

Twistor Newsletter

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In response to various requests from ex-students and colleagues, etc., it is proposed to send out brief accounts (non-periodically) of work on twistor theory and topics which is being carried out at the Mathematical Institute, Oxford. This is not intended in any way to be a substitution for publication of results, but just a means of letting our friends know what has been going on here in twistor theory. Most of the work described here is in the process of being written up for publication. Published work (or in completed theses) will not normally be included in TNT. Little concession to readability is being made here, simply because we just don't care. There is lots of work, rather than nothing at all. And diagrammatic notation for twistors, spinors, etc. will be used freely without apology (e.g. $H = H^+$, $HH = H^+H^- + H^-H^+$).

Notes on contents of TNT 1:

1. Twistor description of low-lying baryon states (R.P.H.), p.①. This shows how the Gell-Mann - Ne'eman hadron classification can be incorporated into the twistor scheme. The description of baryon resonances, however, has led to problems. One way out (suggested by L.P.H.) leads to a modification of the twistor programme. Another, (suggested by G.A.J.S.) leads to a modification of quantum field theory. Neither is described here, since their authors have chosen to stick to less controversial topics!
2. The twisted photon (R.W.), p.②. This does for the photon what the "non-linear graviton" (see 2 articles by R.P. in 1st two 1976 issues of Gen. Rel. Grav.) did for the graviton.
3. Zero-rest-mass fields from twistor functions (R.W.), p.③. Twistor ϕ 's without twists!
4. The Twistor Quadrille (G.A.J.S. et al.), p.④. Shows how the construction 2. above (twisted photon) forces electric charge to come in fixed multipoles of one basic unit. This is related to the famous Dirac argument (which uses magnetic monopoles) but is not quite the same. It should be pointed out that the patching $Z = e^2 z$ gives rise to the standard twistor description $\Psi_{AB} = \frac{1}{(2\pi i)^2} \int d^2z \int d^2\bar{z} f(z) \bar{f}(\bar{z}) d^4\pi$, for zero homogeneity $f(z)$.
5. The non-linear graviton representing the analogue of Schwarzschild or Kerr black holes (G.A.J.S.), p.⑤. An illustration of the general twistor construction. This solution had been previously found by the Pittsburgh group and also by Plebanski. It should be mentioned that the weak field limit of the non-linear graviton is given by a patching relation $Z = Z + e^2 f$ where $f = f(k)$ is hom. deg. +2. ($+L$ is $I^{(2)}$). G.A.J.S. has shown that the curvature of the resulting "univ." agrees with $\Psi_{ABCD} = \frac{1}{(2\pi i)^4} \int d^4z d^4\bar{z} f(z) \bar{f}(\bar{z})$. G.A.J.S. then exponentiates this infinitesimal form to get the general patching $Z = e^{\int f(z) dz}$.
6. Local metric properties of H-space (R.H. et al.), p.⑥. The other way of doing H. The curvature of H-space explicitly obtained. It would be nice to relate this to comments concerning 5.
7. The universal bracket factor (R.P.), p.⑦. Can anyone make this work rigorously?
8. Some elementary twistor integrals with boundary (R.P.), p.⑧. May not all be familiar.
9. Projected lepton cube (R.P. et al.), p.⑨. New leptons of spin $\frac{1}{2}$ might create a problem!
10. The email counter (R.P. et al.), p.⑩. Yes, it's really needed!

1. TWISTOR DESCRIPTION OF LOW-LYING BARYON STATES

D. INTRO.

Particle states, according to the current twistor view (cf. Penrose 1975 a), are represented by holomorphic functions of one or more twistors. Such a function will presumably fit 'intrinsically' into the structure of some sort of curved twistor space (as in, e.g., 'the nonlinear graviton' and 'the twisted photon') but just how that comes about in general (if it is to!) is presently an unsolved problem. The holomorphic functions generally represent the incoming and outgoing states in various types of particle processes. These functions can be 'simple', in order to represent the in and out states in elementary processes such as elastic scatterings, hadron charge-exchange reactions, or neutron beta-decay. Or they can be 'compound', in order to represent the in and out states in many-particle processes involving bound states such as electron capture into an atomic orbital, $2\text{H} \rightarrow \text{H}_2$, or $2\text{H}^2 \rightarrow \text{He}^+$, etc. The amplitudes for these processes are nowadays thought to be represented by contour integral formulas. The differential form to be integrated is in each case an expression which involves a product of all the in and out particle states (i.e. the holomorphic functions) and a certain 'kernel' which represents the interaction. The general rules which would state what the kernel is and how the contour is to be chosen are still largely unknown, though promising and substantial developments have been made in studies of a number of 'prototype' interactions (cf. MacCallum & Penrose, 1973, section 4; Penrose, 1975 b, section 4; Sparling 1974; Sparling 1975, section 3; Hodges 1975; Ryman 1975; Harris 1978). [Clearly, more mathematical techniques and 'physical/geometrical insight' (whatever that is) is needed in this area!] Some preliminary investigations (e.g. MacCallum & Penrose, 1973, p. 292-297) seem to suggest that Regge-pole theory (in the broad sense of the theory) has much to say about the composition of twistor amplitudes. The implications of analyticity and crossing relations, etc., have just barely begun to be explored in the context of twistor theory, and it appears that this will be a most fruitful domain of investigation.

Working at a more primitive level, some progress has been made during the last two years in the representation of leptons and hadrons (independently of the details of their interactions) in twistor terms. These developments have largely been the next logical steps up from the discovery of 'twistor internal symmetry groups' (cf. Penrose, 1975a; Penrose, 1975b, p. 321 - 330; Parjas, 1975). Leptons, apparently, are to be described generally by holomorphic functions of two twistor-type variables, and hadrons by holomorphic functions of three twistor-type variables. The set-up for the two lowest-lying baryon multiplets — the $N(430)$ octet and the $\Delta(1232)$ decimat — turns out to be particularly straightforward, and this is the main topic which will be reviewed here. There are many interconnections between the twistor description of hadrons and the standard phenomenological quark model (for a comprehensive and surprisingly still-up-to-date account of this model, see Delita 1966), and it is quite likely that there is much yet to be gleaned from the study of the 'standard' quark model in its various manifestations (in this connection, also see Feynman et al 1970, Feynman et al 1971; and especially Feynman, 1972, which is a real galaxy of facts and ideas).

2) QUARKS

According to non-standard lore (due, of course, to M. Gell-Mann) there are three kinds of quarks, which are called up, down, and 'sideways' quarks. Each of these quarks has spin $1/2$, so they are represented (relativistically) by a triplet of spinors: u^A, d^A, s^A . The quarks are distinguished according to the various charge-values they carry, and these are listed in the following chart:

	u^A	d^A	s^A
electric charge	$+2/3$	$-1/3$	$-1/3$
hypercharge	$1/3$	$1/3$	$-1/3$
baryon number	$1/3$	$1/3$	$1/3$
total isospin	$1/2$	$1/2$	0

(3)

Since each quark has baryon number $1/3$, in order to manufacture a complete baryon what is required is a state made out of three quarks. There are ten different three-quark combinations of three kinds of quarks. These are

$$d^A d^B d^C \quad u^A d^B d^C \quad u^A u^B d^C \quad u^A u^B u^C$$

$$d^A d^B s^C \quad u^A d^B s^C \quad u^A u^B s^C$$

$$d^A s^B s^C \quad u^A s^B s^C$$

$$s^A s^B s^C$$

Now these combinations aren't all in definite spin-states. The ones on the corners of the triangle are symmetric alright, and represent states of pure spin $3/2$, but the others have skew parts and therefore are mixed states of spin $3/2$ and spin $1/2$. Splitting things into spin eigenstates, two distinct multiplets are obtained:

$$d^{(A} d^B d^{C)} \quad u^{(A} d^B d^{C)} \quad u^{(A} u^B d^{C)} \quad u^{(A} u^B u^{C)}$$

$$d^{(A} d^B s^{C)} \quad u^{(A} d^B s^{C)} \quad u^{(A} u^B s^{C)}$$

$$d^{(A} s^B s^{C)} \quad u^{(A} s^B s^{C)}$$

$$s^{(A} s^B s^{C)}$$

decoupled

$$d^A d_B u^C \quad u^A u_B d^C$$

$$u^A d^B s_C$$

$$d^A d_B s^C$$

$$u^A u_B s^C$$

$$u_B d^B s^A$$

$$s^A s_B d^C$$

$$s^A s_B u^C$$

octet

Note that in the 'octet' multiplet two states are obtained in the middle because there are two linearly independent ways of contracting $u^A d^B s^C$ down to spin $1/2$. One of them is symmetric under $u^A \leftrightarrow d^A$

