

# Twistor Newsletter (no 11: 25, February, 1981)

Mathematical Institute, Oxford, England

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## Concerning a Fourier Contour Integral

There are various places in twistor theory where, given a function  $F(z)$ , holomorphic in some region of  $T$ , one is required to consider a corresponding operator  $F(\partial_w)$  (where  $\partial_w = \frac{\partial}{\partial w}$ ). Moreover, it might be that we are more interested in some cohomology class that  $F(z)$  serves to define, than in the function  $F$  itself. The definition of the corresponding "cohomology class" for  $F(\partial_w)$  would then be needed (cf., for example, R.P. in TN11 "A New Angle..."). Now one approach to the study of expressions like  $F(\partial_w)$  might be to use Fourier transforms. But the normal Fourier transform of a function  $\varphi(x)$  requires  $\varphi$  to be defined along the entire real line, and thus is unsuitable for holomorphic sheaf cohomology. Suppose, instead, that we have  $\varphi(s)$  holomorphic in some annular region in  $C \ni s$ , and define

$$f(z) = \frac{1}{2\pi i} \oint e^{sz} \varphi(s) ds,$$

the contour being taken around the annulus. This provides an

continued  
on page 42

NOTE ON THE  $\phi^*$  EQUATION

N. Buchdahl

In a recent paper entitled "Twistor Description of Classical Yang-Mills Fields" Henkin and Manin give a reformulation of the classical coupled Yang-Mills-Dirac equations in terms of obstructions to extenions of bundles and cohomology classes on regions of ambi-twistor space  $\mathbb{PA}$ . The purpose of this note is to demonstrate how the same game can be played with the  $\phi^*$  equation. (For relevant information concerning  $\mathbb{PA}$ , see Mike Eastwood's article in TN9.)

Let  $U$  be an open subset of compactified complexified Minkowski space  $\mathbb{M}$ . By the  $\phi^*$  equation on  $U$  I mean the coupled system

$$\left. \begin{array}{l} (a) \quad \mu = \lambda \phi \psi \\ (b) \quad \square \phi = \mu \phi \\ (c) \quad \square \psi = \mu \psi \end{array} \right\} \begin{array}{l} \lambda \in \mathbb{C}, \quad \mu \in \Gamma(U, \mathcal{O}_{\mathbb{M}}(-1)) \\ \phi \in \Gamma(U, \mathcal{O}_{\mathbb{M}}(0)), \quad \psi \in \Gamma(U, \mathcal{O}_{\mathbb{M}}(1)) \end{array}$$

Let  $V \subset \mathbb{PA}$  be the open set associated with  $U$  in the usual way. If  $U$  is such that its intersection with every null geodesic is connected and has  $H^1(\mathbb{C}) = 0 = H^2(\mathbb{C})$ , then the following isomorphisms hold:

$$\begin{aligned} H^1(V, \mathcal{O}(-1, -1)) &\simeq \Gamma(U, \mathcal{O}_{\mathbb{M}}(-1)) \simeq H^2(V, \mathcal{O}(-2, -2)) \\ H^1(V, \mathcal{O}(0, -2)) &\simeq \Gamma(U, \mathcal{O}_{\mathbb{M}}(0)), \quad H^1(V, \mathcal{O}(-2, 0)) \simeq \Gamma(U, \mathcal{O}_{\mathbb{M}}(1)) \\ H^2(V, \mathcal{O}(-1, -3)) &\simeq \Gamma(U, \mathcal{O}_{\mathbb{M}}(-1)^{\vee}) \quad , \quad H^2(V, \mathcal{O}(-3, -1)) \simeq \Gamma(U, \mathcal{O}_{\mathbb{M}}(-1)^{\vee}) \end{aligned}$$

If  $\phi, \psi$  correspond to  $\hat{\phi} \in H^1(V, \mathcal{O}(0, 2))$ ,  $\hat{\psi} \in H^1(V, \mathcal{O}(-2, 0))$  respectively, and  $\mu$  corresponds to  $\hat{\mu} \in H^1(V, \mathcal{O}(-1, -1))$  and  $\hat{\mu}' \in H^2(V, \mathcal{O}(-2, 2))$  under these isomorphisms, then equation (a) above becomes

$$\hat{\mu}' = \lambda \hat{\phi} \cup \hat{\psi}$$

where  $\cup$  is the cup product, whilst equation (b) (resp. eqn. (c)) is interpreted on  $V$  as the statement

"The obstruction to the extension of  $\hat{\phi}$  (resp.  $\hat{\psi}$ ) to the first infinitesimal neighbourhood of  $\mathbb{PA}$  in  $\mathbb{P} \times \mathbb{P}^1$  is equal to  $\hat{\mu} \cup \hat{\phi}$  (resp.  $\hat{\mu}' \cup \hat{\psi}$ )."

## Axisymmetric Stationary Fields.

Twistor theory can be applied to several stationary axisymmetric field theories. I'd like to mention three of these: the 3-dim. Laplace equation, the Yang-Mills-Higgs equations, and Einstein's equations.

One can generate axisymmetric solutions of the 3-dim. Laplace equation by specializing the usual contour integral formula, and arriving at the following. Define  $\gamma$  and  $\zeta$  by  $\gamma = i\omega^0/\pi_0 - i\omega^1/\pi_1$ , and  $\zeta = \pi_0/\pi_1$ . Then

$$(A) \quad \varphi(x, y, z) = \frac{1}{2\pi i} \oint f(\gamma) \frac{d\gamma}{\zeta}; \quad f \text{ holomorphic},$$

is a solution of the Laplace equation, axisymmetric about the  $z$ -axis. (I think it's the general such solution.) The inverse twistor function appears to be extremely simple here, because of the fact that  $\gamma = -2z$  on the  $z$ -axis\*, so that  $f(\gamma) = \varphi(0, 0, -\frac{\gamma}{2})$ .

For example, the "Schwarzschild-Weyl" solution  $\varphi = \frac{1}{2} \log \frac{r_+ + r_- - 2m}{r_+ + r_- + 2m}$  [1] is easily seen to arise from  $f = \frac{1}{2} \log \frac{r+2m}{r-2m}$ . Someone might like to investigate all this in more detail.

Notes: (i)  $\varphi$  is real if  $f$  satisfies the reality condition  $\overline{f(\gamma)} = f(\bar{\gamma})$ .

(ii) The formula (A) is essentially due to Whittaker [2].

Now the problem of finding stationary "multi-monopole" solutions of the Y-M-H equations (see [3] for details). Twistor methods give (at least implicitly)

\* My conventions are  $\omega^0 = i(t+z)\pi_0 + i(x-ly)\pi_1$ ,  $\omega^1 = i(x+iy)\pi_0 + i(t-z)\pi_1$ .

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the general such solution, as follows. Take a  $2 \times 2$  <sup>unimodular</sup> matrix  $F(Y, S)$  of holomorphic functions, satisfying the reality condition  $F(\bar{Y}, -\bar{S}^{-1}) = \text{conjugate transpose}$  of  $F(Y, S)$ . This gives rise to a Y-M-H field in space-time which is stationary, real, and a solution of the field equations. The problem (not yet solved) is to ensure that this field has the correct global properties. The only explicit cases so far understood are axisymmetric ones, where the  $S$ -dependence of  $F$  is rather special. The two simplest ones are [4]

$$F = \begin{bmatrix} Y^{-1}(e^Y - e^{-Y}) & -S^{-Y} \\ S^{-1}e^{-Y} & Ye^{-Y} \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} H^{-1}(e^Y + e^{-Y}) & S^2 e^{-Y} \\ S^{-2} e^{-Y} & He^{-Y} \end{bmatrix}$$

where  $H = Y^2 + \pi^2/4$ .

Finally, stationary axisymmetric vacuum space-times. As pointed out by Witten [5], this problem can also in principle be solved by twistor methods : the matrices  $F$  above "encompass" all such space-times. The only case as yet understood (by me) is that of the Weyl solutions, which correspond to  $F = \text{diag}(e^{f(Y)}, e^{-f(Y)})$ .

- References
1. Kramer et al, Exact Solutions, p.201.
  2. Whittaker & Watson §18.3.
  3. Jaffe & Taubes, Vortices and Monopoles.
  4. Ward, YMH Monopole of Charge 2, to appear in CMP.
  5. L. Witten, Phys Rev D19, 718 (thanks to KPT for pointing)  
(this paper out to me)

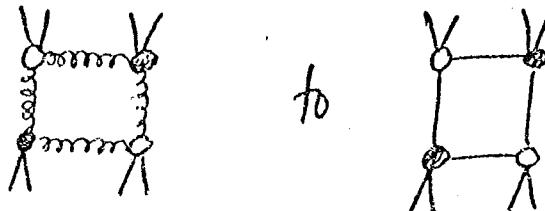
Mystery: how are the twistor methods related to "Bäcklund" methods; cf. Forgács et al, Budapest preprint KFKI-1980-122; Cosgrove, JMP 21 2417.

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Dublin.

## A New Year's Resolution

In [1] RP and I described the three contours for the box diagram. Two further remarks ought to be made.

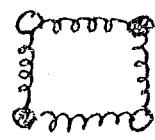
D) I claimed that Leray sequences can be used to map the contours from



This later turned out to be false, so that the required map must be constructed by hand. (For details see [2].)

2) As Nigel Hitchin has pointed out the space  $B$  is not, strictly speaking, a blow up (as defined in [3]) of

the space



If  $p: B \rightarrow \{ \text{point} \}$  were a

blow up along  $p(I)$  the inverse image of a point in  $p(I)$  would have to be a  $\mathbb{CP}^2$  because  $\text{codim}(p(I)) = 3$ .

In other words the space of directions normal to  $p(I)$  in  $\{ \text{point} \}$  is  $\mathbb{CP}^2$ . In fact not all these directions

are separated in the inverse image, which is a  $\mathbb{CP}^1$ .

The map  $p$  is called a resolution of the singular space  $p(I)$ .

Stephen Thurgott

References:

[1] "Three Channels for the Box Diagram": TN 10

[2] My D.Phil. Thesis: Oxford 1980

[3] "Blowing up the Box": Advances in twistor theory

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## THE INVERSE TWISTOR FUNCTION REVISITED

N. Buchdahl

0. In [2] Penrose gave an explicit integral formula for generating homogeneous functions on parts of projective twistor space from massless fields defined on certain regions of compactified, complexified Minkowski space, such that the procedure provided an inverse to the usual  $\Phi$ -transform taking the homogeneous functions to fields. The purpose of this article is to give a cohomological interpretation of this inverse transform (the inverse twistor function).

The notation used here will follow that in [1]. In particular,  $P$  := projective twistor space,  $F$  := projective primed spin bundle,  $M$  := compactified, complexified Minkowski space, and  $\mu: F \rightarrow P$ ,  $\nu: F \rightarrow M$  are the standard 'projections'. For a sheaf  $S$  on a space  $X$ , a resolution  $0 \rightarrow S \rightarrow R^0 \rightarrow R^1 \rightarrow R^2 \rightarrow \dots$  is denoted by  $X: 0 \rightarrow S \rightarrow R^0$ , and the groups  $(\ker \Gamma(X, R^0) \rightarrow \Gamma(X, R^1)) / (\text{im } \Gamma(X, R^0) \rightarrow \Gamma(X, R^1))$  by  $H^0(\Gamma(X, R^0))$ .

Before proceeding, some general definitions and results from sheaf theory will first be collected together in the next section.

1. If  $X, Y$  are topological spaces,  $f: X \rightarrow Y$  is a mapping and  $S$  is a sheaf on  $Y$ , the topological inverse image of  $S$ ,  $f^{-1}S$ , is the sheaf on  $X$  characterised by  $(f^{-1}S)_x \simeq S_{f(x)}$  for all  $x \in X$ . If  $X$  and  $Y$  are smooth manifolds and  $f$  is a smooth mapping of maximal rank (i.e.  $\text{rk}(df) = \dim Y \leq \dim X$  at every point of  $X$ ), then there exists a resolution  $X: 0 \rightarrow f^*F_Y \rightarrow \mathcal{G}_f^0$  (where  $F_Y$  is the sheaf of germs of smooth functions on  $Y$ ). Here  $\mathcal{G}_f^0$  is the sheaf of germs of relative q-forms on  $X$ : locally a relative q-form 'looks like' a q-form on a fibre of  $f$  parameterised by the variables transverse to the fibre. The differentials  $d_f: \mathcal{G}_f^0 \rightarrow \mathcal{G}_f^{q+1}$  are just differentiation along the fibres.

If  $B$  is a smooth bundle on  $Y$ , then because the transition functions of  $f^*B$  are constant along the fibres of  $f$ , one can tensor through the above resolution by  $\mathcal{G}_X(f^*B)$  to obtain a res-

solution  $X : 0 \rightarrow f^* \mathcal{F}_Y(B) \rightarrow \mathcal{F}_f(B)$ , where  $\mathcal{F}_f(B) := \mathcal{F}_f \otimes_{\mathcal{O}_X} \mathcal{F}_X(f^* B)$ .  
 Also, in complete analogy to the smooth case, if  $X$  and  $Y$  are complex manifolds and  $f$  is holomorphic (and of maximal rank), then there is a resolution  $X : 0 \rightarrow f^* \mathcal{O}_Y(V) \rightarrow \mathcal{S}_f(V)$  for any holomorphic vector bundle  $V$  on  $Y$ .

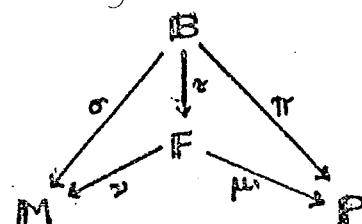
Finally, the following theorem from my D. Phil. Qualifying dissertation will play a central role in subsequent sections of this article:

THEOREM: Let  $X, Y$  be paracompact complex manifolds and  $f: X \rightarrow Y$  be a surjective holomorphic mapping of maximal rank. Let  $V$  be a holomorphic vector bundle on  $Y$ . If, for each  $y \in Y$ ,  $f^{-1}(y)$  is connected and  $H^p(f^{-1}(y), \mathbb{C}) = 0$  for  $p = 1, \dots, N$ , then the canonical mapping  $H^p(Y, \mathcal{O}(V)) \rightarrow H^p(X, f^* \mathcal{O}(V))$  is an isomorphism for  $p = 0, 1, \dots, N$ , and a monomorphism for  $p = N+1$ .

