$ is isomorphic to $\kappa^{-1/4}$, where $\kappa=\Omega^3\mathcal{PT}$ is the canonical bundle. The Killing vector K gives rise to a holomorphic vector field on $\mathcal{PT}$ which preserves the divisor l . Therefore the minitwistor space $\mathcal{Z}$ (the space of trajectories of K in $\mathcal{PT}$) also admits a divisor with values in the $-1/4$ power of the canonical bundle. The minitwistor space $\mathcal{Z}$ satisfies the assumptions of Proposition 5.1 of Ref. [2] and the corresponding EW metric is of the form (4). Therefore $\hat{g}=\Omega^2 g$, where g is given by (3). Both $\hat{g}$ and g are scalar flat (this follows from the spinor Ricci identities and from equation (8) respectively). As a consequence we deduce that $\Omega=\Omega(t)$. Now we can use the coordinate freedom [2] to absorb Ω in the solution to the equation (1).

□

Remarks:

- This Proposition is analogous to a result of LeBrun [5] who constructs all scalar-flat Kähler metrics with symmetry in Euclidean signature from solutions to the $SU(\infty)$ Toda equation and its linearisation.
- If $H=const$ then (2) reduces to the wave equation in $2+1$ dimension, and consequently the metric (3) is the $(++--)$ Gibbons–Hawking solution [3].

---

[2]This is a consequence of the following result: *Let $\Sigma^{A'B'}$ be a basis of real SD two-forms on an ASD scalar-flat manifold such that*

$$\mathrm{d}(\iota_{A'}\iota_{B'}\Sigma^{A'B'})=\mathrm{d}(o_{A'}\iota_{B'}\Sigma^{A'B'})=0. \qquad (10)$$

Then there exists a covariantly constant real section of $S^{A'}$.
Proof. Let $(o_{A'},\iota_{A'})$ be a normalised spin basis. The covariant derivatives of the basis can be expressed as

$$\nabla_a\iota_{B'}=U_a\iota_{B'}+V_a o_{B'},\qquad \nabla_a o_{B'}=W_a\iota_{B'}-U_a o_{B'}.$$

The first condition in (10) can be rewritten as $\nabla_A{}^{A'}(\iota_{A'}\iota_{B'})=0$, which implies $V_a=2U_{AB'}\iota^{B'}\iota_{A'}$. The second condition in (10) yields $U_{AA'}=\alpha_A\iota_{A'}, W_{AA'}=\beta_A\iota_{A'}$ for some α_A,β_A. Therefore

$$\nabla_{AA'}\iota_{B'}=\alpha_A\iota_{A'}\iota_{B'}. \qquad (11)$$

Contracting the RHS of the above equation with $\nabla^A{}_{C'}$, and symmetrising over $(A'B')$ gives 0 because g is ASD and scalar-flat. As a consequence $\nabla^A{}_{(C'}[\iota_{A')}\iota_{B'}\alpha_A]=0$ which gives $\nabla^A{}_{A'}\alpha_A=0$. Consider the real spinor $\hat{\iota}_{A'}:=\iota_{A'}\exp f$, where $\iota^{A'}\nabla_{AA'}f=0$ in order to preserve $\mathrm{d}\Sigma^{0'0'}=0$. Integrability conditions for $\nabla_{AA'}f=\alpha_A\iota_{A'}$ are satisfied, as α_A solves the neutrino equation. Therefore we can find f for each α_A, and equation (11) implies that $\hat{\iota}_{A'}$ is covariantly constant.

□

- Note that $d\Sigma^{1'1'} \neq 0$ unless $W = H_x/2 + f(t)$, in which case

$$d\Sigma^{1'1'} = d(H_{xt} - H_x H_{xx} - H_{yy}) \wedge dy \wedge dt = 0,$$

and we are working in a covariantly constant real spin frame. The metric

$$g = \frac{H_{xx}}{2}(dy^2 - 4dxdt - 4H_x dt^2) - \frac{2}{H_{xx}}(dz - \frac{H_{xx}dy}{2} - H_{xy}dt)^2 \quad (12)$$

is therefore pseudo hyper-Kähler. In [2] it was shown that all pseudo hyper-Kähler metrics with a symmetry satisfying $dK_+ \wedge dK_+ = 0$ are locally given by (12). Here dK_+ is an SD part of dK.

In the split signature we can arrange for one of the complex structures to be real and for the other two to be purely imaginary:

$$-I^2 = S^2 = T^2 = 1, \qquad IST = 1,$$

and S and T determine a pair of transverse null foliations. Now $g(TX, TY) = g(SX, SY) = -g(X, Y)$ for any pair of real vectors X, Y. The endomorphism I endows $\mathcal{M}$ with the structure of a two-dimensional complex Kähler manifold, as does every other complex structure $aI + bS + cT$ parametrised by the points of the hyperboloid $a^2 - b^2 - c^2 = 1$.

- If $W_x \neq H_{xx}/2$ then (3) is not Ricci–flat. This can be verified by a direct calculation. It also follows from more geometric reasoning: The Killing vector $K = \partial_z$ acts on SD two-forms by a Lie derivative. One can choose a basis $\Sigma^{A'B'}$ such that one element of this basis is fixed, and the Killing vector rotates the other two. The components of the SD derivative of K are coefficients of these rotations. Therefore $(dK)_+ = const$ if g is pseudo hyper-Kähler. In our case $dK_+ = (H_{xx}/W_x)dz \wedge dt$. Therefore H_{xx}/W_x must be constant for (3) to be Ricci-flat. An example of a non-vacuum metric is given by $W = H_y/2$.

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Chapter 1: The nonlinear graviton and related constructions

Chapter 2: Spaces of complex null geodesics

Chapter 3: Hypersurface twistors and Cauchy-Riemann Structures

Chapter 4: Towards a twistor description of a general space-times.

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