and so has total isospin $\frac{1}{2}$, while the other is skew and/or has total isospin $\frac{3}{2}$. Using the chart given above the charge, hypercharge, and total isospin of each of these states can be checked, and these of course correspond to the quantum numbers of the two lowest-mass baryon multiplets:

$$\Delta^+ \Delta^0 \Delta^* \Delta^{*+} \dots (1232 \text{ MeV.})$$

$$\Xi^+ \Xi^0 \Xi^* \Xi^{*+} \dots (1385) \quad \frac{3}{2}^+ \Delta(1232)$$

$$\Xi^- \Xi^0 \Xi^* \Xi^{*+} \dots (1530) \quad \text{deciat}$$

$$\Omega^+ \Omega^0 \Omega^* \Omega^{*+} \dots (1672)$$

$$N^0(940) \quad N^+(938)$$

$$\Xi^0(1192)$$

$$\Xi^-(1197) \quad \Xi^*(1189) \quad \frac{1}{2}^+ N(940) \text{ octet}$$

$$\Lambda^0(1115)$$

$$\Xi^0(1321) \quad \Xi^0(1315)$$

3) TWISTOR MODEL

At any given spacetime point x with each quark state a unique twistor can be associated. With the quark u^μ there is the twistor $U_\mu = (u_\mu, -x^\lambda u^\lambda)$, and so on for the other quarks. As we vary the quark state, the twistor varies, and also as we vary the spacetime point the twistor varies. In this way quarks are 'twizzled'.

As mentioned earlier, a baryon state is described in twistor terms by a function of three variables. So for a low-lying baryon we take an arbitrary holomorphic function of the three twistors U_1, U_2, S_3 . The spacetime field corresponding to a particular baryon eigenstate can be projected out by taking a contour integral. The integration is carried out in the 'quark space', and leaves behind a net spacetime field. As a coefficient in the integrand we use one of the baryonic three-quark configurations given above. For a proton, for example, we use

the coefficient $u^a u_d s^b$. All three twistors are put through the same spacetime point x , and you form the contour integral

$$P_\alpha(x) = \frac{1}{(2\pi i)^6} \oint u^a u_d s^b f(u_a, d_u, s_a) \delta g,$$

where δg is the invariant measure on the quark space. An explicit twistor measure δg is given by

$$\frac{1}{2^3} I^{u_1} I^{d_1} I^{s_1} dU_{u_1} dU_{d_1} dD_{s_1} dS_{u_1} dS_{d_1} dS_3,$$

or, more simply, we can think of integrating on a contour over the components of u_a , d_u , and s_a , just leaving behind a net spacetime dependence.

A set of operators can be constructed which will tell you what the quantum numbers of the function are directly without necessarily having to make reference to the formulation of the associated spacetime field. This can be seen most clearly in the case of the charge-like quantum numbers. In order for an integral of the type given above to give a non-trivial value, the total homogeneity of the integrand in any one twistor must be zero (cf. Penrose, 1975 b, p. 337-339). Consider now the twistor u_a . Since δg is homogeneous of degree 2 in u_a , if a total of n quark coefficients is desired in the integrand then f must be homogeneous of degree $-n-2$. This means that the operator which tells you how many u^a type quarks there are is minus the homogeneity operator minus two. Write \tilde{U}^a for the operator D/Du_a . The Euler homogeneity operator then in $u_a \tilde{U}^a$. So, generally now, we can write

$$u = -u_a \tilde{U}^a - 2$$

$$d = -D_u \tilde{D}^u - 2$$

$$s = -S_u \tilde{S}^u - 2$$

for the three 'quark occupation number' operators, and from these you easily obtain:

$$\text{electric charge} = \frac{2}{3}u - \frac{1}{3}d - \frac{1}{3}s$$

$$\text{hypercharge} = \frac{1}{3}u + \frac{1}{3}d - \frac{1}{3}s$$

$$\text{baryon number} = \frac{1}{6}u + \frac{1}{6}d + \frac{1}{6}s$$

The spin, isospin, and mass operators can also be expressed directly in twistor terms. The spin is a difficult one, and it was first obtained by Sparling. The isospin operator is

$$I^2 = (J_1)^2 + (J_2)^2 + (J_3)^2$$

$$J_1 := \pm(D_u \bar{U}^* - U_u \bar{D}^*), \quad J_2 := \pm(D_u \bar{U}^* + U_u \bar{D}^*), \quad J_3 := \pm(U_u \bar{U}^* + D_u \bar{D}^*)$$

And the mass operator is given by

$$\frac{1}{2}M^2 = I^{a\mu} U_a D_\mu I_{\nu\delta} \bar{U}^\nu \bar{D}^\delta + I^{a\mu} D_a S_\mu I_{\nu\delta} \bar{D}^\nu \bar{S}^\delta + I^{a\mu} S_a U_\mu I_{\nu\delta} \bar{S}^\nu \bar{U}^\delta.$$

It can be left as an 'exercise' to demonstrate that if the holomorphic function is in an eigenstate of M^2 , then the associated spacetime field satisfies automatically the corresponding massive wave equation.

- Lane Hughston

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2. The Twisted Photon.

(2)

The complex structure of (flat) projective twistor space \mathbb{PT} is completely determined by the conformal structure of Minkowski space. By deforming the non-projective space \mathbb{T} while preserving \mathbb{PT} , one can code information about zero-rest-mass fields into the global structure of \mathbb{T} .

The General Case. Let \mathcal{B} denote the prised spin-bundle (with coords (x^A, τ_A)) over a neighbourhood M in Minkowski space, and let T be the Euler operator $\pi_{\alpha} \cdot \partial/\partial \pi_{\alpha}$. The elements of \mathbb{T} can be regarded as integral surfaces of the 2-dim. distribution on \mathcal{B} determined by the two vector fields $\{\pi^A \nabla_{\alpha A}, \pi^A \nabla_{\beta A}\}$. [These integral surfaces project down to totally null 2-planes in M .] To twist up \mathbb{T} without changing \mathbb{PT} , replace $\{\pi^A \nabla_{\alpha A}\}$ by $\{\pi^A \nabla_{\alpha A} - \psi_{\alpha} T\}$, where ψ_{α} represents a pair of functions on \mathcal{B} . Applying Frobenius' theorem, one finds that the new distribution is integrable if and only if

$$\pi^A \nabla_{\alpha A} \psi^{\alpha} - \psi_{\alpha} (T \psi^{\alpha}) = 0. \quad (1)$$

So if (1) is satisfied, then there exists a 4-dim. quotient space \mathcal{J} , which has the structure of a fibre bundle over \mathbb{PT} . Now let ψ_{α} have the form $\psi_{\alpha}(x, \tau) = (\Phi_{\alpha}^{A_1 \dots A_p}(x) \tau_{A_1} \dots \tau_{A_p})$. Then (1) $\Leftrightarrow \nabla^A (\partial^{\alpha} \Phi_{\alpha}^{A_1 \dots A_p}) = 0$. (2)

In other words, $\Phi_{\alpha}^{A_1 \dots A_p}$ is a "right-flat" Hertz potential for a zero-rest-mass field given by $\Phi_{AB \dots P} := \nabla_{A_1} \dots \nabla_{A_p} \Phi_{\alpha}^{A_1 \dots A_p}$. [Equation (2) implies that $\nabla^A \Phi_{AB \dots P} = 0$.]