(The relevant proposition in my dissertation is slightly weaker than this, and I am grateful to Nick Woodhouse for his assistance which enables me to state this stronger version.)

Let  $B := \mathbb{P} \times \mathbb{P} \setminus \text{diagonal}$ , and let  $\pi: B \rightarrow \mathbb{P}$  be the projection onto the first factor. An element of  $B$  is a pair of linear 1-subspaces of  $\mathbb{P}$  which, being non-coincident, uniquely determine a linear 2-subspace of  $\mathbb{P}$ , i.e. an element of  $M$ ; let  $\sigma$  denote this map  $B \rightarrow M$ . Similarly, there is a mapping  $\tau: B \rightarrow F$  which is defined so that the following diagram commutes:

(2.1)



If  $U \subset M$  is open, let  $U_0 := \sigma(U)$ , (where, as usual  $U' = \sigma^{-1}(U)$  and  $U'' = \mu(U')$ ). If, moreover,  $U \subset M^*$ , then  $U_0 = U \times D$ ,

the space-time-like coordinates  $(x^{AB'}, \pi_{A'}, \zeta_{A'})$  [where  $x^{AB'} = (x^{AB}, \delta^{AB})$ ] are 'homogeneous' coordinates on  $M^I$ , and  $\pi_{A'}$ ,  $\zeta_{A'}$  are homogeneous coordinates on their respective  $\mathbb{C}\mathbb{P}_1$ 's. In terms of these coordinates, the various mappings of (2.1) are

$$\begin{array}{ccc} & U_0 & \\ & \downarrow \sigma & \downarrow \pi \\ U' & & U'' \\ \swarrow \nu & & \searrow \mu \\ U & & U'' \end{array} \Rightarrow \begin{array}{ccc} (x^{AB'}, \pi_{A'}, \zeta_{A'}) & \leftrightarrow & (U', V') = (x^{AB}, \pi_A, \zeta_A) \\ & \downarrow \sigma & \downarrow \pi \\ (x^{AB}, \pi_A) & & x^{AB}, \pi_A = U'' \\ \swarrow \nu & & \searrow \mu \\ x^{AB'} & & x^{AB}, \pi_A = U'' \end{array}$$

It will be shown subsequently that the inverse twistor function naturally gives rise to elements of the cohomology group  $H^1(U_0, \pi^* \mathcal{O}_{\mathbb{P}}(-m))$ , which, under the conditions for which the  $\Omega$ -transform is an isomorphism explicitly generate representative cocycles for elements of the group  $H^1(U'', \mathcal{O}_{\mathbb{P}}(-n-2))$ .

5. The space  $D = \mathbb{C}\mathbb{P} \times \mathbb{C}\mathbb{P} \setminus \text{diagonal}$  introduced above is of considerable importance and deserves special attention.  $D$  is a fibration over  $\mathbb{C}\mathbb{P}_1$  with fibre  $\mathbb{C}\mathbb{P}_1 \setminus \{\text{point}\} = \mathbb{C}$ , and is therefore of the same homotopy type as  $\mathbb{C}\mathbb{P}_1$ . Moreover,  $D$  is Stein (since it can be embedded as a closed submanifold of  $\mathbb{C}^4$ ), so that the line bundles on it are completely determined by  $H^2(D, \mathbb{Z}) = \mathbb{Z}$ . It is not too difficult to deduce that the restriction mapping  $H^2(\mathbb{C}\mathbb{P} \times \mathbb{C}\mathbb{P}, \mathbb{Z}) \rightarrow H^2(D, \mathbb{Z}) : \mathbb{Z} \oplus \mathbb{Z} \rightarrow \mathbb{Z}$  is given by  $(m, n) \mapsto m-n$ , so it follows that  $\mathcal{O}_{(m,n)} \cong \mathcal{O}_{(m+1, n+1)}$  on  $D$ . (The same isomorphism is thus also true on  $U_0$  (for any topologically trivial Stein  $U \subset M^I$ ).

By using the fact that every section  $f \in \Gamma(D, \mathcal{O}(m,n))$  can be expressed (non-uniquely) as a holomorphic function on  $\mathbb{C}^4 \times \mathbb{C}^*$ , a direct argument with power series shows that  $f$  may be expressed (uniquely) in the form

$$f(\pi_{A'}, \zeta_{A'}) = \sum_{s \geq \max\{-m, -n\}} Q^s C^{A'_1 \dots A'_{m+s} B'_1 \dots B'_{n+s}} \pi_{A'_1} \dots \pi_{A'_{m+s}} \zeta_{B'_1} \dots \zeta_{B'_{n+s}}$$

where  $Q := \pi_{A'} \zeta_{A'}$  and  $C^{A'_1 \dots A'_{m+s} B'_1 \dots B'_{n+s}} = C^{(A'_1 \dots A'_{m+s})}$ .

It follows that every section  $f \in \Gamma(U_0, \mathcal{O}(m,n))$  can be expressed in the form (3.1), but with  $C^{A'_1 \dots A'_{m+s} B'_1 \dots B'_{n+s}}$  now belonging to  $H^1$ .

$$\Gamma(U, \mathcal{O}^{(A_1 \dots A_m, n+s)} \otimes_{\mathbb{S}} \mathbb{S}) .$$

4. Suppose now that  $c^{A_1 \dots A_n} \in \Gamma(U, \mathcal{O}^{(A_1 \dots A_n)} \otimes \mathbb{S})$ ; then

$$f_r := \frac{\partial^s c^{A_1 \dots A_n}}{\partial v^{r+1}} \frac{\partial^{n-s} \pi_{A_1 \dots A_n}}{\partial u^{n+1}} \dots \frac{\partial^{n-s} \pi_{A_1 \dots A_n}}{\partial u^{n+1}} \quad r = 0, \dots, n$$

is a well-defined element of  $\Gamma(U_0, \mathcal{O}^{(r-s, n-r-s)})$ , and a direct but tedious calculation gives

$$(4.1) \quad \frac{\partial f_r}{\partial v^s} + \frac{\partial f_{r+1}}{\partial u^s} = i \epsilon^s y_u^\alpha \nabla_{v^\alpha} c^{A_1 \dots A_n} \pi_{A_1 \dots A_n} \dots \pi_{A_{r+1} A_{r+1}} \dots \pi_{A_n A_n} \\ + \epsilon^s (n+1-s) \epsilon_{A_1 \dots A_{r+1}} c^{A_1 \dots A_{r+1}} \pi_{A_1 \dots A_{r+1}} \pi_{A_{r+1} A_{r+1}} \dots \pi_{A_n A_n}$$

where  $y_u^\alpha := (\delta_{\alpha}^0, -i \omega^{\alpha 0})$  and  $\epsilon_{A_1 \dots A_n} := (0, \delta_{\alpha}^0)$ . (Since  $U^*$  and  $V^*$  are homogeneous coordinates, equation (4.1) does not really make sense, and moreover, the equation tacitly assumes that  $U \subset \mathbb{M}^*$ ; however, there is an obvious meaning to the equation and the conclusions to be drawn from it will be manifestly invariant, so these ambiguities will be ignored for the sake of transparency and simplicity.)

Taking  $s = n+1$  in (4.1), one sees that each of the equations

$$(4.2) \quad \frac{\partial f_r}{\partial v^s} + \frac{\partial f_{r+1}}{\partial u^s} = 0 \quad r = 0, \dots, n-1$$

is equivalent to  $c^{A_1 \dots A_n}$  satisfying the zero-rest-mass equations. Furthermore, (4.2) implies

$$(4.3) \quad \frac{\partial^2 f_r}{\partial v^s \partial v^{s+1}} = 0 \quad r = 0, \dots, n.$$

(In fact, each of the  $n+1$  equations (4.3) is also equivalent to  $c^{A_1 \dots A_n}$  satisfying the zero-rest-mass equations, although to see this directly requires a very lengthy calculation.)

The section  $f_0 \in \Gamma(U_0, \mathcal{O}(-n-1, -1))$  is particularly important – it is the null datum for the field  $c^{A_1 \dots A_n}$ . Setting  $f := \frac{\partial f_0}{\partial u^s} dv^s \in \Gamma(U_0, \Omega_{\mathbb{M}}(-n-2))$ , equation (4.3) says precisely that the relative 1-form  $f$  is  $d_{\mathbb{M}}$ -closed. Thus there is a mapping

$$(4.4) \quad \{ \text{massless fields of helicity } -\frac{1}{2}n \text{ on } U \} \rightarrow H(\Gamma(U_0, \Omega_{\mathbb{M}}(-n-2)))$$

It will now be shown (briefly) that (4.4) is in fact an isomorphism.

Injectivity: Suppose  $f$  is  $d_{\pi}$ -exact; i.e.  $\frac{\partial f}{\partial v^a} = \frac{\partial w}{\partial v^a}$  for some  $w \in \Gamma(U_0, \mathcal{O}(-n-2, 0))$ . Then  $-(n+1)f_0 = R w$ , where  $R := v^a \frac{\partial}{\partial v^a} = \pi_{A'} \frac{\partial}{\partial v_{A'}^{(0)}}$ . However,  $f_0$  does not involve  $\pi_{A'}$  whereas  $R w$  does, so the uniqueness of the form (3.1) implies  $f_0 = 0$ .

Surjectivity: Suppose  $g = g_a dV^a$  is a  $d_{\pi}$ -closed relative 1-form on  $U_0$ . If  $U^a g_a = \sum_{s \geq n+1} \zeta^s c^{A_1 \dots A_{s-n}, B_1 \dots B_s} \pi_{A_1} \dots \pi_{A_{s-n}}, \zeta_{B_1} \dots \zeta_{B_s}$ , then by setting

$$h := \sum_{s \geq n+2} \zeta^s c^{A_1 \dots A_{s-n-1}, B_1 \dots B_s} \pi_{A_1} \dots \pi_{A_{s-n-1}}, \zeta_{B_1} \dots \zeta_{B_s}$$

and using the operator  $R$  defined above, it is easy to check that  $c^{A_1 \dots A_s}$  must satisfy the zero-rest-mass equations, and that  $g_a = \frac{\partial h}{\partial v^a} - \frac{1}{n+1} \frac{\partial c_0}{\partial v^a}$ , where  $c_0$  is the null datum for the field  $c^{A_1 \dots A_n}$ .

5. The isomorphism (4.4) is the first in a chain of such, relating cohomology classes on  $U_0$ ,  $U'$  and  $U$ , and, in the appropriate circumstances, on  $U''$ . The next step relates classes on  $U_0$  and  $U'$ .

The mapping  $\pi : U_0 \xrightarrow{\text{onto}} U'$  defines a fibration over  $U'$  with fibre  $\mathbb{C}P^1 \setminus \{\text{point}\} \cong \mathbb{C}$ . Thus by the theorem of Section 0, one has  $H^p(U_0, \pi^* \Omega_{\mu}^q(m)) \cong H^p(U', \Omega_{\mu}^q(m))$  for all  $m$  and all  $p, q \geq 0$ . Since the resolution  $\mathbb{F} : 0 \rightarrow \mu^* \mathcal{O}_{\mu}^p(m) \rightarrow \Omega_{\mu}^p(m)$  generates a resolution  $\mathbb{B} : 0 \rightarrow \pi^* \mu^* \mathcal{O}_{\mu}^p(m) \rightarrow \pi^* \Omega_{\mu}^p(m)$ , one may apply these isomorphisms to the (two) pairs of long exact cohomology sequences which these resolutions induce. Then a multiple application of the Five Lemma shows that  $H^p(U_0, \pi^* \mu^* \mathcal{O}_{\mu}^p(m)) \cong H^p(U', \mu^* \mathcal{O}_{\mu}^p(m))$  for all  $p \geq 0$ . But  $\mu \circ \pi = \pi'$ , so  $\pi^* \mu^* \mathcal{O}_{\mu}^p(m) = \pi'^* \mathcal{O}_{\mu}^p(m)$ , and one therefore concludes that  $H^p(U_0, \pi^* \mathcal{O}_{\mu}^p(m)) \cong H^p(U', \mathcal{O}_{\mu}^p(m))$  for all  $p \geq 0, m \in \mathbb{Z}$ .

It is a relatively simple exercise to check that the diagram

$$(5.1) \quad \begin{array}{ccc} H^*(\pi(U_0, \Omega_{\pi}^{n-2})) & \longrightarrow & H^*(U_0, \pi^* \mathcal{O}(-n-2)) \\ \downarrow & & \downarrow \\ \text{Helicity } \frac{1}{2} n \text{ massless fields } \xrightarrow{\sim} & H^*(U', \mathcal{O}(-n-2)) \end{array}$$

on  $U$

commutes, (again one uses the operator  $R$  and the uniqueness of the form (3.1)), and therefore the canonical inclusion  $H^1(\Gamma(U_0, \Omega_{\pi}^{n-2})) \rightarrow H^1(U_0, \pi^*\mathcal{O}(-n-2))$  is actually an isomorphism.