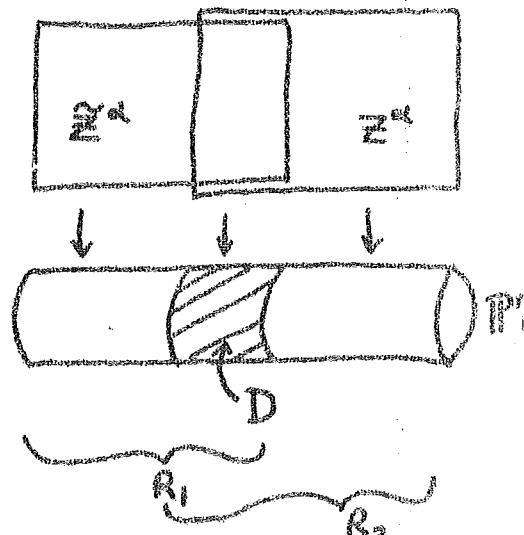
The Photon Case. Take $\psi_{\alpha} = i \Phi_{\alpha}^{A_1}(x) \tau_{A_1}$, with $\nabla^A (\partial^{\alpha} \Phi_{\alpha}^{A_1}) = 0$. (3)

The flat space \mathbb{T} is a principal fibre bundle over \mathbb{PT} with group \mathbb{C}^* , and the deformation given by (3) will preserve this particular kind of bundle structure, whereas in the general case the fibres ~~will~~ have a different structure. In this sense, the construction applies more naturally to photons than to other zero-rest-mass particles.

George Sparling has shown how to build the photon bundle in a more twistorial way, as follows. Think of \mathbb{PT} as the neighbourhood of a line in \mathbb{CP}^3 and let $f(z^a)$ be homog. of degree 0 and holomorphic on an annular region D . Take two pieces of flat twistor space (with coords. z^a and \bar{z}^a respectively) and "patch" them over the region D by

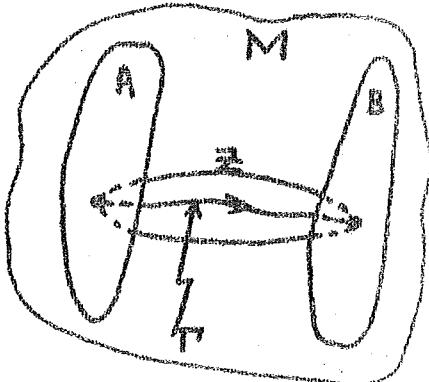
$$z^a = e^{if(z)} \bar{z}^a.$$

This patching builds a 4-dim. manifold \mathcal{J} , and \mathcal{J} has the structure of a principal fibre bundle with group \mathbb{C}^* , as required.



We now show how the construction using $\Phi_A^{A'}(x)$ is related to the one using $f(Z)$, by deriving $\Phi_A^{A'}$ from f and vice versa. So suppose we are given $f(Z)$. For fixed x , the function $f(ix^{AA'}\pi_{A'}, \pi_B)$ is effectively a function of one complex variable and can be "split" as follows: $f(ix^{AA'}\pi_{A'}, \pi_B) = f_2(x^A, \pi_{A'}) - f_1(x^A, \pi_{B'})$, (4) where f_i is holomorphic on the regions R_i , $i = 1, 2$. Now $0 = \pi^{A'} \nabla_{AA'} f = \pi^{A'} \nabla_{AA'} f_2 - \pi^{A'} \nabla_{BB'} f_1$ so $\pi^{A'} \nabla_{BB'} f_2 = \pi^{A'} \nabla_{AA'} f_1$. The left-hand side of this equation is holomorphic on R_2 and the right-hand side on R_1 , so both sides are globally holomorphic, and consequently linear in $\pi^{A'}$. Thus we can define a vector $\bar{\Phi}_a(x)$ by $\pi^{A'} \nabla_{AA'} f_1(x, \pi) = +\bar{\Phi}_{AA'}(x) \pi^{A'}$. There is some freedom in the decomposition (4), namely $f_2 \mapsto f_2 + \lambda(x)$, $f_1 \mapsto f_1 + \lambda(x)$. This leads to the gauge freedom $\bar{\Phi}_a(x) \mapsto \bar{\Phi}_a(x) + \nabla_a \lambda(x)$.

Conversely, suppose we're given $\bar{\Phi}_a(x)$ satisfying (3). Pick two twistors $A^a \in R_1$ and $B^a \in R_2$. If $Z^{A'}$ is a third twistor whose corresponding null 2-plane meets the null 2-planes A and B in M , then put $f(Z^{A'}) = \int_{T'} \bar{\Phi}_a dx^a$, where T' is some path in Z from A to B (see figure). Using Stokes' theorem and equation (3), one can show that the integral is independent of the particular path T' chosen. The freedom in the choice of A and B corresponds to the freedom of being able to add into $f(Z)$ a twistor function which is holomorphic all over R_1 or all over R_2 .



There is a third way of visualizing the twisting-up procedure. A twistor determines a null 2-plane, together with a $\pi_{B'}$ -spinor propagated over the 2-plane. The usual propagation law is the parallel one, i.e. $\pi^{A'} \nabla_{AA'} \pi_{B'} = 0$. But in the twisted photon case we take the propagation equation to be

$$\pi^{A'} (\nabla_{AA'} + i \bar{\Phi}_{AA'}) \pi_{B'} = 0.$$

The integrability condition for this equation is precisely (3).

Finally we note that it is not necessary for M to be flat: the construction generalizes without too much trouble to right-flat space-times.

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3. Zero-Rest-Mass Fields from Twistor Functions.

There is a way of deriving fields from twistor functions without explicitly using contour integration. It is based on a 'splitting' formula due to George Sparling: if $g(\pi_A)$ is of degree -1, with singularities in two disconnected regions S_1 and S_2 , then $g = g_1 + g_2$, where $g_i(\pi) := \frac{i}{2\pi i} \oint_{\Gamma_i} \phi(\rho_B, \pi^B)^{-1} g(\rho_B) \rho_B d\rho^B$, $i = 1, 2$.

Note that g_1 is holomorphic on the complement of S_2 (i.e. over the 'left-hand side' of π -space) and g_2 over the other side.

Suppose now that $F(z^A)$ is of degree $n > -2$, and put

$$\Psi_{A\dots p'}(x^c, \pi) := \underbrace{\pi_{A'} \dots \pi_p}_{\Gamma} F(i x^{cc'} \pi_{c'}, \pi_{c'}) .$$

This spinor field $\Psi_{A\dots p'}$ has degree -1 in π_B , and so, for fixed x^c ,

$$\text{it can be split: } \Psi_{A\dots p'}(x, \pi) = \Psi_{A\dots p'}(x, \pi) - \tilde{\Psi}_{A\dots p'}(x, \pi). \quad (1)$$

(In fact, ψ and $\tilde{\psi}$ depend holomorphically on x .) Define a spinor field $\Phi_{A'\dots p'}$ by

$$\Phi_{A'\dots p'} := \pi^{A'} \tilde{\Psi}_{A''\dots p'} .$$

Claim: this field $\Phi_{A'\dots p'}$ is a zero-rest-mass free field on spacetime. There are two ways of seeing this; the first is to use the splitting formula to verify that $\Phi_{A'\dots p'}$ is the 'usual' field: $\Phi_{A'\dots p'} = \pi^{A'} \left\{ \frac{i}{2\pi i} \oint_{\Gamma} \phi(\rho_{A'}, \pi^B)^{-1} \rho_{A'} \rho_{A''} \dots \rho_p F(i x^{cc'} \rho_{c'}, \rho_{c'}) \rho_B d\rho^B \right\}$

$$= \frac{i}{2\pi i} \oint \rho_{A'} \dots \rho_p F(i x^{cc'} \rho_{c'}, \rho_{c'}) \rho_B d\rho^B .$$

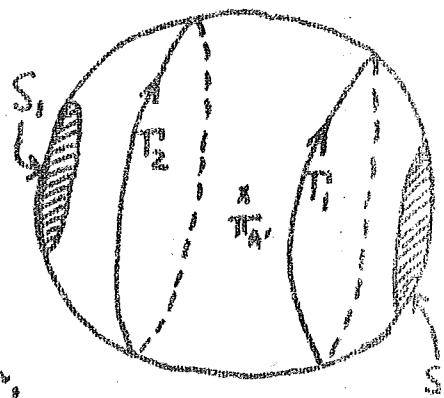
The second way is to use 'globality' arguments. For example, transvecting (1) with $\pi^{A'}$ gives $\tilde{\Psi}_{A\dots p'} \pi^{A'} = \tilde{\Psi}_{A\dots p'} \pi^{A'}$. The left-hand side of this equation can only be singular in S_2 , and the right-hand side only in S_1 , so both sides must be globally holomorphic. Being homogeneous of degree zero, they must therefore be independent of $\pi^{A'}$, i.e. functions of x only. Thus $\tilde{\Psi}_{A\dots p'}$ is a function of x only. By similar arguments one can deduce that $\Phi_{A'\dots p'}$ is symmetric and satisfies the field equations.