6. So far, this cohomological interpretation of the inverse twist-or function has not related fields on  $U$  to cohomology classes on  $U''$ ; in order to do this, it is clear from (5.1) that it will be necessary to impose on  $U$  the conditions for which the  $\mathfrak{Q}$ -transform is an isomorphism. In fact, in addition to (5.1), one has the commutative diagram

(6.1)

$$\begin{array}{ccc} H^1(U_0, \pi^*\mathcal{O}(-n-2)) & & \\ \downarrow s & \swarrow & \searrow \\ H^1(U'', \mathcal{O}(-n-2)) & & \\ H^1(U', \mu^*\mathcal{O}(-n-2)) & & \end{array}$$

which makes these restrictions obvious. However, the restrictions may also be seen directly as follows: the fibre of  $\pi|_{U_0}$  over a point  $u \in U''$  is the same as the inverse image under  $\mathfrak{C}$  of the  $\alpha$ -plane in  $U$  corresponding to  $u$ ; this set is therefore a fibration over that  $\alpha$ -plane with fibre  $D$ , so it follows from a spectral sequence argument that  $H^p(\pi^*\omega) \cap U_0, \mathbb{C}) \cong H^p(\{\alpha\}-\text{plane corresponding to } u \cap U, \mathbb{C})$  for  $p = 0, 1$ . So by the theorem again, one has  $H^1(U_0, \pi^*\mathcal{O}(m)) \cong H^1(U'', \mathcal{O}(m))$  under precisely the same conditions for which the  $\mathfrak{Q}$ -transform is an isomorphism; that is, when  $U$  is  $\mathfrak{Q}$ .

7. The analysis of Section 5 did not require the homogeneity to be  $\leq -2$ , so one might expect that right-handed massless fields can also be dealt with within this framework. This is indeed the case: if  $c_A^{A_1 \dots A_m} \in \Gamma(U, \mathcal{O}_A^{(A_1 \dots A_m)} \wedge \mathbb{V})$ , then  $f := \mathfrak{Q}^* y_A c_A^{A_1 \dots A_m} \pi_{A_1} \dots \pi_{A_m} dv^A$  belongs to  $\Gamma(U_0, \Omega_{\pi}^1(n-2))$  and satisfies

$$d_{\pi} f = \mathfrak{Q}^* y_A c_A^{A_1 \dots A_m} \nabla^A (A' c_{A'}^{A_1 \dots A_m}) \pi_{A_1} \dots \pi_{A_m} \pi_{A'} dv^A dv^{A'}$$

simply

Thus  $d_{\pi} f = 0 \Leftrightarrow c_A^{A_1 \dots A_m}$  is a potential for a right-handed massless field on  $U$ . Moreover, it is easy to see that  $f = d_{\pi} g$  for some  $g \in \Gamma(U_0, \Omega_{\pi}^1(n-2))$  if and only if  $c_A^{A_1 \dots A_m} = \nabla_A^{(A_1 \dots A_m)} g$  for some  $g^{A_1 \dots A_m} \in \Gamma(U, \mathcal{O}^{(A_1 \dots A_m)})$ . Thus  $H^1(\Gamma(U_0, \Omega_{\pi}^1(n-2))) \leftarrow \{\text{belicity } \frac{1}{2} \text{ in pots/gauge on } U\}$

$A \in \Omega_{\pi}^1(n-2)$

is injective, and in the same vein one can easily prove its surjectivity. Hence the relevant commutative diagram is now

$$(7.1) \quad \begin{array}{ccc} H^1(\Gamma(U_0, S_{\pi}^{(n-2)})) & \longrightarrow & H^1(U_0, \pi^{-1}\mathcal{O}(n-2)) \\ \uparrow & & \uparrow \\ \{ \text{belicity } \frac{1}{2} \text{ potentials/} \} & \xrightarrow{\quad} & H^1(U', \mu^{-1}\mathcal{O}(n-2)) \\ \text{gauge on } U & & \end{array}$$

3. When  $U$  is  $\mathfrak{G}_1$ , (i.e. the fibres of  $\pi|_{U_0}$  are connected and simply-connected), one can construct a representative cocycle on  $U'$  for a given massless field explicitly, the method being essentially the original integral formula given in [2]: if  $a \in U'$ , choose  $b \in U'$  such that  $(a, b) \in U_0$ ; then there is a neighbourhood  $W \subset U'$  of  $a$  such that  $W \times \{b\} \subset U_0$ . Given a  $d_{\pi}$ -closed 1-form  $f$  on  $U_0$ , set

$$g(u, v) := \int_{(u, b)}^{(u, v)} f \quad u \in W$$

where the integral is taken over any path in  $\pi^{-1}(W) \cap U_0$  from  $(u, b)$  to  $(u, v)$ , (path independence is guaranteed as  $U$  is  $\mathfrak{G}_1$ ). By construction,  $d_{\pi} g = f$  in  $\pi^{-1}(W) \cap U_0$ , and if one chooses a locally finite cover of  $U'$  of this form,  $\{a_i, b_i, W_i\}$  say, the cocycle is then given by

$$g_{ij}(u) = \int_{(u, b_j)}^{(u, b_i)} f \quad u \in W_i \cap W_j$$

\* \* \*

I should like to thank Mike Eastwood for his critical and perceptive comments on this work.

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## Massive Particle States and $n$ -Point Massless Fields

Lane Hughston & Tom Hurd

In TNio ('Mass, Cohomology, and Spin') it was noted that corresponding to any normalizable positive frequency state  $\Phi_{a..b}(r)$  of mass  $m$  and spin  $s$  there exists a unique two-point field  $\phi(x,y)$  of zero rest mass and zero helicity in each variable separately, with total mass  $m$  and total spin  $s$ . One of the reasons why this correspondence is interesting is that it leads to a description of massive states as elements of the sheaf cohomology group  $H^2(P_3^+ \times P_2^+, \mathcal{O}_{sm}(a,b))$ . According to twistor particle ideas, the values of  $a, b, s$ , and  $m$  are then to be interpreted as the quantum numbers of the state.

The basic correspondence indicated above can be seen much more simply by using a Fourier transform technique, which we shall describe in what follows. This enables us to set out the general relationship between massive fields (of any spin) and  $n$ -point massless fields (for all  $n$ ). The resulting picture is readily seen to be (formally) identical to the twistor particle classification scheme.

Massive Scalar Field. Let  $\Phi(r)$  denote a positive frequency massive scalar field with Fourier transform  $\tilde{\Phi}(\beta)$ . We introduce a measure  $\mu$  defined by

$$\mu(x, y, \alpha, \beta) = (2\pi)^{-8} e^{-i\alpha \cdot x} \left( e^{-i\beta \cdot y} \delta^+(x^i) \delta^+(y^i) d\alpha^i d\beta^i \right).$$

Then the two-point field<sup>1</sup>

$$(1) \quad \phi(x, y) = \int \tilde{\Phi}(\alpha + \beta) \mu(x, y, \alpha, \beta)$$

satisfies the zero rest mass equations in each variable, has total spin 0, and has total mass  $m$ , with  $\phi(r, r) = \Phi(r)$ , and it is the unique field with these properties.<sup>2</sup>

Integral Spin. Suppose a massive field of integral spin  $s$  has the Fourier transform  $\tilde{\Phi}_{a..b}(\beta)$ , with  $\beta^2 \tilde{\Phi}_{a..b} = 0$ . Then the corresponding spin  $s$  two-point field is:

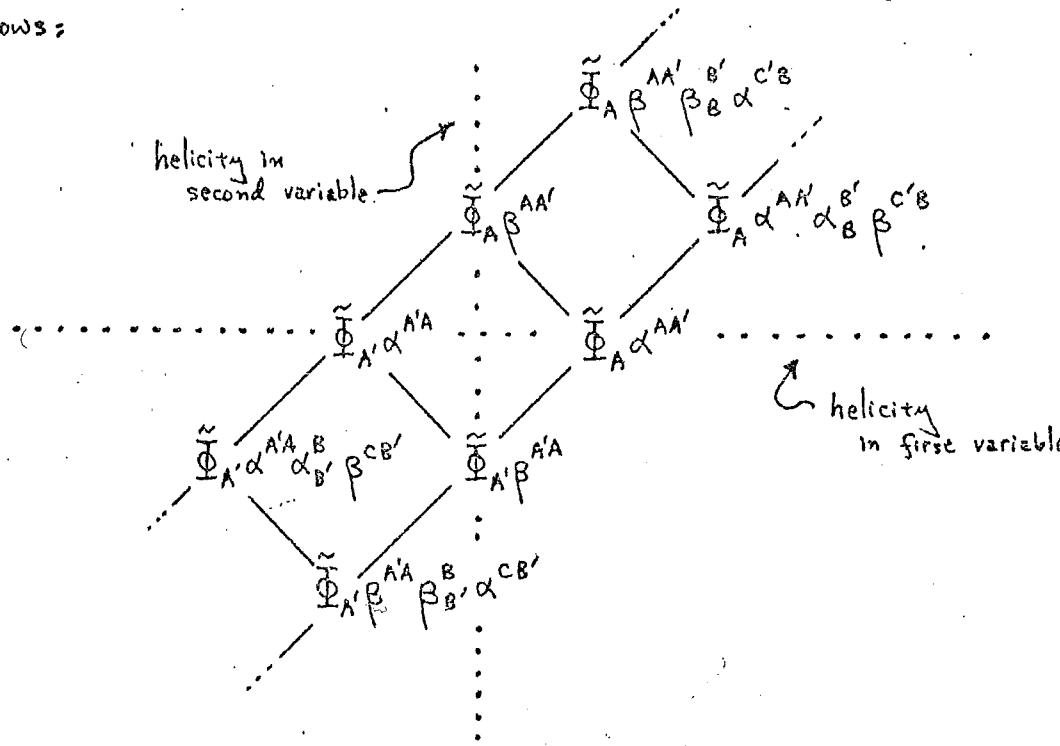
$$\phi(x, y) = \int k^a \dots k^b \tilde{\Phi}_{a..b}(\alpha + \beta) \mu(x, y, \alpha, \beta), \quad k^2 = \alpha^2 - \beta^2.$$

Half-Integral Spin. Suppose that the Dirac field  $\{\tilde{\Phi}_A(r), \tilde{\Phi}_{A'}(r)\}$  of mass  $m$  has the Fourier transform  $\{\tilde{\Phi}_A(p), \tilde{\Phi}_{A'}(p)\}$ . Then the two-point field<sup>3</sup>

$$\phi^{A'}(x, y) = \int d^{AA'} \tilde{\Phi}_A(\alpha + \beta) \mu(x, y, \alpha, \beta)$$

is of helicity  $1/2$  in the first variable and helicity  $0$  in the second variable. Its total spin is  $1/2$ , and its total mass is  $m$ , with  $\phi^{A'}(r, r) = \tilde{\Phi}^A(r)$ .

More generally given a Dirac field we can construct the complete two-twistor 'lepton ladder' of two-point states to which it corresponds, the relevant Fourier integrands being as follows:



One point that should be noted is that in the case of a field such as

$$\phi^{A'B'C'}(x, y) = \int d^{AA'} d^{BB'} d^{CC'} \tilde{\Phi}_A(\alpha + \beta) \mu(x, y, \alpha, \beta)$$

although the total helicity (sum of  $x$  and  $y$  helicity) is  $3/2$ , the total spin is  $1/2$ . In particular  $\phi^{A'B'C'}(r, r) = \tilde{\Phi}^{(A')}(r) \tilde{\Phi}^{(B')}(r) \tilde{\Phi}^{(C')}(r) / m^2$ .

The difference in the helicities must lie in range  $[-s, s]$ . (In model-building one tries to interpret the sum and difference of the helicities in terms of lepton number, electric charge, etc..)

Three-Point Fields. A similar set-up is available corresponding to three-twistor states. Suppose, for example, we wanted to describe a spin  $3/2$  particle as a  $\Delta^+$  resonance

within the old 3-twistor scheme for baryons. Then if  $\tilde{\Phi}_{ABC}(p)$  denotes the Fourier transform we have

$$\phi^{AB,C}(x,y,z) = \int \alpha^{AA'} \alpha^{BB'} \beta^{CC'} \tilde{\Phi}_{A'B'C'}(d\beta + \gamma) u(x,y,z, \alpha, \beta, \gamma),$$

with  $u$  defined in the obvious way. The field  $\phi^{AB,C}(x,y,z)$  has helicity 1 in the 1<sup>st</sup> variable, helicity 1/2 in the 2<sup>nd</sup> variable, and helicity 0 in the 3<sup>rd</sup> variable; the helicity in each variable corresponds to the 'number of quarks' present of that type. In addition to having spin 3/2,  $\phi^{AB,C}(x,y,z)$  satisfies three further differential equations, showing that its  $\hat{C}_2$  and  $\hat{C}_3$  eigenvalues are appropriate for an  $SU(3)$  10 state, and that it has isospin 3/2. All the familiar twistor particle lone can be similarly treated.<sup>4</sup>

Two-Photon States. In conclusion we mention the slightly more down-to-earth problem of constructing the general two-photon state with  $s=0$ ,  $P=-1$ , and mass  $m$ . (The consideration of such states is of interest in connection with  $\pi^0 \rightarrow 2\gamma$  decay.) A two-photon state  $F_{ab,cd}(x,y)$  satisfies Maxwell's equations on the first two indices with respect to  $x$  and on the second two indices with respect to  $y$ . Bose statistics implies<sup>5</sup>  $F_{ab,cd}(x,y) = F_{cd,ab}(y,x)$ . The general solution with mass  $m$  and spin 0 is:

$$F_{ab,cd}(x,y) = \int [\xi \alpha_A^P \alpha_B^Q \Sigma^{A'B'} \beta_{C'P} \beta_{D'Q} \varepsilon_{C'D'} + \gamma \alpha_A^P \alpha_B^Q \Sigma^{AB} \beta_{C'P} \beta_{D'Q} \varepsilon_{CD}] \tilde{\Phi}(d\beta) u(x,y, \alpha, \beta),$$

where  $\xi$  and  $\gamma$  are constants. A short calculation shows that

$$F_{ab,cd}(r,r) = [\frac{1}{e}(\xi + \gamma) g_{abcd} + \frac{1}{e}(\xi - \gamma) \varepsilon_{abcd}] \tilde{\Phi}(r),$$

and thus for a pseudo-scalar field we take  $\xi = 1$ ,  $\gamma = -1$ . Our treatment (which, insofar as we know, is new) is considerably simpler than what one finds in standard texts on relativistic quantum mechanics.

### NOTES

1. The two-point field  $\phi(x,y)$  defined in eq.(1) can be expressed in various interesting alternative ways. Let us write  $p^2 = \alpha^2 + \beta^2$ ,  $k^2 = \alpha^2 - \beta^2$ ,  $r^2 = \frac{1}{2}(x^2 + y^2)$ , and  $q^2 = \frac{1}{2}(x^2 - y^2)$ . Then the  $k^2$  integration in eq.(1), which for each value of  $p^2$  is over a compact region, can be carried out explicitly so as to yield the

formula:

$$\phi(x, y) = \frac{1}{(2\pi)^2} \int j_0(\Lambda) \tilde{\Phi}(p) e^{-ip \cdot r} d^4 p,$$

where  $\Lambda = [(p \cdot q)^2 - m^2 q^2]^{1/2}$ , and  $j_0$  is the spherical Bessel function of order zero. When the  $p$ -integration is carried out we recover the operator expression for  $\phi(x, y)$  given in TN10:  $\phi(x, y) = \mathcal{V}_0 \tilde{\Phi}(r)$ , where  $\mathcal{V}_0 = j_0(\hat{\Lambda})$ , with  $\hat{\Lambda}^2 = (i R \cdot q)^2 - m^2 q^2$  and  $R = \partial/\partial r$ .

2. Some interesting features arise when  $\tilde{\Phi}(r)$  is a plane wave, i.e.  $\tilde{\Phi}(p) = \delta(p - p')$ . In this case one sees that  $2\Lambda/m$  is the distance between  $x$  and  $y$  when they are projected into any spacelike hyperplane orthogonal to  $p'$ . Now  $j_0(\Lambda)$  vanishes when  $\Lambda = n\pi$  for any integer  $n > 0$ . Thus for a plane wave the split field  $\phi(x, y)$  has a node whenever  $x$  and  $y$  are separated by an integral multiple of the Compton wavelength  $h/m$  (our units are such that  $\hbar = h/2\pi = 1$ , and  $m$  is 'really'  $m/h$ ).

3. A space-time representation for  $\phi^A(x, y)$  can be given as follows:  $\phi^A(x, y) = i X^{AA} \mathcal{V}_0 \tilde{\Phi}_A(r)$ , where  $X = \partial/\partial x$ . Note that  $\mathcal{V}_0 \tilde{\Phi}_A(r)$  satisfies the massless wave-equation in each of  $x$  and  $y$ , and thus acts (in the  $x$  variable) as a Hertz potential. In the case of the field  $\phi^{A'B'C'}(x, y)$  the corresponding formula is given by  $\phi^{A'B'C'}(x, y) = -i X^{AA'} X_B^{B'} Y^{C'B'} \mathcal{V}_0 \tilde{\Phi}_A(r)$ .

4. The  $\hat{C}_2$  and  $\hat{C}_3$  operators, as well as the  $\hat{S}^2$  operator, have been calculated for triple 0-helicity systems by A.S. Popovich (cf. TN10 and his D.Phil. thesis).

5. Cf., for example, W.E. Thirring Principles of Quantum Electrodynamics, Academic Press (1958), Chapter 6.

Gratitude is expressed to APH, RP, and ASP for helpful discussions. RP prompted us to look at the 28 problem, and we also benefited in this connection from conversations with T.B. and M.C.S.

### Twistor Diagrams and Relative Cohomology

The purpose of this note is to describe a generalization of the twistor propagators introduced in (1) and (2). The twistor propagators  $\phi_n$  belong to  $H^2(P^- \times P^*; \Omega(-n-2, -n-2))$  and, for  $n \geq 0$ , have the exact form (2)

$$\phi_n = (w \cdot z)_{n+2} \cdot \log \left( \frac{w \cdot A \bar{B} \cdot z}{w \cdot B \bar{A} \cdot z} \right), \text{ where } A, B \in P^- \text{ are arbitrary.}$$

If we define  $\Omega^- = P^- \times P^{*-}$   $\cap \{w \cdot z = 0\}$ , then

$$\log \left( \frac{w \cdot A \bar{B} \cdot z}{w \cdot B \bar{A} \cdot z} \right) \in H^1(\Omega^-, \Omega), \quad (*)$$

since  $\frac{w \cdot A \bar{B} \cdot z}{w \cdot B \bar{A} \cdot z} \in H^0(\Omega^-, \Omega^*)$  is easily seen to be a line

bundle over  $\Omega^-$  with zero Chern class. Since  $\frac{w \cdot A \bar{B} \cdot z}{w \cdot B \bar{A} \cdot z}$

does not have zero Chern class when considered as a line bundle over  $P^- \times P^{*-}$ , the logarithm in (\*) does not extend to  $P^- \times P^{*-}$ , and the dot product in the definition of the  $\phi_n$  is therefore nonzero.

The following is a generalization of this sort of construction:

Theorem. Let  $X$  be a complex manifold such that the Chern mapping

$$c: H^{m-1}(X; \Omega^*) \longrightarrow H^m(X; \mathbb{Z})$$

is surjective for some fixed  $m$ . Let  $\xi$  be a line bundle over  $X$ , and suppose  $A \subset X$  is the zero set of some section  $f \in \Gamma(X; \xi)$ . If  $H^{m-1}(A; \mathbb{Z}) = 0$ , then for any integer  $n \geq 0$ , there is a natural map

$$P_n: H^m(X, A; \mathbb{Z}) \rightarrow H^m(X; \Omega(\xi^{-n-1})).$$

Proof. We have the exact segment

$$0 = H^{m-1}(A; \mathbb{Z}) \rightarrow H^m(X, A; \mathbb{Z}) \rightarrow H^m(X; \mathbb{Z}) \xrightarrow{p} H^m(A; \mathbb{Z}),$$

so that  $H^m(X, A; \mathbb{Z}) \cong \ker p: H^m(X; \mathbb{Z}) \rightarrow H^m(A; \mathbb{Z})$ . Let

$g \in H^m(X, A; \mathbb{Z}) \subset H^m(X; \mathbb{Z})$ ; we have the commutative diagram

$$\begin{array}{ccccccc} & & H^m(X; \theta) & \rightarrow & H^{m-1}(X; \theta^*) & \xrightarrow{c} & H^m(X; \mathbb{Z}) \rightarrow 0 \\ & \swarrow p & & & \downarrow p & & \\ 0 = H^{m-1}(A; \mathbb{Z}) & \rightarrow & H^{m-1}(A; \theta) & \xrightarrow{\exp} & H^{m-1}(A; \theta^*) & \xrightarrow{c'} & H^m(A; \mathbb{Z}) \end{array}$$

and can therefore find  $\tilde{g} \in H^{m-1}(X; \theta^*)$  such that  $c\tilde{g} = g$ .

$c'p\tilde{g} = pg = 0$ , so there is a unique  $h \in H^{m-1}(A; \theta)$  such that  $\exp h = pg$ .

It is easy to see that  $h$  is well defined on a neighborhood of  $A$  in  $X$ , and we can consider

$$P_n(g) = (\ell)_{n+1} \cdot h \in H^m(X; \theta(\mathfrak{s}^{-m-1})).$$

Since  $H^{m-1}(X; \theta) \subset \ker [\cdot(\ell)_{n+1}]: H^{m-1}(A; \theta) \rightarrow H^m(X; \theta(\mathfrak{s}^{-m-1}))$ ,

$P_n(g)$  is independent of the choice of  $g$  made in defining  $h$ .  $\square$

It is interesting that the key geometrical group in this theorem is the relative cohomology group  $H^m(X, A; \mathbb{Z})$ , rather than  $H_m(X-A; \mathbb{Z})$ , which has been more usually investigated in the analysis of twistor diagrams. As we shall see, however, this construction enables us to evaluate a variety of twistor diagrams which have previously escaped a cohomological formulation, and this suggests that the groups  $H^m(X, A; \mathbb{Z})$  may be the more fundamental.

As an application of this construction, suppose that  $X = P^* \times P^{*-}$  and  $f = w \cdot z$  ( $s = O(1,1)$ ), so that  $A = \mathbb{S}^2$ . It is not hard to check that  $H^4(X, A; \mathbb{Z}) \cong \mathbb{Z}$ , and

$$P_{n+1}(1) \in H^4(P^* \times P^{*-}; O(-n-z, -n-z))$$

are the twistor propagators.

Alternatively, consider the diagram

$$B \xrightarrow{n} \bullet \xrightarrow{-n} C,$$

where  $B \cdot C \neq 0$ . Writing  $\overset{\text{dual}}{B}$  for the plane in  $P$  which is dual to  $B \in P^*$ , etc., it is not hard to see that in order to evaluate this diagram cohomologically, we need to interpret  $\bullet$  as an element of  $H^4(B^\perp \times C^\perp; O(n-3, n-3))$ .

Setting  $X = B^\perp \times C^\perp$  and  $f = w \cdot z$ , it is clear that since  $C \not\subset B^\perp$ ,  $A$  is a fiber bundle with base  $B^\perp$  (i.e., a  $\mathbb{CP}^2$ ) and fiber  $\mathbb{CP}^1$ . Therefore  $\dim H^4(X; \mathbb{Z}) = 3$  and  $\dim H^4(A; \mathbb{Z}) = 2$ , so that  $H^4(X, A; \mathbb{Z}) \cong \mathbb{Z}$ .

$$P_{2-n}(1) \in H^4(B^\perp \times C^\perp; O(n-3, n-3))$$

are the desired cohomology elements.

Other diagrams have been evaluated using this theorem, such as  $\bullet$  and the easy channel of the box diagram. This work is currently being written up for publication elsewhere.

Matt Ginsberg

- (1) Ginsberg, "A Cohomological Scalar Product Construction," ATT, pp. 293-300
- (2) Eastwood and Ginsberg, "Duality in Twistor Theory," Duke Math. J. (1981)

### Extracting Eigenvalues from Homogeneous Twistor Functions

In the application of twistor theory to particle physics, we frequently require that homogeneous functions of  $n$  twistors,  $F(z_i^*)$  where  $i = 1, 2, \dots, n$ , be in eigenstates of certain operators. This condition ensures that the appropriate quantum numbers are encoded in the fields  $\psi^{A_1 \dots A_n}(x)$  generated by the functions  $F(z_i^*)$  via the contour integral formulation

$$\psi^{A_1 \dots A_n}(x) = \oint_{\rho_x} [\{\pi\}] F(z_i^*) \Delta \quad 1$$

where  $\Delta$  denotes the appropriate projective form, namely

$z_1 dz_1 \wedge z_2 dz_2 \wedge \dots \wedge z_n dz_n$ ,  $\rho_x$  denotes the restriction to twistors passing through the space-time point  $x^*$  and where  $\{\pi\}$  is an appropriate spinor coefficient in which the number of occurrences of each of the spinors  $\{\underline{z}\}$  and  $\{\underline{\pi}\}$  is determined by the homogeneity degree of the function  $F(\{\underline{z}\})$  in each of the twistors  $\{\underline{z}\}$ .

We are primarily interested in operators associated with the  $n$  twistor internal symmetry group  $IU(n)$  and many of these can be written in a form involving  $E_j^i$ , the trace-free generators of the  $SU(n)$  subgroup given by

$$E_j^i = \{\underline{z}\} = \{\underline{z}\} - \frac{1}{n} \{\underline{\pi}\} \quad 2$$

The eigenvalues of any operator of the form

$$\hat{O} = E_j^i R \quad 3$$

can be evaluated in a fairly straightforward way. We will need, first of all, the following results:

Lemma 1 (Due to GAJS)

If we define  $\tilde{E}_j^i$  by

$$\tilde{E}_j^i = \eta_j^{A'} \frac{\partial}{\partial \pi_k^{A'}} - \frac{1}{n} \delta_j^i \pi_k^{A'} \frac{\partial}{\partial \pi_k^{A'}} \quad 4$$

then by an application of the differential chain rule we have the following result

$$R \cdot E_j^i f(z_k^*) = \tilde{E}_j^i \rho_x f(z_k^*) \quad 5$$

where of course,  $\rho_x f(z_k^*) = f(ix^{AA'} \pi_k^{A'}, \pi_k^{A'})$  for some  $\pi_k^{A'}$ .

Lemma ii

$$\oint \frac{2}{\partial \pi_{e'}} F \pi_{e'} d\pi_{e'} = 0$$

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where  $F$  is homogeneous of degree -1. The proof follows by an application of Stokes' Theorem to  $d(\pi^{e'} F)$  where we have

$$\begin{aligned} d(\pi^{e'} F) &= \frac{\partial}{\partial \pi_{e'}} (\pi^{e'} F) d\pi_{e'} \\ &= \epsilon^{e'e'} F d\pi_{e'} + \pi^{e'} \frac{\partial}{\partial \pi_{e'}} F d\pi_{e'} \\ &= F d\pi^{e'} + \pi^{e'} \frac{\partial}{\partial \pi_{e'}} F d\pi_{e'} + \pi^{e'} \frac{\partial}{\partial \pi_{e'}} F d\pi_{e'} \\ &= \left( \frac{\partial}{\partial \pi_{e'}} F \right) \pi^{e'} d\pi_{e'} \end{aligned}$$

Consider now a twistor function  $F(z_i^*)$  which is in an eigenstate of some operator  $\hat{O}$  of the form given in (3) with eigenvalue  $e$ . Associated with  $F(z_i^*)$  is a spinor coefficient  $\{\pi\}$  giving rise to a non-vanishing contour integral. Acting on  $F(z_i^*)$  with the operator  $\hat{O}$  we have

$$\oint_A \{\pi\} \hat{O} F(z_i^*) \Delta = e \oint_A \{\pi\} F(z_i^*) \Delta \quad 7$$

However, we may also write

$$\begin{aligned} \oint_A \{\pi\} \hat{O} F(z_i^*) \Delta &= \oint_A \hat{O} \{\pi\} F(z_i^*) \Delta \\ &\quad + \oint_A [\{\pi\}, \hat{O}] F(z_i^*) \Delta \quad 8 \end{aligned}$$

Applying Lemmas i and ii to the first term on the right hand side of (8) we see that the term must vanish so that, comparing (3) and (8) we have

$$[\{\pi\}, \hat{O}] \simeq e \{\pi\} \quad 9$$

where  $\simeq$  denotes equality inside the contour integral.