Finally, the procedure can be generalized to functions $F(z^A)$ of degree $n > -2$,

as follows. Put $\Psi_{A\dots p'}(x, \pi) := \pi_{A'} \underbrace{\frac{\partial}{\partial \omega^A} \dots \frac{\partial}{\partial \omega^p}}_{n+2} F(i x^{cc'} \pi_{c'}, \pi_{c'})$.

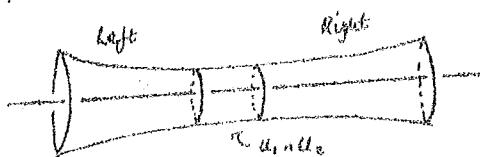
Now split ψ into $\psi - \tilde{\psi}$ and define $\Phi_{A\dots p'} := \pi^{A'} \tilde{\Psi}_{A''\dots p'}$.

Then, as before, we can either use globality to show that $\Phi_{A\dots p'}$ is a zero-rest-mass field, or we can check that $\Phi_{A\dots p'} = \frac{i}{2\pi i} \oint \frac{\partial}{\partial \omega^A} \dots \frac{\partial}{\partial \omega^p} F(i x^{cc'} \pi_{c'}, \pi_{c'}) \pi_B d\pi^B$.



The Twistor Quadrille

The aim of this article is to show explicitly how to construct a twistor space which encapsulates the global structure of a Coulomb Maxwell field. The construction dovetails neatly with an analogous construction for the Schwarzschild-like graviton, which will be given later. We carry out the usual construction for a line bundle over part of projective twistor space incorporating the Coulomb field in any sufficiently small four-dimensional neighbourhood, of topology \mathbb{C}^4 , of complexified Minkowski space, on which the field is non-singular. We then try to extend this line bundle to handle the Coulomb field in a complex four dimensional neighbourhood of a two-dimensional real sphere, surrounding the source singularity. We see that our construction breaks down unless a certain integrality condition holds, reflecting the condition that charge should be integral in a twistorial framework. The construction goes as follows. We first recall the building of the line bundle over that portion of projective twistor space which contains all the projective twistors which pass through points belonging to a neighbourhood of \mathbb{C}^4 topology in Minkowski space. This portion may be regarded as a certain neighbourhood U , say, of a line in projective twistor space:-



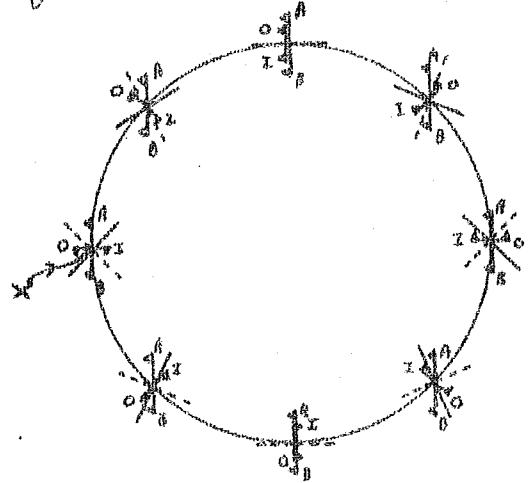
As usual we split the neighbourhood into two overlapping regions: U_1 ; the 'left' with projective co-ordinates $\{\bar{z}_1\}$ and U_2 , the 'right', with projective co-ordinates $\{\bar{z}_2\}$ such that the overlap region $(U_1 \cap U_2)$ contains an annular region of each compact projective line in U . We also use co-ordinates z_i for that part of $\mathbb{C}^4 - \{0\}$ which projects down to U_i under the canonical map $\mathbb{C}^4 - \{0\} \rightarrow \mathbb{CP}^3$, $i = 1, 2$. We may assume that the projective co-ordinates $\{\bar{z}_1\}$ for a point on $U_1 \cap U_2$ are the same as $\{\bar{z}_2\}$. Then if we identify z_1 with z_2 on the overlap region we would obtain just the usual non-projective twistors which project to U , as a principal bundle of a line bundle over U . However here we identify the point with co-ordinates z in the U_2 chart with the point with co-ordinates $e^{f(z)} z$ in the U_1 chart, (i.e. $z_1 = z$ corresponds to $z_2 = e^{f(z)} z$), where we take

$$f_0(z) = \log\left(\frac{z_2}{z_1 z_0}\right) \quad \text{and} \quad f \text{ is hermitian. So our overlap region}$$

has to be such that it avoids the singularities $z_2 = 0$, $z_1 = 0$ or $z_0 = 0$, appropriately. Our choice of $f_0(z)$ then guarantees that we have encapsulated the Coulomb

field in the appropriate region of Minkowski space, if, say, we take \mathbf{f}^{μ} to be spinless. Otherwise we move our origin into the complex until \mathbf{f}^{μ} becomes effectively spinless.

To study the overlap region more carefully we picture the singularities of the twistor function in Minkowski space. We may restrict our attention to a real two sphere surrounding the charge, as the picture will be essentially unchanged in a sufficiently small complex neighbourhood of that sphere. We may assume that the sphere lies in a $t = \text{constant}$ plane with centre on the charge worldline in a frame in which the charge is at rest.



At each point x^{AB} of the two spheres there are (generically) four spinors n_{AB} , up to proportionality such that if $\hat{z} = (x^{AB}n_{AB}, n_{AB})$ then $f_0(\hat{z})$ is singular. We project the null vectors n_{AB} into the surface $t = \text{constant}$ and obtain the above picture, the flag-tipped lines representing the singular directions at each point. The vectors A and B at each point correspond to the singularities $\sum h = 0$ and $\sum b$ respectively. The vectors O and I correspond to the singularities $\sum z = 0$, -o pointing outward and z pointing inward to the centre. For simplicity and with no real loss of generality we may assume A and B point exactly oppositely and that A coincides with O at the north pole (and B with I, there). Then the whole picture is rotationally symmetric about the north-south axis. We portray the overlap region at various points of the sphere by drawing a line/ to represent the orientation, in the sphere of directions at each point, of the central great circle of the overlap region, for the sphere's worth of twistors that represent the given point in twistor space. (In this construction the overlap region may always be taken to consist of an annular neighbourhood of a great circle of the sphere of directions, for each point). At the north pole the annular region may be taken to be equatorial, —, as indicated. As we move away from the north pole O moves away from A and the annular region gradually has to be up to allow for this. Eventually when we reach the south pole, the region becomes

punctured. We may then define our region U_1 by saying that for each point on a partial sphere of shape  (a neighbourhood of the south pole is deleted), we have a hemi-sphere of directions for the twistors through those points, the hemi-sphere (actually it must be strictly larger than a hemi-sphere!) being tipped up like this  at the point X on the equator (given the tipping up at X we may work it out everywhere else by continuity).

So for shorthand U_1 is denoted by



Similarly U_2 consists of



Then the transition function $c_{12}(z) = e^{f_1(z)} \cdot \frac{z_1}{z_2}$ is to be used on the overlap region $U_1 \cap U_2$ which consists of



This finishes the construction as far as the unitless hemi-sphere is concerned and produces a twistor space carrying the structure of the Coulomb field in a small neighbourhood of that hemi-sphere. However we could equally well start from the south pole, using regions, indicated with a dotted line above,

$U_3 = \circlearrowleft$ points \circlearrowright allowed directions at X , co-ordinates $\frac{z_1}{z_2}$

$U_4 = \circlearrowright$ points \circlearrowleft allowed directions at X , co-ordinates $\frac{z_1}{z_2}$

and identifying $z_4 = \frac{z_1}{z_2}$ in U_4 with $z_3 = c_{34}(z)$ \neq in U_3 where $c_{34}(z) = \frac{z_1}{z_2}$

on the overlap region $U_{34} = U_1 \cap U_2$. Now can we patch this together to give a line bundle corresponding to a neighbourhood of the whole two-sphere? We can do so if we can find transition functions $c_{ij}(z)$ defined on $U_{ij} = U_i \cap U_j$ such that $c_{ij}c_{jk} = c_{ik}$ on $U_i \cap U_j \cap U_k = U_{ijk}$, $\forall i, j, k$ (with $c_{ii} = 1$, $c_{ij} = c_{ji}$), identifying $z_j = z$ in U_j with $z_i = c_{ij}(z)$ in U_i .