We can now, by way of an example, use the above technique to show that the eigenvalues of the n-twistor version of the  $J^2$  operator, given by

$$J^2 = S_a S^a \quad 10$$

where  $S_a$  is the spin vector, take the form  $-m^2 j(j+1)$  where  $j$  is an integral multiple of  $\frac{1}{2}$  and where we impose the condition that all the

functions we are dealing with are mass eigenstates. (The mass operator is associated with the inhomogeneous part of  $IU(n)$ .) We need to show first of all, that  $J^2$  has an expression involving  $E_j^k$ . We have<sup>1</sup>

$$S_a = \boxed{\square} - \frac{1}{2} \theta \boxed{\square} + 2 \boxed{\square} = \boxed{\square} \quad 11$$

where  $\theta = \boxed{\square}$  so that  $S_a$  may be written in the form

$$\boxed{\square} = \boxed{\square} + \alpha \boxed{\square}$$

$$= \boxed{\square} + \alpha' \boxed{\square}$$

$$= \boxed{\square} + \alpha'' \boxed{\square} \quad 12$$

Also, making use of the fact that the spin operator is orthogonal to the momentum operator, we have

$$S_a P^a = \boxed{S} \boxed{\square} = 0$$

$$\Rightarrow \alpha'' = \frac{1}{m^2} \boxed{\square}$$

13

where  $m^2$  is the eigenvalue of  $\boxed{\square} \approx \boxed{\square}$ . Thus  $S_a$  can be expressed in the form  $E_j^k R$  as required.

Consider now a function  $F(\boxed{\square})$  homogeneous of degree  $-h_1-2, -h_2-2, \dots, -h_n-2$ , in each of the  $n$  twistors  $\{\boxed{\square}\}$  respectively. For simplicity, we shall consider the case where all the  $h_i$  are positive. The general case with some  $h_i$  negative (so that the contour integral involves  $\oint_{\partial\omega}$  term) may be dealt with by a straightforward extension of the following discussion.

The field generated by the function  $F(\boxed{\square})$  is given by

$$\Psi^{A \dots D}(x) = \oint_P \boxed{\square} \dots \boxed{\square} F(\boxed{\square}) \Delta \quad 14$$

where  $\tau = \sum_{i=0}^n h_i$ . We have not yet specified the symmetry properties of the spinor indices on  $\Psi^{A \dots D}(x)$ . Looking first at the case of  $\Psi^{(A \dots D)}(x)$ , we have

$$[\underbrace{\{ \dots \}}_r, s^a s_a] \simeq [\underbrace{\{ \dots \}}_r, s^a] s_a \simeq [\underbrace{\{ \dots \}}_r, s^a] s_a \quad 15$$

Now

$$[\underbrace{\{ \dots \}}_r, s^a] = r [\{, s^a\}] \underbrace{\{ \dots \}}_{r-1} \quad 16$$

using symmetry and the fact that

$$[[\{, s^a\}], \{]] = 0 \quad 17$$

so that

$$\begin{aligned} [\underbrace{\{ \dots \}}_r, s^a s_a] &= r [\{, s^a\}] (\underbrace{r-1}_{r-2}) [\{, s_a\}] \underbrace{\{ \dots \}}_{r-2} \\ &\quad + r [[\{, s^a\}], s_a] \underbrace{\{ \dots \}}_{r-1} \end{aligned} \quad 18$$

Using the following commutation relation

$$[\{, \square\}] = - \square \{ \quad 19$$

it is straightforward to see that

$$[\{, \square] s = - \square s \quad 20$$

and

$$\begin{aligned} [[\{, \square], s] &= \frac{3}{2} \square \{ s \\ &= -\frac{3}{4} \square \{ s = -\frac{3}{4} m^2 \{ s \end{aligned} \quad 21$$

where we have used the identity <sup>2</sup>

$$\square \{ \{ = -\frac{1}{2} \{ \{ \square \quad 22$$

Also we have

$$[\{, s] [\{, s] = -\frac{1}{4} m^2 \{ \{ \quad 23$$

so substituting (20), (21) and (22) into (18) we obtain

$$[\underbrace{\{ \dots \}}_r, s^a s_a] \simeq -\frac{r}{2} (\frac{r}{2} + 1) \circled{m^2} \underbrace{\{ \dots \}}_r \quad 24$$

Thus, by our previous argument we see that if  $\psi^{(A..D)}(x)$  given by (14) is non-vanishing, then  $F(z_i^*)$  satisfies

$$j^* F(z_i^*) \simeq -j(j+1)^{m^2} F(z_i^*) \quad 25$$

where  $j = \frac{r}{2}$ .

In the case where  $m$  pairs of indices are antisymmetrized (corresponding to  $m$  terms of the form  $\{\}$  in (14)) the eigenvalues of  $J^2$  are found to be  $-j(j+1)m^2$  where

$$j' = \frac{r}{2} - m \quad 26$$

A similar discussion for the case where some of the homogeneity degrees  $h_i$  are negative, so that we have fields of the form:

$$\Psi^{(r)}(x) = \oint_{\infty} \left\{ \left\{ \dots \left[ \left[ \right] \right] \right\} \dots \right\} F(\zeta) \Delta \quad 27$$

yield eigenvalues  $-p_1(p_2+1)m^2$  where  $p$  is just the number of symmetrized spinor indices.

Clearly, this technique can be used to evaluate eigenvalues of various other operators such as the Casimir operators

$$C_2 = E_j^i E_i^j, \quad C_3 = E_i^{[i} E_j^{j]} E_k^{k]} \quad \text{etc.}$$

as well as, for example, in the base of functions of three twistors, eigenvalues of  $Q$ ,  $Y$ ,  $I^2$ ,  $I_3$ ,  $B$  ( $B = 2m^2 \left[ \begin{smallmatrix} \zeta \\ \bar{\zeta} \end{smallmatrix} \right] + 2$ ). Also, given momentum eigenstates we can evaluate the eigenvalues of a chosen  $z$ -component of the spin vector  $s_z$ . It should also be mentioned that this technique may be used to evaluate transition coefficients for operators such as  $I_{\pm}$  or  $J_{\pm}$  for which the functions  $F(z_i)$  are not eigenstates.

M. Sheppard.

refs.

- 1 G.A.J.S. to be pub. see also Quantum Gravity
- 2 G.A.J.S. to be pub.

### Non-Projective Propagators

In this note, we will present a new description of the twistor propagators, based on the geometry of non-projective twistor space. It seems possible that the ideas involved can be extended to describe propagators coupled to electromagnetic or self-dual Yang-Mills fields.

Let  $\pi: T \rightarrow P$  be the usual projection, and for  $U \subset P$ , denote  $\pi^{-1}(U)$  by  $\tilde{U}$ . Then there are always maps

$$H^p(U; \mathcal{O}(r)) \xrightarrow{\pi^*} H^p(\tilde{U}; \mathcal{A}(r)) \xrightarrow{i^*} H^p(\tilde{U}; \mathcal{O}),$$

where  $\mathcal{A}(r)$  is the subsheaf of  $\mathcal{O}$  of germs of functions homogeneous of degree  $r$ , so that we have the short exact sheaf sequence on  $T$ :

$$0 \rightarrow \mathcal{A}(r) \xrightarrow{i} \mathcal{O} \xrightarrow{\pi^*} \mathcal{O} \rightarrow 0.$$

$\frac{\partial}{\partial z^k} - r$

This sequence and the maps  $\pi^*$  have been discussed by Eastwood in (1). We will abuse notation and also write

$$\pi^*: H^p(U; \mathcal{O}(r)) \rightarrow H^p(\tilde{U}; \mathcal{O}).$$

To evaluate twistor diagrams on  $P$ , we use the isomorphism of Serre duality

$$H^3(P; \mathcal{O}(-4)) \cong \mathbb{C}.$$

Similarly, we may represent non-projective integration by a map

$$I: H^3(T; \mathcal{O}) \rightarrow \mathbb{C}.$$

This map is given as follows: Let  $\omega$  be the canonical  $(4,0)$ -form on  $T$ ,  $\omega = \epsilon_{\alpha\beta\gamma\delta} dz^\alpha \wedge d\bar{z}^\beta \wedge dz^\gamma \wedge d\bar{z}^\delta$ . For  $f \in H^3(T; \mathcal{O})$  a  $(0,3)$ -form,  $\omega \wedge f$  is a 7-form on  $T$  which can be integrated over an  $S^7$  surrounding the origin in  $\mathbb{C}^4$  to give a complex number. This procedure is described by Penrose in (2), where it is shown that the following diagram commutes:

$$\begin{array}{ccc} H^3(P; \mathcal{O}(-4)) & \simeq & \mathbb{C} \\ \downarrow \pi^* & & \downarrow i \\ H^3(T; \mathcal{O}) & \xrightarrow{\text{Int}} & \mathbb{C} \end{array}$$

Penrose also shows that  $i\pi^*: H^3(P; \mathcal{O}(n)) \rightarrow \mathbb{C}$  is the zero map for  $n \neq -4$ .

Lemma.  $\pi^*\phi \neq 0 \in H^1(\widetilde{\mathcal{V}}; \mathcal{O})$ , where  $\widetilde{\mathcal{V}} = P \times_{P^*} \{n \leq w \cdot z \leq 0\}$ , and  $\phi \in H^1(\mathcal{V}; \mathcal{O})$  is used to construct the twistor propagators

$$\phi_n = (w \cdot z)_{n+2} \cdot \phi \in H^2(P \times_{P^*} \{n \leq w \cdot z \leq 0\}; \mathcal{O}(-n-2, -n-2))$$

(see (3). or (4)).

Proof. Suppose  $f \in H^1(\widetilde{P^+}; \mathcal{O}(-3))$  and  $g \in H^1(\widetilde{P^{*+}}; \mathcal{O}(-3))$  correspond to spacetime fields with non-vanishing inner product, so that

$$f \cdot g \cdot \phi \cdot \frac{1}{w \cdot z} \neq 0 \in H^0(P \times_{P^*} \{n \leq w \cdot z \leq 0\}; \mathcal{O}(-4, -4)).$$

It follows that

$$\pi^*(f \cdot g \cdot \phi \cdot \frac{1}{w \cdot z}) \neq 0 \in H^0(T \times T^*, \mathcal{O}),$$

since  $\pi^*$  and dot product clearly commute, we see that

$$\pi^k f \cdot \pi^k g \cdot \pi^k \phi \cdot \pi^k \frac{1}{w \cdot z} \neq 0$$

and hence that  $\pi^k \phi \neq 0 \in H^1(\tilde{U}; \mathcal{O})$ .  $\square$

Next, recall that on  $\tilde{U}$ ,  $\phi$  is defined by

$\exp(\phi) = \mathcal{O}(1, -1) \in H^0(\tilde{U}; \mathcal{O}^*)$ . The key point in the definition of  $\phi$  is that  $\mathcal{O}(1, -1)$ , when restricted to  $\tilde{U}$ , has vanishing Chern class  $(3, 4)$ . Transition functions for  $\mathcal{O}(1, -1)$  may be taken to be

$$f_{12} = \frac{w \cdot A \bar{B} \cdot z}{w \cdot B \bar{A} \cdot z},$$

defined on  $U_1 \cap U_2$ , where  $U_1 = \{w \cdot A \bar{B} \cdot z \neq 0\}$  and  $U_2 = \{w \cdot B \bar{A} \cdot z \neq 0\}$  cover  $\tilde{U}$ . Since  $w \cdot A \bar{B} \cdot z \in H^0(U_1; \mathcal{O}^*)$ , we have:

Lemma.  $\pi^* \exp \phi = 0 \in H^1(\tilde{U}; \mathcal{O}^*)$ .  $\square$

Theorem. Given the usual short exact sequence on  $\tilde{U}$ :

$$0 \rightarrow \mathbb{Z} \xrightarrow{i} \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \rightarrow 0,$$

there is a  $k \in H^1(\tilde{U}; \mathbb{Z})$  such that

$$\pi^* \phi = ik \in H^1(\tilde{U}; \mathcal{O}).$$

Writing  $i$  for the natural injection  $\mathbb{Z} \rightarrow \mathcal{O}(0, 0)$  as well, we have

$$\pi^* \phi = ik \in H^1(\tilde{U}; \mathcal{O}(0, 0))$$

also.

Proof. This follows immediately from an examination of the commutative diagram:

$$\begin{array}{ccc}
 H^1(\Omega^{\pm}; \theta) & \xrightarrow{\exp} & H^1(\Omega^{\pm}; \theta^*) \\
 \downarrow \pi^* & & \downarrow \pi^* \\
 i \swarrow & H^1(\tilde{\Omega}^{\pm}; \mathcal{H}(0,0)) & \\
 & \downarrow i^* & \\
 H^1(\tilde{\Omega}^{\pm}, \mathbb{Z}) & \xrightarrow{i} & H^1(\tilde{\Omega}^{\pm}; \theta) \xrightarrow{\exp} H^1(\tilde{\Omega}^{\pm}; \theta^*)
 \end{array}$$

along with the lemma.  $\square$

In fact, it is not hard to see that  $H^1(\tilde{\Omega}^{\pm}, \mathbb{Z}) = \mathbb{Z}$ , and  $\phi$  can be taken to correspond to the generator of this group. Just as  $\phi \in H^1(\Omega^{\pm}; \theta)$  does not extend to either  $P^- \times P^{*-}$  or to all of  $\Omega$  (ambitwistor space), we also have

$$H^1(\tilde{\Omega}^{\pm} \times \tilde{\Omega}^{*\pm}, \mathbb{Z}) = 0 = H^1(\tilde{\Omega}^{\pm}, \mathbb{Z}).$$

The appearance of the nonprojective space  $\tilde{\Omega}^{\pm}$  in this construction is very suggestive. It seems possible that, by investigating the topology of similar subvarieties in the product  $V^- \times V^{*-}$  for  $V^-$  an arbitrary vector bundle over  $P^-$ , the propagators  $\phi_n$  can be coupled to electromagnetic or self-dual Yang-Mills fields.