We list all the various intersections as follows:-

$U_{12} =$		\oplus		$U_{123} =$	
$U_{13} =$		\oplus		$U_{124} =$	
$U_{14} =$		\oplus		$U_{134} =$	
$U_{23} =$		\oplus		$U_{234} =$	
$U_{24} =$		\oplus		$U_{1234} =$	
$U_{34} =$		\oplus			

Now consider, for instance the relation $C_{12} = C_{14} C_{42}$ or U_{124} , so

$\frac{\partial}{\partial h} = C_{14} C_{42}$. Now U_{14} contains the direction $\frac{\partial}{\partial h} = 0$, so $C_{14}(\frac{\partial}{\partial h})$ cannot be singular there. However U_{14} does not contain directions where $\frac{\partial}{\partial h} = 0$ nor $\frac{\partial}{\partial k} = 0$. Also U_{24} contains a direction where $\frac{\partial}{\partial k} = 0$ but not where $\frac{\partial}{\partial h} = 0$ nor $\frac{\partial}{\partial l} = 0$. So we may solve for

$$C_{14}(\frac{\partial}{\partial h}) = \frac{\frac{\partial}{\partial h}}{(\frac{\partial}{\partial k})^2} \quad \text{and} \quad C_{42} = \frac{\frac{\partial}{\partial k}}{\frac{\partial}{\partial l}^2}$$

Next consider $C_{12} \rightarrow C_{13} C_{32}$ or U_{13} . By similar analysis we solve by

$$C_{32} = \frac{\frac{\partial}{\partial k}}{(\frac{\partial}{\partial A})^2}, \quad C_{13} = \frac{\frac{\partial}{\partial k}}{\frac{\partial}{\partial B}}. \quad \text{Then we have } C_{32} C_{24} = C_{34} \text{ or } U_{234}$$

and $C_{31} C_{46} = C_{34}$ or U_{134} , so we have a complete solution.

The four singularities are just right that they can dance out of trouble. The solution is unique up to multiplication or division as appropriate by a single overall constant. To see the integrality of charge showing up, we see the effect of replacing $f_0(z)$ by $\lambda f_0(z)$ some $\lambda \neq 0$, $\lambda \in \mathbb{C}$. Then $C_{ij}(z) = -f_0(z))^\lambda$ and again this is the only possible solution up to an overall constant. But then $(C_{13})^\lambda$ is not single-valued on U_{13} nor is $(C_{34})^\lambda$ on U_{24} , unless λ is integral. So the construction fails to work globally around the two-sphere unless λ is integral (although we do not prove that no other construction will work, this will follow from a more sheaf-theoretic discussion).

This is joint work of Roger Penrose and George Sparling based on Richard Ward's construction.

5. The Non-linear graviton representing the analogue of Schwarzschild or Kerr black hole

We use the Hamiltonian method to define the Heavens. We have, to start with, two co-ordinate patches with co-ordinates \hat{z}_1 on the 'left' and \hat{z}_2 on the right. We identify the point \hat{z} in the \hat{z}_1 patch, U_1 , with the point \hat{z}' in the \hat{z}_2 patch, U_2 , obtained from \hat{z} by moving a distance λ along the Hamiltonian vector field defined by the function $f_a(\hat{z})$ via the Poisson bracket structure for the non-linear graviton ($\frac{\delta}{\delta z} \frac{\delta}{\delta p}$). To get the 'Schwarzschild' or 'Kerr' Heavens, we try as an ansatz

$$f_a(\hat{z}) = \hat{z} \frac{\partial}{\partial \hat{z}} \log \left(\frac{\hat{z} \hat{P}}{\hat{z} \hat{B}} \right) - \frac{\hat{z} \hat{P}}{\hat{z} \hat{B}}, \text{ which guarantees the correct linearized limit for a suitable patching region.}$$

$$\text{Then } \hat{z}' = \hat{z} + \lambda \frac{\partial}{\partial \hat{z}} \log \left(\frac{\hat{z} \hat{P}}{\hat{z} \hat{B}} \right).$$

Under $\hat{z} \rightarrow \hat{z} + \lambda \frac{\partial}{\partial \hat{z}}$, $\hat{z}' \rightarrow \hat{z}' + \mu \frac{\partial}{\partial \hat{z}'}$, so we may identify \hat{z}_1 with $\hat{z}_1 + 2\pi i \lambda \frac{\partial}{\partial \hat{z}_1}$, \hat{z}_2 with $\hat{z}_2 + 2\pi i \lambda \frac{\partial}{\partial \hat{z}_2}$ and ignore henceforth the fact that the logarithm may not be single-valued.

Our patches U_1, U_2, U_3, U_4 will be almost the same patches as we used for the construction of the Coulomb twistor space. We use the same patches of neighbourhood of real two-sphere & partial sphere of directions at each point, except that we now use a complex three-dimensional neighbourhood of the real two-sphere, lying in the plane $t = \text{constant}$ and then spread the neighbourhood by time translation, $t \rightarrow t + 2\pi k \tau$, together with parallel translation of the partial spheres of directions, where τ occupies a closed strip in its Argand plane bounded by the straight lines $\text{Re}\tau = 0$ and $\text{Re}\tau = 1$. Note that \hat{E}_2^1, \hat{E}_4^1 and \hat{E}_6^1 are left invariant under time translation. Also each partial sphere of directions is made as large as possible, so merits just two directions and each partial sphere of points is made as large as possible, so merits just one point, the north or south pole, the south for U_1 and U_2 , the north for U_3 and U_4 .

Then the patching goes as follows (compare with Coulomb patching)

$$\hat{z}_1 = \hat{z} \text{ is identified with } \hat{z}_2 = \hat{z} + \lambda \frac{\partial}{\partial \hat{z}} \log \left(\frac{\hat{z} \hat{E}_2^1}{\hat{z} \hat{E}_2^0} \right) \text{ on } U_{12},$$

$$\text{with } \hat{z}_3 = \hat{z} + \lambda \frac{\partial}{\partial \hat{z}} \log \left(\frac{\hat{z} \hat{E}_4^1}{\hat{z} \hat{E}_4^0} \right) \text{ on } U_{13},$$

$$\text{with } \hat{z}_4 = \hat{z} + \lambda \frac{\partial}{\partial \hat{z}} \log \left(\frac{\hat{z} \hat{E}_6^1}{\hat{z} \hat{E}_6^0} \right)^2 \text{ on } U_{14}.$$

$\dot{z}_1 = \dot{z}$ is identified with $\dot{z}_1 = \dot{z} + \lambda \frac{\sqrt{z_1}}{z_2} \log\left(\frac{\sqrt{z_1}}{\sqrt{z_2} \sqrt{z_3}}\right)$ on U_{123} ,

with $\dot{z}_3 = \dot{z} + \lambda \frac{\sqrt{z_1}}{z_2} \log\left(\frac{\sqrt{z_1}}{(\sqrt{z_2})^2}\right)$ on U_{234} ,

with $\dot{z}_4 = \dot{z} + \lambda \frac{\sqrt{z_1}}{z_2} \log\left(\frac{\sqrt{z_1}}{\sqrt{z_3} \sqrt{z_4}}\right)$ on U_{134} .

$\dot{z}_2 = \dot{z}$ is identified with $\dot{z}_1 = \dot{z} + \lambda \frac{\sqrt{z_1}}{z_2} \log\left(\frac{\sqrt{z_1}}{\sqrt{z_3}}\right)$ on U_{13} ,

with $\dot{z}_3 = \dot{z} + \lambda \frac{\sqrt{z_1}}{z_2} \log\left(\frac{(\sqrt{z_1})^2}{\sqrt{z_2}}\right)$ on U_{23} ,

with $\dot{z}_4 = \dot{z} + \lambda \frac{\sqrt{z_1}}{z_2} \log\left(\frac{\sqrt{z_1} \sqrt{z_2}}{\sqrt{z_3}}\right)$ on U_{34} .

$\dot{z}_3 = \dot{z}$ is identified with $\dot{z}_1 = \dot{z} + \lambda \frac{\sqrt{z_1}}{z_2} \log\left(\frac{\sqrt{z_1}}{(\sqrt{z_2})^2}\right)$ on U_{14} ,

with $\dot{z}_2 = \dot{z} + \lambda \frac{\sqrt{z_1}}{z_2} \log\left(\frac{\sqrt{z_1}}{\sqrt{z_3}}\right)$ on U_{134} ,

with $\dot{z}_4 = \dot{z} + \lambda \frac{\sqrt{z_1}}{z_2} \log\left(\frac{\sqrt{z_1}}{\sqrt{z_3} \sqrt{z_4}}\right)$ on U_{34} .