Matt Ginsberg

- (1) Eastwood, "On Non-Projective Twistor Cohomology," TN10
- (2) Penrose, in Quantum Gravity
- (3) Eastwood and Ginsberg, "Twistor Propagators," TN9
- (4) Eastwood and Ginsberg, "Duality in Twistor Theory," Duke Math. J., 1981

## Helicity Raising Operators and Conformal Supersymmetry

Lane Hughston & Tom Hurd

Twistors and dual twistors have a natural interpretation as helicity raising and lowering operators acting on ZRM fields [see e.g. Penrose §2.13 and Eastwood §2.14 in Advances in Twistor Theory]. In what follows we hope to show that the natural extension of this interpretation to the second quantized picture is just what people call conformal supersymmetry.

Consider, for a start, the Fock space  $F_0$ , of free ZRM scalar many-particle states, and its associated free field operators:  $\Psi^-(x)$  (creation) and  $\Psi^+(x) = \overline{(\Psi^-(x))}$  (annihilation). The action on one-particle states of the helicity lowering operator parametrised by a dual twistor  $Q_d$  is such that:

$$\begin{array}{ccc} f_{-2}(z^*) & \xrightarrow{Q_d z^*} & g_1(z^*) \\ \uparrow \text{P-transform} & & \downarrow \text{P-transform} \\ |\phi(x)\rangle & \xrightarrow{Q_d R^d} & |\phi_n(x)\rangle \end{array}$$

is a commutative diagram, where  $R^d$  is the generator of the transformation. The action of  $R^d$  on single-particle states is therefore given by:

$$(1) \quad Q_d R^d : |\phi(x)\rangle \longrightarrow | -i q^{B^1}(x) \nabla_{BB^1} \phi(x) + q_B \phi(x) \rangle$$

where  $q^{B^1}(x) = q^{B^1} + i x^{BB^1} q_B$  and  $Q_d = (q_B, q^{B^1})$ .

From this we see that  $R^d$  will generate transformations out of  $F_0$ , onto the Fock space  $F_{-1/2}$  of helicity  $-1/2$  ZRM states. We are led, then, to consider  $F_0 \otimes F_{-1/2}$  as the total Fock space and to introduce the field operators:  $(\Psi_{A^1}^-(x))$  (creation) and  $(\Psi_A^+(x)) = \overline{(\Psi_{A^1}^-(x))}$  (annihilation). This is the minimal non-trivial representation space for the action of  $R^d$ . (continued from previous page)

The action (1) (and the natural action on many-scalar-particle states) will be induced if the commutation relation:

$$(2) \quad [Q_d R^d, \Psi^-(x)] = -q^{B^1}(x) \Psi_{B^1}^-(x)$$

is assumed, together with the specification  $R^d |0\rangle = 0$ . For example, the action on a one-particle state is got as follows: (continued from previous page)

$$\begin{aligned} Q_d R^d |\phi\rangle &= Q_d R^d \cdot \frac{i}{2} \int d^3 z^{AA'} \overleftrightarrow{\Psi^-(x)} \nabla_{AA'} \phi(x) |0\rangle \quad (\text{defn of } |\phi\rangle) \\ &= \frac{i}{2} \int d^3 z^{AA'} [\overleftrightarrow{Q_d R^d}, \overleftrightarrow{\Psi^-}] \nabla_{AA'} \phi |0\rangle \\ &= \frac{i}{2} \int d^3 z^{AA'} \left( -q^{B^1}(x) \overleftrightarrow{\Psi_{B^1}^-(x)} \right) \nabla_{AA'} \phi |0\rangle \end{aligned}$$

$$\begin{aligned}
 &= -i \int d^3x \delta^{AA'} \nabla_{A'} (\bar{\Psi}_A^\dagger(x) \Psi_B^-) \phi |0\rangle \quad (\text{integrate by parts}) \\
 &= -i \int d^3x \delta^{AA'} [i \delta_A^{B'} q_B \bar{\Psi}_{B'} + q^{B'}(x) \nabla_{AB'} \bar{\Psi}_{A'}] \phi |0\rangle \quad (\nabla_A [\delta_B \bar{\Psi}_A] = 0) \\
 &\approx -i \int d^3x \delta^{AA'} \bar{\Psi}_{A'} [q_A \phi - i q^{B'}(x) \nabla_{AB'} \phi] |0\rangle \quad (\text{integrate by parts}) \\
 &= [-i q^{B'}(x) \nabla_{BB'} \phi + q_B \phi] |0\rangle
 \end{aligned}$$

The natural action of  $R^a$  on states containing helicity  $\pm \frac{1}{2}$  particles is determined if we impose the anticommutation relation:

$$\{R^a, \Psi_A^\dagger\} = 0.$$

The anticommutation relations

$$\{R^a, R^b\} = 0$$

are now derived from (2) and the assumed statistics of the field operators.

The adjoint operator  $\bar{R}_a$ , defined such that  $\langle \bar{R}^a | \phi \rangle$   $= (\langle \Psi | \bar{R}_a) | \phi \rangle = \langle \Psi | (\bar{R}_a | \phi \rangle)$  for all  $|\phi\rangle$  and  $|\Psi\rangle$ , maps states as follows:

$$P^a \bar{R}_a : |\phi_A(x)\rangle \longrightarrow |-\bar{p}^a(x) \phi_A(x)\rangle$$

$$P^a \bar{R}_a : |\phi(x)\rangle \longrightarrow 0 \quad \text{for fixed } P^a \leftrightarrow p^a(x).$$

Summarizing, we have:

$$[R^a, \Psi^\dagger] = -\sum_A \delta^{AB'} \bar{\Psi}_{B'}$$

$$\{R^a, \Psi_{A'}^\dagger\} = 0$$

$$[\bar{R}_a, \Psi^\dagger] = 0$$

$$\{\bar{R}_a, \Psi_{A'}^\dagger\} = -i \bar{p}_{\beta A'} \bar{\Psi}^\dagger \quad \text{where } \bar{p}_{\beta A'} = \left[ i \nabla_{BA'} \right]$$

together with the adjoint equations  $[\bar{R}_a, \bar{\Psi}^\dagger] = \bar{p}_a^A \Psi_A^\dagger$  etc.

The conformal group,  $SU(2,2)$ , is generated by the tracefree part of the anticommutator:

$$\frac{1}{2} \{R^a, \bar{R}_b\} = E_\rho^a = \begin{bmatrix} p_A^B + \frac{1}{2} \delta_A^B (s+d) & k^{AB'} \\ p_A^B & p_A^B + \frac{1}{2} \delta_A^B (s-d) \end{bmatrix}$$

One can check that

$$[E_\rho^a, R^Y] = \delta_\rho^Y R^Y \quad \text{and} \quad [E_\rho^a, \bar{R}_Y] = -\delta_\rho^Y \bar{R}_Y$$

and from these we derive the commutator for the generators of  $U(2,2)$ :

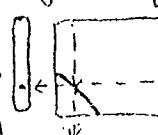
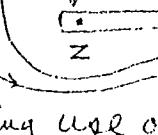
$$[E_\rho^a, E_\sigma^Y] = -\delta_\rho^Y E_\sigma^a + i \delta_\sigma^a E_\rho^Y.$$

The trace  $s = \frac{1}{2} E_\alpha^a$  is the total helicity operator, and is changed by  $\pm \frac{1}{2}$  by  $R^a$  and  $\bar{R}_a$ .

Finally we note that the algebra of Poincaré supersymmetry is got from the above by forgetting the  $W^A, B^{A'}$  parts of  $R^a$  and  $\bar{R}_a$  and the  $k^a$  and  $d$  parts of  $E_\rho^a$ .

## - A New Angle on the Googly Graviton.

Recall (TN 8,9; A.T.T. pp 168–176) the “googly” idea for describing self-dual (as opposed to anti-self-dual) solutions of Einstein’s vacuum equations in terms of some sort of deformed twistor space  $\mathcal{T}$ . Here  $\mathcal{T}$  is “flat” as a “non-linear graviton” (of the old-fashioned type) but now the information of the self-dual curvature is to be stored in the way that the line  $\mathbf{I}$  sits in  $\mathbb{P}\mathcal{T}$ , or rather, in the way that  $\mathbb{S}$  (the  $w^A$  spin-space) sits in  $\mathcal{T}$ , in  $0 \rightarrow \mathbb{S} \rightarrow \mathcal{T} \rightarrow \mathbb{S}^* \rightarrow 0$  (in some sense). This is to be such as to determine a 4-parameter family of maps  $\alpha : \mathcal{T} \rightarrow \mathbb{S}$  (the “googly maps”) which generalize the flat-space googly maps  $\alpha : (w^A, \pi_A) \mapsto w^A - ix^{A'}\pi_{A'}$ , the regularity conditions on such maps being defined, at  $\mathbb{S}$ , by “the way  $\mathbb{S}$  sits in  $\mathcal{T}$ ”.

Unfortunately, no good idea seemed forthcoming about how actually to fix these regularity conditions and, in particular, no relation had emerged for determining them, in the weak field limit, from a twistor function  $f(z^\alpha)$  homogeneous of degree -6. In the standard “non-linear graviton” a +2-homogeneous twistor function  $g$  plays a role as an infinitesimal transition function between two patches:  $x = z + \epsilon \int d_\alpha g(z)$ . Our “new angle” is to view  $f$  not as providing a transition function between patches, but somehow as providing a co-transition function between co-patches! The guiding idea is that we must reverse directions of maps (since the “n.l.g.” setup  $\mathcal{T} \xrightarrow{\alpha} \mathbb{S}^*$  is to dualize to the “googly” setup  $\mathbb{S} \xleftarrow{\alpha} \mathcal{T}$ ). A “patching” may be thought of in terms of a subspace of the product space of  $\{x\}$  with  $\{z\}$ :  ; and we need to use the two canonical projections  $(x, z) \mapsto x$  and  $(x, z) \mapsto z$ . So, dually, a “co-patching” may be thought of in terms of a factor space of this product space  where now we seem to be making use of the two canonical injections  $x \mapsto (x, 0)$  and  $z \mapsto (0, z)$ .

To specify such a co-patching we require some sort of foliation of  $\{x\} \times \{z\}$ , the 4-dimensional leaves of which integrate to the 4-surfaces that we factor out by to get  $\mathcal{T}$ . This might possibly relate to earlier googly ideas involving non-Hausdorff manifolds since such 4-surfaces might sometimes bifurcate.

How can we specify such a foliation, bearing in mind that

in the weak-field limit a -6-homogeneity function  $f$  must be involved? The answer seems to lie in the twistor quantization procedure  $\tilde{f} \mapsto -\partial_z$ . Thus, instead of a dual-twistor transition relation  $\psi = w + \epsilon [\tilde{\partial}_w \tilde{f}(w)]$ , we have  $\partial_x = \partial_z - \epsilon [\tilde{\partial}_z, \tilde{f}(\partial_z)]$ . The reason for the Lie bracket arising on the right here is that quantum mechanically we would think of the  $\tilde{\partial}_w$  of  $\epsilon \tilde{\partial}_w \tilde{f}(w)$  as acting on everything to the right of it, including some unmentioned ket  $|\psi\rangle$  which is understood to be sitting over on the right, so that term should, instead, be written as  $\epsilon [\tilde{\partial}_w, \tilde{f}(w)]$ , to make sure that only the  $\tilde{f}$  is differentiated.

A function  $\Phi(x, z)$  is deemed to be defined on the googly space  $T$  if it satisfies the equation  $\partial_x \Phi = \partial_z \Phi - \epsilon [\tilde{\partial}_z, \tilde{f}(\partial_z)] \Phi$ , this being the statement that it is constant along the foliation. Now this is clearly a bit of a mess, because  $\tilde{f}(w)$ , which is the twistor transform of the -6-function  $f(z)$ , is a +2-function which is nothing like a polynomial, so  $\tilde{f}(\partial_z)$  is not simple to define. We can get some insights by considering the case of an elementary state. If we take

$$\tilde{f}(w_\alpha) = \frac{(w_0)^p (w_1)^q}{p! q! (w_2)^{r+1} (w_3)^{s+1}}, \text{ then } f(z^\alpha) = \frac{(z^2)^r (z^3)^s (-1)^{r+s}}{r! s! (z^0)^{p+1} (z^1)^{q+1}}$$

where the coordinates  $z^0, z^1, z^2, z^3$  and dual coordinates  $w_0, w_1, w_2, w_3$  refer to some arbitrary basis (bearing no special relation to  $I^{\alpha\beta}$ ). We need to interpret

$$\tilde{f}\left(\frac{\partial}{\partial z^\alpha}\right) = \frac{\left(\frac{\partial}{\partial z^0}\right)^p \left(\frac{\partial}{\partial z^1}\right)^q}{p! q! \left(\frac{\partial}{\partial z^2}\right)^{r+1} \left(\frac{\partial}{\partial z^3}\right)^{s+1}}$$

and for this we can use

$$\int_0^u \left( \int_0^{u_1} \left( \int_0^{u_2} \cdots \left( \int_0^{u_{n+1}} \varphi(u_{n+1}) du_{n+1} \right) \cdots du_3 \right) du_2 \right) du_1 = \int_0^u (-1)^n \frac{(x-u)^n}{n!} \varphi(x) dx$$

(integrate repeatedly by parts) for the denominator terms and

$$\frac{1}{p!} \left( \frac{d}{du} \right)^p \varphi(u) = \frac{1}{2\pi i} \oint \frac{\varphi(x)}{(x-u)^{p+1}} dx$$

for the numerator terms. This gives

$$\tilde{f}\left(\frac{\partial}{\partial z^\alpha}\right) \psi(z^\alpha) = \frac{1}{(2\pi i)^2} \oint f(Q^\alpha - Z^\alpha) \psi(Q^\alpha) d^4 Q,$$

using some suitable contour-with-boundary in  $Q$ -space. This

formula now makes no reference to the fact that we are using an elementary state, so we can extend by linearity to an arbitrary pair  $\tilde{f}, f$  which are twistor transforms of one another. The condition for  $\Phi$  to be defined on  $T$  now becomes

$$\begin{aligned} (\partial_x - \partial_z) \Phi(x, z) &= \frac{-\epsilon}{(2\pi i)^2} \oint \left\{ \bar{z} f(\bar{z}) - f(\bar{z}) \bar{z} \right\} \Phi(x, \bar{z}) d^4 Q \\ &= \frac{\epsilon}{(2\pi i)^2} \oint \bar{k} f(\bar{k}) \Phi(x, z+k) d^4 R \end{aligned}$$

where the substitution  $k = \bar{z} - z$  has been made, and the contour boundary correspondingly moved.