It is easy to verify that these relations are consistent on U_{123} , U_{124} , U_{134} and U_{234} , using the fact that $\sqrt{z_1}$, $\sqrt{z_2}$, $\sqrt{z_3}$ and $\sqrt{z_4}$ are all preserved under the identifications. This completes the global definition of the space. The difference between 'Kerr' and 'Schwarzschild' is merely a change of co-ordinates.

Symmetries:

We have a four parameter group of symmetries given by

$$\dot{z}_1 \rightarrow \frac{1}{z_1} + \lambda \frac{\sqrt{z_1}}{z_2} \log \frac{\sqrt{z_1}}{\sqrt{z_2} \sqrt{z_3}}, \quad (1)$$

$$\dot{z}_1 \rightarrow \frac{1}{z_1} + \lambda \frac{\sqrt{z_1}}{z_2} \log \frac{\sqrt{z_1}}{\sqrt{z_2}}, \quad (2)$$

$$\dot{z}_3 \rightarrow \frac{1}{z_3} + \lambda \frac{\sqrt{z_3}}{z_4} \log \left(\frac{\sqrt{z_3}}{\sqrt{z_4}} \right), \quad (3)$$

$$\dot{z}_4 \rightarrow \frac{1}{z_4} + \lambda \frac{\sqrt{z_4}}{z_3} \log \left(\frac{\sqrt{z_4}}{\sqrt{z_3}} \right), \quad (4)$$

where λ satisfies $\lambda^2 = 17$, $\lambda \bar{\lambda} = 11$, $\bar{\lambda} \lambda = 19$. Note that it is possible that $\log(\sqrt{z_1})$ is singular in U_1 region for suitable \dot{z} and λ , so (1) breaks down. This is just because the point \dot{z} moves outside the U_1 region, so cannot be described by co-ordinate transformations in U_1 co-ordinates only. However the transformation can still be described in terms of suitable co-ordinates.

(16)

We note that the twistor space possesses a four parameter set of globally defined functions homogeneous of degree two, namely the set $\{\alpha \tilde{z}_1 + \beta \tilde{z}_2\}$, where α and β are arbitrary. Every holomorphic curve representing a point of the Heaven then lies entirely on a 'quadratic' of equation $\alpha \tilde{z}_1 + \beta \tilde{z}_2 = 0$ for some α, β . Each symmetry of the space leaves each quadratic invariant.

We next describe the holomorphic curves lying in $U_{1,2,2}$ representing the points of the space-time. The patching identifies $\tilde{z}_2 = (\omega^A, n_A)$ with $\tilde{z}_1 = (\tilde{\omega}^A, n_A)$, where

$$\tilde{\omega}^A = \omega^A - \lambda n_A P^{AB} \log \left(\frac{n_A \cdot n_B P^{AB}}{(\eta_A \cdot \alpha^B)(\eta_A \cdot P^B)} \right) \quad \text{on } U_{1,2}$$

where $\begin{pmatrix} \tilde{z}^B \\ \omega^B \end{pmatrix} \leftrightarrow \begin{pmatrix} B_A^B & P_{AB} \\ S^{AB} & A^B{}_B \end{pmatrix}$.

Let $\omega^A = P_{AB} \omega^{B1} / (\tfrac{1}{2} \rho^2)$ $\tilde{\omega}^A = P_{AB} \tilde{\omega}^{B1} / (\tfrac{1}{2} \rho^2)$, $\mu = 2(-\tfrac{1}{2} P^B)$,
and then let $\chi^{A'} = \omega^{A1} + A^{A'}{}_{B1} n^{B1}$ (shift of origin into the complex, making ψ effectively spinless) and $\tilde{\chi}^{A'} = \tilde{\omega}^{A1} + A^{A'}{}_{B1} n^{B1}$.

Then we have

$$\tilde{\chi}^{A'} = \chi^{A'} + \mu n^{A1} \log \left(\frac{\chi_A \cdot n^{A1}}{(\eta^A \cdot \alpha^B)(\eta^{B1} \cdot P^{AB})} \right)$$

For a holomorphic curve, $\tilde{\chi}^{A'} n_A = \chi^{A'} n_A = -Q^{A'B'} n_A \cdot n^{B1}$ for some $Q^{A'B'} = Q^{B'A'}$.
Put $Q_{A'B'} = \delta_{AB} \eta_{B1}$. Then the holomorphic curves are given by

$$\chi^{A'} = -\mu n^{A1} \log \left(\frac{\chi_A \cdot n^{A1}}{\alpha_A \cdot n^{A1}} \right) - Q^{A'B'} n_B + t n^{A1}$$

$$\tilde{\chi}^{A'} = \mu n^{A1} \log \left(\frac{\chi_A \cdot n^{A1}}{P_{AB} \cdot n^{A1}} \right) - Q^{A'B'} n_B + t n^{A1},$$

provided ϵ_A and α_A are on the 'left' and η_A and P_A on the right.

Note that we apparently have a five parameter system of curves. However (S^{A1}, η^{A1}, t) and $(\lambda S^{A1}, \tfrac{1}{2} \eta^{A1}, t + \mu \log \lambda)$ give the same curve, for any λ , so we must identify (S^{A1}, η^{A1}, t) with $(\lambda S^{A1}, \tfrac{1}{2} \eta^{A1}, t + \mu \log \lambda)$ for all λ . Then if $\mu \neq 0$ (non-flat case) we may gauge t away completely.
We assume $\mu \neq 0$, so we may take for the holomorphic curves,

$$\omega^{A'} = -\mu \eta^{A'} \log \left(\frac{\beta_{A'} \eta^{A'}}{\alpha_{A'} \eta^{A'}} \right) = (\Omega^{A'B'} + A^{A'B'}) \eta_{B'}$$

$$\text{And } \tilde{\omega}^{A'} = \mu \eta^{A'} \log \left(\frac{\gamma_{A'} \eta^{A'}}{\beta_{A'} \eta^{A'}} \right) = (\Omega^{A'B'} + A^{A'B'}) \eta_{B'}.$$

Then the metric turns out to be given by

$$(ds)^2 = -\frac{4}{\mu} \left[\mu d\bar{s}_{A'} dy^{A'} - \mu ((\bar{s}_{A'} ds_A - s_A dy^{A'}))^2 + \frac{1}{4} d\Omega_{E^F} d\Omega^{E'F'} \right]$$

$$\text{and } \frac{1}{2} d\Omega_{E^F} d\Omega^{E'F'} = -\frac{1}{4} (\bar{s}_{A'} dy^{A'} - y^{A'} ds_A)^2 - ds_{A'} dy^{A'} (\bar{s}_{B'} y^{B'}), \quad \text{so}$$

$$(ds)^2 = -\frac{4}{\mu} \left[(\mu - \bar{s}_{A'} y^{A'}) ds_A dy^{A'} - \frac{1}{4} \left(\frac{2\mu - \bar{s}_{A'} y^{B'}}{\mu - \bar{s}_{A'} y^{A'}} \right) (\bar{s}_{A'} ds_A - y^{A'} dy^{A'})^2 \right]$$

Clearly we may extend our co-ordinate patch so that all $\bar{s}_{A'} \in \mathbb{C}^1 - \{0\}$, $y_A \in \mathbb{C}^1 - \{0\}$ such that $\bar{s}_{A'} y^{B'} - \mu \neq 0$ are allowed. We do not examine the 'unrec' $\bar{s}_{A'} y^{B'} = \mu$, here.

We have the four parameter group of killing symmetries given by

$$s_{A'} \rightarrow t_{A'}{}^{B'} s_{B'}, \quad y_{A'} \rightarrow \bar{t}_{A'}{}^{B'} y_{B'}, \quad \text{where } t_{A'}{}^{B'} \bar{t}^{A'}{}_{C'} = \delta^{B'}_{C'},$$

the group being $GL(2, \mathbb{C})$. The corresponding Lie algebra of killing vectors is given by $\xi_{A'} \mu_{A'}{}^{B'} \frac{\partial}{\partial s_{B'}} + \eta_{A'} \mu^{B'}{}_{A'} \frac{\partial}{\partial y_{B'}}$, for arbitrary $\mu_{A'}{}^{B'}$.