Note that  $[\partial_x \Phi] = [\partial_z \Phi]$ , which is to say that any such  $\Phi$  is defined on two-tailed twistor space, i.e. the 6-dimensional space of pairs  $(x, z)$  factored out by the equivalence relation:

$$(x, z) \equiv (x + \epsilon, z - \epsilon), \quad \epsilon \text{ arbitrary.}$$

Thus, a two-tailed twistor has two "IT-parts", but only one "W-part" namely the sum of the W-parts of  $x$  and of  $z$ . This is the dual concept to that of a two-headed twistor, which is a pair  $(x, z)$  subject to the restriction  $|x| = |z|$ , so that there are two W-parts now, but only one IT-part. The standard (left-handed) non-linear graviton arises as a subspace of two-headed twistor space, while the googly (right-handed) one arises as a factor space of two-tailed twistor space (at least in the simplest case when only two patches or copatches are needed).

How do we define the googly maps? The motivating idea (which seems not to be quite right — although this is not absolutely clear) is as follows. We seek, for each googly point  $x$ , a map  $\tau: T \rightarrow S$  which is the sum of a function of  $x$  and a function of  $z$ , i.e.

$$\tau^A(x, z) = \sigma^A(x) + \kappa^A(z), \quad [\sigma \text{ and } \kappa \text{ each homogeneous of degree one.}]$$

We require  $\tau^A$  to satisfy the equation that ensures

that it be defined on  $\mathcal{T}$ ; also we require ("identity on  $\mathbb{I}$ ")

$$\sigma^A(X^B, 0) = X^A, \quad \kappa^A(Z^B, 0) = Z^A.$$

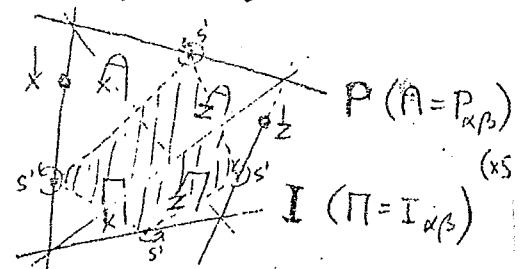
These conditions ensure that, in the case  $f = 0$ , we get the standard flat-space googly maps. Interpreting the spinor index as a "hooked lower twistor index", we would obtain  $\sigma^A(k) = -\lfloor A$  and  $\kappa^A(z) = \lfloor A$  in the flat space case, where  $A = P_{\alpha\beta}$  represents the point defining the googly map, described in the standard way ( $P_{\alpha\beta} = -P_{\beta\alpha}$ ,  $P_{\alpha\beta}P_{\gamma\delta} = 0$ ,  $P_{\alpha\beta} I^{\alpha\beta} = 2$ ). In the non-flat case we would have an additional part to  $\sigma^A$  and  $\kappa^A$  of order  $\epsilon$ , given by a contour-with-boundary integral expression. This appears to be not quite consistent. However, a slight modification of the integral expression works!

For this, we consider not quite  $\sigma^A$ ,  $\kappa^A$  and  $\tau^A$ , but slight generalizations  $\sigma(x)$ ,  $\kappa(z)$ ,  $\tau(x, z)$ , and the googly maps are obtained by appropriately "hooking" these expressions. We take  $\sigma(x) = \bar{x} + O(\epsilon)$ ,  $\kappa(z) = \bar{z} + O(\epsilon)$  and  $\tau(x, z) = \bar{x} + \bar{z} + O(\epsilon)$ . Substituting  $\tau$  for  $\Phi$  in the above integral,

$$\bar{\partial}_x \sigma(x) - \bar{\partial}_z \kappa(z) = \frac{\epsilon}{(2\pi i)^2} \int \int_R \int_R f(R) d^4 R$$

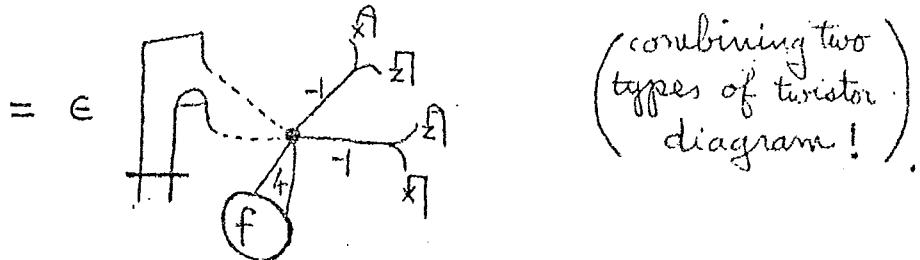
is the expression that we come up with. We have to find an appropriate location for the boundary surfaces for the contour, anticipating (from the above) that the contour topology is, appropriately,  $S^1 \times S^1 \times \mathbb{R} \times \mathbb{R}$  with a boundary probably looking like  $\boxed{\square} \times S^1 \times S^1$  (i.e.  $S^1 \times S^1 \times S^1$ ) (... fill in the square for the contour itself).

What suggests itself is that the boundary lies successively in the planes  $x\bar{x}, z\bar{z}, \bar{x}\bar{z}, x\bar{z}$  and then back to  $x\bar{x}$  again, the contour being a quadrilateral area ( $\times S^2 \times S^2$ ) spanning these planes. Then the integral yields a sum of a function of  $x$  and a function of  $\bar{z}$ , as it must do, for consistency with the left-hand side, but unfortunately it



- fails the second test, which is that applying a further  $\partial_x$  on the left and skewing with the next "leg" we should get zero. This problem is remedied by replacing our expression by its skew part:

$$\boxed{\partial_x \sigma(k) - \partial_z \kappa(z)} = \frac{e}{(2\pi i)^2} \oint \boxed{R} \bar{f} R f(k) d^4 R$$



To obtain the googly maps we can evaluate  $\boxed{\partial_x \sigma}$  (or  $\boxed{\partial_z \kappa}$ ), where the freedom in  $\sigma$  may be supposed restricted by the condition  $\overset{x}{\sigma} = 0$  (and  $\overset{z}{\kappa} = 0$ ) if desired;  $\sigma$  (and  $\kappa$ ) being homogeneous of degree one, so  $\boxed{\partial_x \sigma} = -\boxed{\sigma} = \sigma^A$  in this case;

Though the motivation for this expression has become somewhat muddy just at the end (!), one can check the formula in the case of the linearized Eguchi-Hanson elementary state, for which  $f(z) = \frac{1}{(z^3)^2}$  (cf. G.B.-S. & R.P. in TN9) and find (cf. also A.P.H. in TN11) that

$$\boxed{\partial_x \sigma} = \frac{\left( \boxed{B} \overset{A}{\cancel{x}} \overset{B}{\cancel{x}} - \boxed{A} \overset{B}{\cancel{x}} \overset{A}{\cancel{x}} \right) \overset{A}{\psi} \overset{B}{\psi}}{(\boxed{AB} \overset{B}{\cancel{x}} \overset{A}{\cancel{x}})^2} \quad (\times \text{numerical factor?})$$

which actually is the correct answer (as worked out by P.L., TN11)! Since every linear helicity +2 field is linearly dependent on such elementary states, this shows that the final formula works for a general  $f$ .

[Note: in the above, I have used  $\phi = -\boxed{A} = I^{\alpha\beta} P_{\alpha\beta} (= A_\alpha^\mu \text{ of P.L.'s article})$  and  $\psi = -\boxed{B}$ , so  $A_\alpha = \phi$ ,  $\Gamma\psi = \Pi$ ,  $\Gamma\phi = 0 = \Gamma\Pi$ ,  $\Theta = 2$ ,  $\Psi = \psi$ ,  $\Phi = 0 = \Psi$ ,  $\Phi + \Psi = 1$ .]

Unfortunately it is not yet clear how to work all this when  $T$  is not merely infinitesimally co-deformed. The muddy logic at the end doesn't help matters. Further work is in progress.  
(Thanks to K.P.T.)

Roger Penrose

The Googly Maps for the  
Eguchi-Hanson/Sparling-Pod Graviton

*S. Sen*

In flat Twistor space a finite point of  $\mathbb{C}\mathbb{M}$  is represented by the googly map  $\tilde{\mathcal{S}}_x^A : \mathbb{T} \rightarrow S^*$  given by

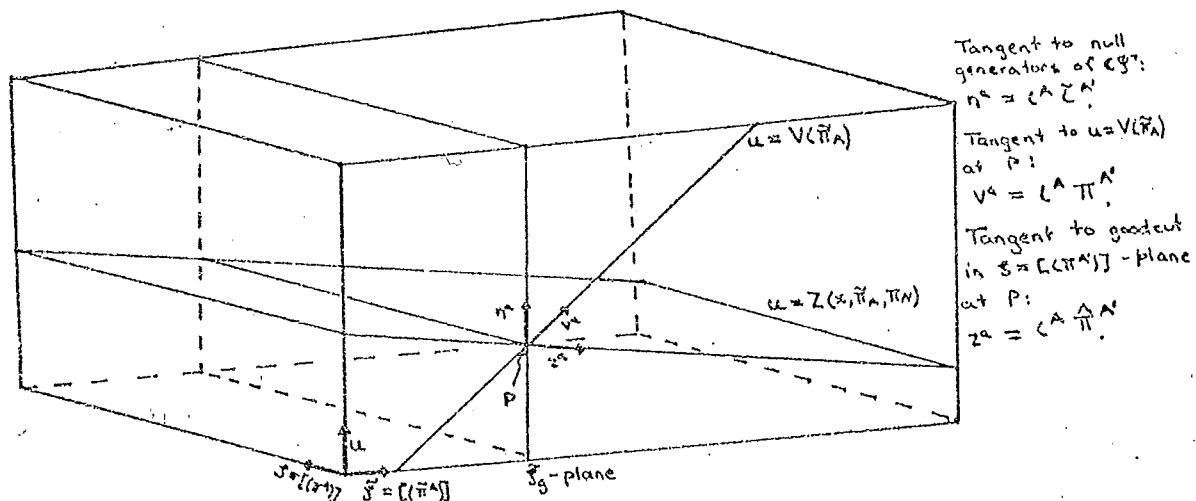
$$(\omega^A, \pi_A) \xrightarrow{\tilde{\mathcal{S}}_x^A} \omega^A - ix^{AA'}\pi_{A'}$$

If  $P_{\alpha\beta}$  (normalized by  $I^{\alpha\beta} P_{\alpha\beta} = 2$ ) represents  $x$  (via the Klein correspondence) then the googly map  $\tilde{\mathcal{S}}_x^A$  can be written

$$Z^A \mapsto I^{\alpha\beta} P_{\alpha\beta} Z^A,$$

( $A_a^b = I^{\alpha\beta} P_{\alpha\beta}$  is  $\dagger$  in R.P.'s diagrammatic notation) where  $S^*$  is identified with  $\mathbb{I}$ . Geometrically,  $A_a^b Z^a$  is the intersection (in  $\mathbb{PT}$ ) of the line  $I$  with the plane  $(P_{\alpha\beta} Z^a)$  containing the line  $P_{\alpha\beta}$  and the point  $[Z^a]$ . These planes, for a given  $x$ , are of course the dual twistors  $[(\sigma_A, -ix^{AA'}\sigma_A)]$ ,  $\sigma^A \in S^*$ .

The googly map  $\tilde{\mathcal{S}}_x^A$  can also be described in terms of the asymptotic twistor structure of  $\mathbb{C}\mathbb{M}$ . The situation is depicted in the following standard diagram of  $\mathbb{C}\mathbb{J}$ .



The function  $u = Z(x, \pi_A, \pi_A')$ , for fixed  $x$ , describes the good cuts of  $\mathbb{C}\mathbb{J}$ . Now, the surface  $u = Z(x, \pi_A, \pi_A')$  is ruled by the dual twistor lines corresponding to the dual twistors  $[(\sigma_A, -ix^{AA'}\sigma_A)]$ ,  $\sigma^A \in S^*$ . An arbitrary twistor  $(\omega^A, \pi_A)$  determines a twistor line  $u = V(\pi_A) = -i\omega^A \pi_A$ .

Suppose the twistor line  $V(\tilde{\pi}_A)$  meets the good cut  $Z(x, \tilde{\pi}_A, \pi_{A'})$  at the point  $P$  lying in the  $\tilde{J}_g = [(\sigma^A)]$ -plane, some  $\sigma^A$ . Then the dual twistor line lying on the good cut and in the  $\tilde{J}_g$ -plane corresponds to the dual twistor  $[(\sigma_A, -ix^{AA'}\sigma_A)]$  which, as a plane in  $\text{IP}\mathbb{R}$ , contains the point  $[(\omega^A, \pi_N)]$ . Thus, the projective value  $\tilde{J}_g$  of  $\tilde{J}_x((\omega^A, \pi_N))$  is given by the  $[(\tilde{\pi}^A)]$  value at which the twistor line  $V(\tilde{\pi}_A)$  intersects the good cut  $Z(x, \tilde{\pi}_A, \pi_{A'})$ .

To achieve a complete description of  $\tilde{J}_x^A$  a scaling for  $\tilde{J}_g$  is required. This can be done by using local twistor methods. The local twistor description of  $A^*Z^*$  at  $P$  is  $(0, \lambda_g \tilde{\zeta}_A)$  where  $\lambda_g = \frac{\tilde{\pi}^A \pi_{A'}}{\tilde{\pi}^B \zeta_B}$ , thus providing a scaling for  $\tilde{J}_g$  obtained above. This local twistor description can be used to obtain the googly maps for the Sparling-Tod (=Eguchi-Hanson, cf. Burnett-Stuart in TIN9)  $\mathcal{H}$ -space. The good cut equation here is

$$(1) \quad \frac{\partial Z}{\partial \pi_A \pi_{A'}} = \sigma \pi^A \pi^{A'}$$

where  $\sigma = \lambda (\alpha \cdot \tilde{\pi})^2 (\beta \cdot \tilde{\pi})^2 Z^{-3}$ ,  $\alpha \cdot \tilde{\pi} \equiv \alpha^A \tilde{\pi}_A$ ,  $\alpha \cdot \beta = 1$  and  $\lambda$  is a deformation parameter. The solutions can be described as follows:

$$\text{Define, } Q^{ABA'B'} \equiv x^{AA'} x^{BB'} + 2(1-\kappa) \beta_{(c} \kappa_{b)} x^{ca} x^{cb'} \beta^{(a} \alpha^{b)}$$

$$\text{where } \kappa = (1 - \frac{\lambda}{\Delta})^{1/2} \quad \text{and} \quad \Delta = \det(x^{ab}), \quad \text{then} \quad Z^2 = Q^{ABA'B'} \tilde{\pi}_A \tilde{\pi}_{B'} \pi_{A'} \pi_{B'}$$

$$\text{With } \{\beta^A, \alpha^A\} \text{ as a basis so that } \beta \cdot \tilde{\pi} = \tilde{\pi}_0, \alpha \cdot \tilde{\pi} = \tilde{\pi}_1 \text{ and } x^{AA'} = \begin{pmatrix} u & x \\ y & v \end{pmatrix}$$

Then, the metric is

$$ds^2 = 2(dudv - dxdy)\kappa + 2\frac{\lambda}{\Delta} \kappa' (udv - xdy)(vdv - ydx).$$

(These expressions differ from those given elsewhere, e.g. Burnett-Stuart in TIN9 by a coordinate transformation).

An inversion and conformal rescaling yield a finite metric in the neighbourhood of the vertex of the  $\mathcal{S}^*$  of the

1 for local twistors and asymptotic twistors cf. Penrose and MacCallum.

$\mathbb{H}$ -space. Coordinates are chosen so that the dual twistor lines are given by solutions of (1) for fixed  $\tilde{\pi}_A$ . The  $\mathbb{H}$ -space is, of course, left-flat and so the asymptotic twistor space is flat, the twistor lines being solutions of  $\frac{\partial^2 Z}{\partial \tilde{\pi}_A \partial \tilde{\pi}_B} = 0$  for fixed  $\tilde{\pi}_A$ .

For an arbitrary twistor line  $V(\tilde{\pi}_A)$  and good cut  $Z(x, \tilde{\pi}_A, \pi_A)$  the value  $\tilde{s}_g$  is calculated by setting  $V-Z=0$ . The value  $\lambda_g$  is then calculated (the relevant quantities  $\tilde{\pi}_A$ ,  $\pi_A$ ,  $\xi_A$ , being obtained in terms of the coordinates on  $\mathbb{H}^+$ ). The only subtlety in this somewhat lengthy calculation is that the inversion reverses orientation.). By checking in the case  $\lambda=0$ ,  $\lambda_g$  can be seen to scale  $\tilde{s}_g$  as follows:

$$\tilde{s}_x^A : (\omega^A, \pi_A) \longleftrightarrow \begin{pmatrix} \lambda_g \tilde{s}_g \\ \lambda_g \end{pmatrix}.$$

After some manipulation this can be written in the form

$$(2) \quad \tilde{s}_x^A : (\omega^A, \pi_A) \longleftrightarrow (Q^{AB} \omega_A \omega_B)^{-1} (\omega^A \tilde{z}^2 - i Q^{AB} \omega_B \tilde{z})$$

where

$$Q^{AB} \equiv Q^{ABAB'} \pi_{A'} \pi_{B'}$$

$$\tilde{z}^2 \equiv Q^{AB} \omega_A \omega_B - (k-1)(k+1) (\omega \cdot \alpha)^2 (\omega \cdot \beta)^2$$

$$\omega^A \equiv x^{AA'} \pi_{A'}.$$

The expression (2) can be expressed in powers of  $\lambda$ , (using binomial expansions),

$$\tilde{s}_x^A : (\omega^A, \pi_A) \longleftrightarrow \omega^A - i u^A + \frac{\lambda}{\Delta^2} \left[ \frac{i}{2} \frac{\omega \cdot \omega'}{(\omega \cdot \omega')^2} \left( (\omega - i \omega')^2 \right) \right] + \dots$$

The first term in this expansion is just the flat-space googly map, while the second term is (up to sign) the expression obtained by R.P. using his recently developed contour integral formula for the linearized googly maps, in the Eguchi-Hanson case (cf. R.P. in this issue), thus validating this formula. Also, the expression for the googly maps given by K.P.T. in TN9 (which, on the face of it, does not necessarily give the correct scaling of the googly map) is seen, by direct calculation, in fact to give the above answer (2) in the Eguchi-Hanson case.

Some more elementary twistor integrals with boundary

A.P. Hodges

The results given by R.P. (in § 5.4 of Advances in Twistor Theory) can easily be extended to certain other simple cases of boundary integrals. We have

$$\oint \frac{\log \left( \frac{z \cdot F}{z \cdot E} \right) dz}{(z \cdot A)^2 (z \cdot B)(z \cdot C)}$$

which is most easily evaluated as

$$\left\{ \frac{1}{ABCE} \right\} + \left\{ \log \left( \frac{z \cdot F}{z \cdot E} \right) \right\} + \left\{ \frac{\partial_z B C E}{(z \cdot A)(z \cdot B)(z \cdot C)} \right\} dz$$

This integrating by parts, gives

$$\frac{BCEF}{ABCE ABCF}$$

This can be thought of as integrating  $(z \cdot A)^{-2}$  on the line  $\{z | z \cdot B = z \cdot C = 0\}$  from the point  $Z \cdot E = 0$  to the point  $Z \cdot F = 0$

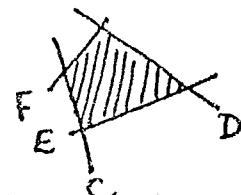
Similarly

$$\frac{(BCDE)^2}{ABCD ABCE ABDE}$$

can be thought of as integrating  $(z \cdot A)^{-3}$  on the plane  $\{z | z \cdot B = 0\}$  over the triangle bounded by lines  $Z \cdot C = 0$ ,  $Z \cdot D = 0$ ,  $Z \cdot E = 0$ .

We now consider

which is the same but this time with the region of integration being the quadrilateral



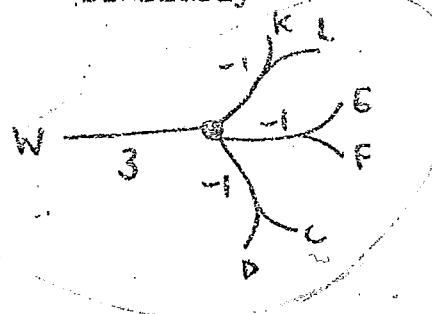
This can be evaluated as the difference of two triangles (CDE and CDF) - R.P.'s method. Or else more analytically as

$$\begin{aligned} & \oint \frac{\log\left(\frac{z_f}{z_e}\right) \log\left(\frac{z_d}{z_c}\right)}{(z_a)^2 (z_b)} dz \\ = & \frac{1}{ABC E} \oint \log\left(\frac{z_f}{z_e}\right) \log\left(\frac{z_d}{z_c}\right) \left\{ \frac{1}{\partial_2 BCE} \frac{1}{(z_a)^2 (z_b)} \right\} dz \end{aligned}$$

Integrating by parts and using earlier result gives

$$\frac{1}{ABC E ABD F} \left\{ \frac{BCDE BDEF}{\partial_2 AED} + \frac{BCEF BEOF}{\partial_2 ABC F} \right\}$$

Similarly



can be thought of as integrating  $(z_w)^{-4}$  over the hexahedron in  $\mathbb{Z}^3$ -space with opposite faces given by planes K,L; E,F; C,D. This volume can be reduced to a sum of tetrahedra and the formula

$$= \frac{(ABCD)^3}{WABC WABD WACD WBOD}$$

applied.

But it's easier to write the hexahedron integral as

$$\frac{1}{WDFL} \oint \log\left(\frac{z_L}{z_K}\right) \log\left(\frac{z_F}{z_E}\right) \log\left(\frac{z_D}{z_C}\right) \left\{ \frac{\partial_2 DFL}{\partial_2 DFL} \frac{1}{(z_w)^3} \right\}$$

The result is

$$\begin{aligned}
 & \left\{ \begin{array}{c} \text{DFKL} \\ \text{WDFIK} \\ \text{WDFL} \\ \text{CDFL} \end{array} \right\} \left\{ \begin{array}{c} \text{CEFK CDFK} \\ \text{WCEFK} \\ \text{WCEL} \\ \text{WCFK} \end{array} \right\} + \left\{ \begin{array}{c} \text{DEFKL CDEL} \\ \text{WDEKL} \\ \text{DGKL CDEKL} \end{array} \right\} \\
 & + \left\{ \begin{array}{c} \text{DEFKL CDEL} \\ \text{WDEKL} \\ \text{DGKL CDEKL} \end{array} \right\} + \left\{ \begin{array}{c} \text{CEFL CEKL} \\ \text{WCFL} \\ \text{CEFK CPKL} \\ \text{WCFK} \end{array} \right\} \\
 & - \left\{ \begin{array}{c} \text{CDFL} \\ \text{WCFL} \end{array} \right\} + \left\{ \begin{array}{c} \text{CEFL CEKL} \\ \text{WCFL} \end{array} \right\}
 \end{aligned}$$

Thought of as a function of  $W_\alpha$ , this is the twistor transform of  $\log\left(\frac{z \cdot F}{z \cdot E}\right) \log\left(\frac{z \cdot D}{z \cdot C}\right)$

More precisely, this is a 1-function of  $W_\alpha$ , the arbitrariness in K,L representing a cohomological freedom. The contour used for it must surround all (four) K-poles together, and separate them from all (four) L-poles.

Application is to R.P.'s infinitesimal googly graviton

$$T(x, z) = \sigma(x) + \kappa(z) = \epsilon \oint \left\{ \frac{\partial f_\alpha}{x} - \frac{\partial f_\alpha}{z} \right\} \delta_{\alpha\beta}(kQ) \log\left(\frac{x \cdot \alpha}{z \cdot \alpha}\right) \log\left(\frac{z \cdot \alpha}{x \cdot \alpha}\right) DQ$$

By twistor-transforming we can get another version in which the X-Z splitting is made explicit. It is easily checked that the transform of

$$\log\left(\frac{x \cdot \alpha}{z \cdot \alpha}\right) \log\left(\frac{z \cdot \alpha}{x \cdot \alpha}\right)$$

using the hexahedron formula above, is of a form which splits explicitly. By choosing

$$\kappa = A^R, \quad L = \nabla^S$$

and by using a certain amount of cohomological juggling, the transform can be represented as

$$\frac{\int_U^S \int_{\Gamma}^R d\theta}{\{w^S w\}_{\infty} \{w^R w\}_{\infty}} = \begin{cases} \text{same expression} \\ \text{only with } Z \text{ for } X \end{cases}$$

and hence

$$f(x) = \oint_{\partial W} \frac{\int_U^S \int_{\Gamma}^R d\theta}{\{w^S w\}_{\infty} \{w^R w\}_{\infty}} \times \frac{\partial w}{\partial x} f_2(w)$$

But the splitting is dependent upon the choice of R and S. The sum  $f(x) + k(z)$  will be independent of R and S, but each of f and K will be undetermined up to a constant (in X, Z) depending on R, S.

(from P.11)  
entire function f, while φ must have singularities within the contour in order that f be non-zero. Note that if

$$\varphi(\zeta) = \dots + a_2 \zeta^{-3} + a_1 \zeta^{-2} + a_0 \zeta^{-1} + a_1 + a_2 \zeta + \dots, \text{ then } f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \dots$$

so that the information contained in  $a_{-1}, a_{-2}, a_{-3}, \dots$  gets lost in the passage to f. Note also that if  $\varphi \mapsto \zeta \varphi$  then  $f \mapsto \frac{d}{dz} f$  and that if  $\varphi \mapsto \frac{d}{d\zeta} \varphi$  then  $f \mapsto -\frac{d}{dz} f$ .

The operation  $\varphi \mapsto \zeta^{-1} \varphi$  corresponds to  $f \mapsto \int f dz$ , the coefficient  $a_{-1}$  now emerging as the constant of integration. Likewise  $\varphi \mapsto \zeta^{-2} \varphi$  corresponds to  $\int (\int f dz) dz$ , and  $a_{-1}, a_{-2}$  both play roles as constants of integration. So we may think of the freedom in φ, in the passage to f, as referring to all the potential constants of integration that might arise.

For an inverse transformation we can write

$$\varphi(\zeta) = \int_{P(\zeta)}^{+\infty} e^{-\zeta z} f(z) dz$$

where  $P(\zeta)$  is holomorphic on and within the annulus, and where "∞" refers to an approach to infinity along a direction chosen to make the integral converge (which requires suitable growth restrictions on f). The freedom in  $P(\zeta)$  relates to the freedom in  $a_{-1}, a_{-2}, a_{-3}, \dots$ . Taking  $P(\zeta) = 0$  yields  $a_{-1} = a_{-2} = \dots = 0$ . It is possible to show directly by moving contours around (modulo holomorphic pieces) that the double transform  $f \mapsto \varphi \mapsto f$  is the identity, a new feature (for twistor theorists) arising here being integration into an essential singularity (at ∞).

No doubt all this is very well known to the right people — but I don't know who they are. And the relevance to cohomological  $F(\partial_W)$  is yet obscure. (Page 172)

| $f(z)$    | $\varphi(z)$ |
|-----------|--------------|
| $z^n/n!$  | $z^{-n-1}$   |
| $e^{az}$  | $(z-a)^{-1}$ |
| $J_0(az)$ | $e^{-az}$    |