The particular case $s_{A'} \rightarrow \lambda s_{A'}$, $y_{A'} \rightarrow \frac{1}{\lambda} y_{A'}$ corresponds to 'time' translation and is the transformation of the space-time determined by the global function λ on the twistor space via the Poisson bracket structure for twistor space ($\frac{1}{2} \delta_{A'B'} \epsilon_{AB}$) : if λ gives rise to the Hamiltonian vector field $\int \xi_{A'} \frac{\partial}{\partial s_{A'}}$ with integral curves $z \rightarrow z + i\lambda \int \xi_{A'} dz$, leading to the 'time' translation of the Heavens. Every 'quadratic' in the four parameter set $\{\alpha, \beta + \frac{1}{2}\eta\}$ such that $\beta + \frac{1}{2}\eta \neq 0$ corresponds to an orbit of the 'time' translation killing vector, so each such quadratic is ruled by a one parameter system of compact holomorphic curves representing the points of the orbit of the killing vector. When $\beta + \frac{1}{2}\eta = 0$ the holomorphic curves disappear (the singularity).

Each quadratic also possesses a one-parameter family of (non-compact) holomorphic curves with a natural linear structure : if z is a point on such a curve then the others are given by $z + \beta \int \xi_{A'} dz$ as β varies. They are the integral curves of the Hamiltonian vector field $\int \xi_{A'} \frac{\partial}{\partial s_{A'}}$. They appear to represent families of points of the conformal infinity of the spacetime. The space-time is type (2,2).

6. LOCAL METRIC PROPERTIES OF H-SPACE

Let u^a be a tangent vector at a point x^a in H-space; then u^a determines a shearfree cut of \mathcal{J}^+ infinitesimally separated from the cut corresponding to x^a . There is a (unique) infinitesimal supertranslation U which takes the cut corresponding to x^a to the infinitesimally separated one, satisfying

$$\mathcal{J}^2 U = \dot{\phi}^0 U,$$

for some Bondi scaling of \mathcal{J}^+ . Let the first cut be given by $u = Z(x^a, J, \bar{J})$; then the infinitesimally separated one is given by

$$u = Z(x^a, J, \bar{J}) + Z_{\mu}(x^a, J, \bar{J}) dx^{\mu}.$$

If λ is an affine parameter along a curve with tangent vector u^a at x^a , we have

$$Z_{\mu} dx^{\mu}/d\lambda = Z_{\mu} u^{\mu} = dZ/d\lambda = U.$$

Thus, $Z_{\mu} u^{\mu} = U$. This relation defines a linear mapping $Z_{\mu}(x^a, J, \bar{J})$ from tangent vectors to the corresponding infinitesimal supertranslations.

If $W(x^a, J, \bar{J})$ is a function on $S^2 \times S^2$ at each point of H-space, we define $\nabla_a W$ to be that object whose components are $W_{,a}$ (this is clearly well-defined), and $\nabla_a Z_b$ by

$$\nabla_a U = \nabla_a (Z_{\mu} u^{\mu}) = u^{\mu} \nabla_a Z_{\mu} + Z_{\mu} \nabla_a u^{\mu}$$

for U an infinitesimal supertranslation between neighboring good cuts of \mathcal{J}^+ , and u^a the corresponding tangent vector in H-space. It can be verified that the usual properties of derivative operators are satisfied. In particular, $\nabla_a [Z_b] = 0$, since $Z_a = \nabla_a Z$. Local geometrical properties of H-space are completely determined by Z_a and its derivatives.

The metric g_{ab} of H-space is given by

$$g_{ab} u^a u^b = \Omega(\int ds U^{-2} T^4),$$

differentiating functionally with respect to u^a twice, we obtain

$$g_{ab} = \Omega \{ 4(\int ds U^{-2})^{-3} [\int ds U^{-3} Z_a] [\int ds U^{-3} Z_b] - 3[\int ds U^{-2}]^{-2} [\int ds U^{-3} Z_a Z_b] \}.$$

This last expression is independent of u^a . It is convenient to substitute for U the quantity $V_0(x^a, J, \bar{J})$, where V_0 is a solution of $\mathcal{J}^2 V_0 = \dot{\phi}^0 V_0$ which is nowhere-vanishing and scaled so that $\int ds V_0^{-2} = 4\Omega$. [Thus, the corresponding vector field v_0^a has norm 2 everywhere.] We can write $Z_a = V_0^{-1} \pi_a$; then $\mathcal{J}(V_0^{-2} \mathcal{J}^2 \pi_a) = 0$. We also write $\pi_a = V_0^{-2} \mathcal{J}^2 \pi_a$, so that $\mathcal{J} \pi_a = 0$.

Using $\nabla_a g_{bc} = 0$, and differentiating the integral expression for g_{ab} under the integral signs, it can be shown that $\nabla_a Z_b$ takes the form

$$\nabla_a Z_b = V_0 (\Lambda \pi_a \pi_b + 2B \pi_a \pi_b + C \pi_a \pi_b),$$

where Λ , B , and C are coefficients satisfying

$$\mathcal{J}(V_0^{-2} \mathcal{J}^2 \Lambda) = V_0^{-3} \mathcal{J}^2 \Lambda,$$

$$V_0^{-2} \mathcal{J}^2 B = -2A,$$

$$V_0^{-2} \mathcal{J}^2 C = -3B.$$

The second derivative of \mathcal{Z}_g may be calculated analogously; the resulting expression is somewhat lengthy. From that expression, the curvature tensor of H-space can be evaluated directly, using $V_{[\alpha}V_{\beta]}Z_{\gamma]} = \frac{1}{2}R_{\alpha\beta\gamma}{}^{\delta}Z_{\delta}$, and integrating over the sphere to obtain R_{abcd} . A rather long calculation gives

$$R_{abcd} = -1/6\pi G(r_0) \left[\begin{aligned} & \bar{n}^a \bar{n}^b \bar{n}^c \bar{n}^d \\ & + n_0^3 A^0 I^a [e^b]^{cd} \{e^c d\} \\ & + r_0^3 n_0^3 [I^a [e^b]^{cd} (I^b [e^c]^{ad} + n [e^c d]) + (I^a [e^b]^{cd} + n [e^b]^{cd}) I^b [e^c]^{ad}] \\ & + r_0^3 c^0 [I^a [e^b]^{cd} [e^c d]] \\ & + (I^a [e^b]^{cd} + n [e^b]^{cd}) (I^b [e^c]^{ad} + n [e^c]^{ad}) \\ & + n [e^b]^{cd} [e^c d] \end{aligned} \right],$$

where n_a and \bar{n}^a are chosen so that $(I^a, n_a, \bar{n}^a, n_b)$ satisfy the usual orthogonality relations for a null tetrad when evaluated at each fixed \mathcal{Z} and $\bar{\mathcal{Z}}$. It is easily verified from this expression that the curvature tensor is trace-free ($I_a{}^a, R_{ab} = 0$; Einstein's equations!) and anti-self dual ($\Phi_{ABCD} = 0$).

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(research by R. S. Rizzo, Ezra T. Newman, and Roger Penrose)

7. The Universal Bracket Factor

Standard bracket factor for twistor integrals:

$$(x)_n = \begin{cases} \frac{(-n)!}{2\pi i (-x)^n} & \text{if } n = 1, 2, 3, \dots \quad (\oint) \\ \frac{x^{-n}}{(-n)!} & \text{if } n = 0, -1, -2, -3, \dots \quad (\oint) \end{cases}$$

means that expression
is to be integrated in
integral with closed
contour surrounding $x=0$
 means that expression
is to be integrated over
open contour with
boundary in $\mathbb{R}^{1,3}$

Universal bracket factor is $[x] = \sum_{n=-\infty}^{\infty} (x)_n$

expression must be treated as formal at the moment (and a rigorous definition from someone would be welcome) because

$$2\pi i \sum_{n=1}^{\infty} (x)_n \stackrel{(\oint)}{=} \frac{1}{x} + \frac{1!}{x^2} + \frac{2!}{x^3} + \frac{3!}{x^4} + \dots \quad \text{which diverges for all } x.$$

But Euler derived (formally) — see Hardy's "Divergent Series"

$$\frac{1}{x} + \frac{1!}{x^2} + \frac{2!}{x^3} + \frac{3!}{x^4} + \dots = e^x \left\{ \gamma - \log x + \frac{1}{x} - \frac{1}{x^2 2!} + \frac{1}{x^3 3!} - \dots \right\}$$

Euler's constant

$$= \int_0^\infty \frac{e^{-w}}{x+w} dw \quad \text{(valid with l.h.s. as asymptotic series)}$$

But, unfortunately, $x=0$ is now a branch point and not a pole so (\oint) of this expression is not strictly meaningful as it stands.

Note that $\sum_{n=-\infty}^{\infty} (x)_n \stackrel{(\oint)}{=} 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = e^x = \frac{1}{2\pi i} \oint \frac{e^{-w}}{x+w} dw$

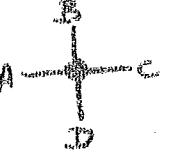
So (formally) $[x] = \{(\oint) \int_0^\infty + (\oint) \phi \} \frac{e^{-w}}{x+w} dw$. (C.A.T. Sparling's observation)

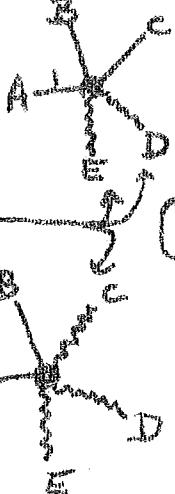
We have (formally) $\frac{d[x]}{dx} = [xz]$ (with suitable contour), twistor transform: $\int [f][g] d^4 z d^4 w$

$$\langle f | g \rangle = \frac{1}{(2\pi i)^4} \oint f(z) [\oint] g(w) dz dw \leftarrow \text{irrespective of homogeneity}$$

The universal bracket $[\cdot]$ has many (formal) uses, e.g. in twistor diagrams. E.g.

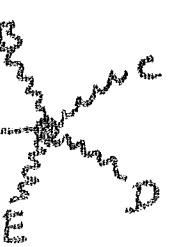
3. Some elementary twistor integrals with boundary (mostly ancient history)

Standard vertex:  $\propto f \frac{Dz}{\overline{1}\overline{1}\overline{1}\overline{1}} \propto \frac{1}{\overline{1}\overline{1}\overline{1}\overline{1}}$ (see R.P. BOHM
Phys. Rev.)

Method of writing:  $\propto f \frac{Dz}{(1)^2 \overline{1}\overline{1}} \propto \frac{4\text{PPE}}{\overline{1}\overline{1}\overline{1}\overline{1} \overline{1}\overline{1}\overline{1}\overline{1}} \propto f \frac{\log(B/A) Dz}{(1)^2 \overline{1}\overline{1}}$

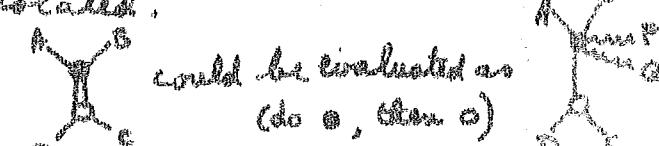
Means: take boundary where this vanishes.

 $\propto f \frac{Dz}{(1)^2 \overline{1}\overline{1}} \propto \frac{(B\text{PPE})^2}{\overline{1}\overline{1}\overline{1}\overline{1} \overline{1}\overline{1}\overline{1}\overline{1}}$

 $\propto f \frac{Dz}{(1)^4} \propto \frac{(\text{PPE})^3}{\overline{1}\overline{1}\overline{1}\overline{1} \overline{1}\overline{1}\overline{1}\overline{1}}$

Note that the "4 lines at a vertex" rule is violated.

G.A.T.S. had noticed several years ago that



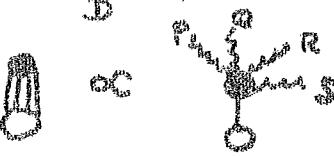
could be evaluated as

(do a, then a)

Similarly:



and



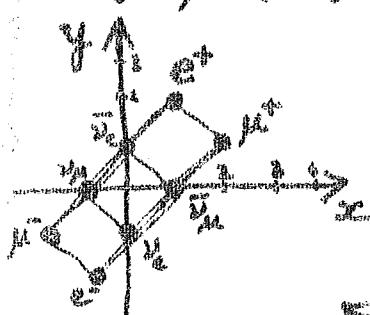
Roger Penrose
etc.

4. Projected lepton cube

Arrange the known spin-1 leptons on a cube thus:

A suggestion for classifying these particles as 2-twistor particles with twistor function $F(X^\alpha, Y^\alpha)$ of homogeneity $(x-2, y-2)$ is to project this cube

to the (x, y) plane
in one of the 48
different possible
ways, e.g.,



still holds as necessary condition

Then all leptonic conservation laws
can be expressed as

Centre of gravity of "in" particles, on figure,
Centre of gravity
of "out" particles

Work by: Lane Hughston, George Spurrier
and Roger Penrose

10. The Snail Contour (Roger Penrose & Brian Bassett)

This is in aid of an understanding of the inverse twisting function integral (due to B.P.B., G.A.J.S. & R.P.) described in outline on pp. 311 - 316 of "Quantum Gravity". But the snail contour would seem to have relevance in other contexts too.

The basic problem is as follows: we have an analytic function F defined on the real sphere S^2 and we wish to re-express the integral $\oint_{S^2} F \, dS$ ($dS = 2\pi$ surface area element) as a contour integral (1-dim, over S^1) of a holomorphic function f which is obtained by a 1-dimensional complex definite integral of F . More explicitly:

$$I = \oint_{S^2} F \, dS = \oint_{S^2} F(\xi, \bar{\xi}) \frac{4 \, d\xi \wedge d\bar{\xi}}{(1 + \xi \bar{\xi})^2} = \oint_{S^1} G(\xi, \bar{\xi}) \, d\xi \wedge d\bar{\xi} = \oint_{S^1} G(\xi, \eta) \, d\xi \wedge d\eta$$

and the question is, can we express

$$I = \oint_{S^1} \int_{P(\xi)}^{S(\xi)} G(\xi, \eta) \, d\eta \, d\xi = \oint_{S^1} f(\xi) \, d\xi,$$

γ is some suitable S^1 contour
on the S^2 sphere

$$\text{with } f(\xi) = \int_{P(\xi)}^{S(\xi)} G(\xi, \eta) \, d\eta?$$

over contour homologous to γ

$$G(\xi, \bar{\xi}) = \frac{4F(\xi, \bar{\xi})}{(1 + \xi \bar{\xi})^2} \quad \text{with} \\ F, F \text{ real-analytic; } G(\xi, \eta), \\ F(\xi, \eta) \text{ holomorphic.}$$

Now if $F=1$ it's easy. We can take $p=0$ and $q=\infty$, and all is simple. This works because the singularities of G on the complexified sphere $(CS^2 \times S^1)^2$ are "a long way away" from the real slices S^2 . To represent a point of CS^2 , we take two independent points $\xi, \eta \in S^2$. The real points of S^2 are given when $\eta = \xi$. The particular function $G(\xi, \eta) = 4(1 + \xi \eta)^{-2}$ (con. $F=1$) has its singularities given when $\xi = -1/\eta$, i.e. when ξ and η are antipodal points. For "rough" functions G , the singularities occur when ξ and η are closer together on S^2 . For a closer than ϵ on the sphere S^2 . We can think of ϵ as indicating a degree of "calmness" of the 2-dimensional contour Σ , homologous to the real S^1 in CS^2 - singularity set of G ; also:

$$\Sigma = D_1 \cup C \cup D_2$$



D_1 & D_2 are two disc-like regions on Σ and C is the remaining annular

D_1 is "in" "in" "the other" "in" the other "in" "along" Σ and $\eta = p(\xi)$ follows it about (anti-holomorphic)

We have $I = \oint_{\Sigma} G(\xi, \eta) \, d\xi \wedge d\eta = \int_{D_1} G \, d\xi \wedge d\eta + \int_C G \, d\xi \wedge d\eta + \int_{D_2} G \, d\xi \wedge d\eta$. But $d\xi \wedge d(p(\xi)) = 0 = d\xi \wedge d(q(\xi))$.

$\int_{D_1} G = 0 = \int_{D_2}$, so $I = \int_C G \, d\xi \wedge d\eta = \oint_{\Sigma} f(\xi) \, d\xi$, as required, provided that G is regular on D_1 , D_2 and C , i.e. provided that $\bar{p}(\xi)$ is closer to ξ than ϵ on D_1 and that $\bar{q}(\xi)$ is closer to ξ on D_2 . If ϵ is small, we need an elaborate winding Σ (from below) magnified which winds clearly all over S^2 , such as