

# LPT Twistor Newsletter (no 13 : 8, December, 1981)

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## Abstracts

Twistor Newsletter will in future include abstracts of twistor-related articles appearing elsewhere. The editors would appreciate it if the authors of such articles would send us abstracts thereof, and we will include those which seem appropriate.

LH

## 2. A theory of 2-surface ("superficial") twistors

As had been noted some years back (R.P., Int. J. Theor. Phys. I (1968) 61), the spin-lowering property of a solution, in  $M^I$ , of the symmetric 2-inde~~s~~ twistor equation

$$\nabla_{A'}^{(A} \gamma^{Bc)} = 0, \quad \gamma^{AB} = \gamma^{BA} \quad (1)$$

can be used to reduce the expressions for the 10 conservation laws of linear gravity theory (field  $\phi_{ABCD}$  subject to  $\nabla^{AA'} \phi_{ABCD} = 0$  in the source-free region) to the charge conservation law of Maxwell theory (field  $\phi_{AB}$  subject to  $\nabla^{AA'} \phi_{AB} = 0$  in source-free space), where we set

$$\frac{G}{2} \phi_{AB} = \phi_{ABCD} \gamma^{CD}. \quad \text{in } \text{Epsilon} \quad (2)$$

Putting (in source-free region only)

$$K_{abcd} = \phi_{ABCD} \epsilon_{A'B'} \epsilon_{C'D'} + \text{c.c.} \quad \text{and} \quad F_{ab} = i \phi_{AB} \epsilon_{A'B'} + \text{c.c.},$$

so that

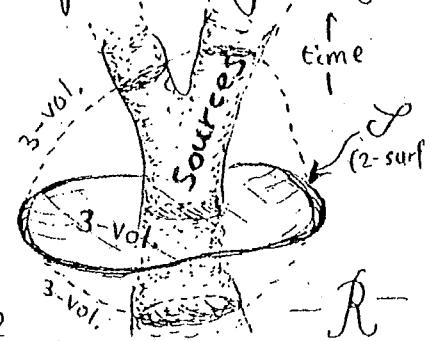
$$F_{ab}^* = \frac{1}{2} \epsilon_{abcd} F^{cd} = \phi_{AB} \epsilon_{A'B'} + \text{c.c.}, \quad (3)$$

we can express the conservation laws for  $K_{abcd}$ , for a region  $R$  surrounding a source distribution, as the fact that

$$\eta = \oint_S F_{ab}^* dx^a \wedge dx^b \quad (S \subset R) \quad (4)$$

measures the total source for  $F_{ab}$  intercepted by any compact 3-volume whose boundary is the 2-surface  $S$ . As  $\gamma^{AB}$  varies, subject to (1),  $\eta$  will range over the various components of the source for  $K_{abcd}$ .

Let  $G^{\alpha\beta}$  be the twistor  $(\epsilon) \Pi^{(\alpha\beta)}$  whose primary part is  $\gamma^{AB}$ .  $G^{\alpha\beta}$  has 10 independent complex components, corresponding to the fact that  $\Pi^{(\alpha\beta)}$  is  $\frac{4 \times 5}{2 \times 1} = \frac{20}{2} = 10$ -dimensional, so there are 10 independent (complex) solutions of (1). The  $\eta$  of (4) is real, so we get 20 independent real "charges" for  $K_{abcd}$  (10 of which will be zero whenever  $K_{abcd}$  arises in the normal way from a potential (=linearized metric)). The sources



time

2-surf

3-vol

3-vol -

R-

themselves are described by the various conserved currents

$$J^a = T^{ab} k_b \quad (\nabla_a J^a = 0; \nabla_a T^{ab} = 0) \quad (5)$$

where  $k^a$  is a killing vector, related to  $\gamma^{AB}$  by

$$k^a = \frac{4}{3} (i \nabla_B^A \bar{\gamma}^{A'B'} - i \nabla_B^{A'} \bar{\gamma}^{AB}). \quad (6)$$

$T^{ab}$  is the usual energy-momentum tensor source for  $K_{abcd}$ :

$$\left[ \begin{array}{l} K_{abcd} = K_{[cd]} [ab] \\ K_{[abc]d} = 0 \end{array} \right] \quad K_{acB}^{\quad c} - \frac{1}{2} g_{ab} K_{cd}^{\quad cd} = -8\pi G T_{ab} \quad (7)$$

( $G$  = grav. const.).

In tensor terms, putting

$$S^{ab} = \gamma^{AB} (e^{A'B'} + e^{AB} \bar{\gamma}^{A'B'}) \quad (8)$$

we have

$$\nabla^{(a} S^{b)c} - \nabla^{(a} S^{c)b} + g^{ab} \nabla_d S^{cd} = 0 \quad (9)$$

as the translation of (1) (so (9) has 20 independent real solutions) and

$$e_{abcd} \nabla^b S^{cd} = \frac{3}{2} k_a \quad (10)$$

as the translation of (6). Eqs. (7), (9) & (10) imply

$$\nabla_{[a} (K_{bc]q} S^{qd}) = \frac{4\pi G}{3} e_{abcd} T^{df} k_f \quad (11)$$

which becomes

$$\nabla_{[a} F_{bc]}^* = \frac{4\pi}{3} e_{abcd} J^d \quad (12)$$

(i.e. the Maxwell equation  $\nabla_a F^{ab} = 4\pi J^b$ ) when

$$F_{ab}^* = \frac{1}{G} K_{abcd} S^{cd} \quad (13)$$

which reduces to (2) in source-free space.

In twistor terms, the relation between  $\gamma^{AB}$  and the killing vector  $k^{AB'}$  (taken to be the primary part of the Hermitian twistor  $K^\alpha_\beta \in \mathbb{T}^\alpha_\beta$ ) is given by

$$K^\alpha_\beta = 4 \bar{G}_{\beta\gamma} I^{\alpha\gamma} + 4 G^{\alpha\gamma} I_{\beta\gamma}. \quad (14)$$

(This is equation (6) in twistor form.) Note that the  $K^\alpha_\beta$  of (14) is trace-free, and with this restriction  $K^\alpha_\beta$  is determined by its primary part  $k^{AB'}$ . (14) is part of the "kinematic sequence", which is (in the simplified form due to L.P.H.)

$$0 \rightarrow \mathbb{T}\mathbb{R} \rightarrow \mathbb{Q}^{[\alpha\beta]} \rightarrow \mathbb{H}^{\alpha\beta}_\beta \rightarrow \mathbb{T}^{(\alpha\beta)} \rightarrow \mathbb{H}^{\alpha\beta}_\beta \rightarrow \mathbb{Q}^{[\alpha\beta]} \rightarrow \mathbb{T}\mathbb{R} \rightarrow 0 \quad (15)$$

where these are considered as real vector spaces, of respective dimensions 0, 1, 6, 15, 20, 15, 6, 1, 0,  $\mathbb{H}^{\alpha\beta}_\beta$  consisting of trace-free Hermitian twistors  $\mathbb{H}^\alpha_\beta$  and  $\mathbb{Q}^{[\alpha\beta]}$  consisting of twistor-real (i.e.

4.

$\bar{Q}_{\alpha\beta} = \frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} Q^{\gamma\delta}$  elements  $Q^{\alpha\beta} \in \mathbb{T}^{[\alpha\beta]}$ . (14) is the  $\mathbb{H}_\beta^\alpha \xrightarrow{\gamma^{\alpha\beta}} \mathbb{H}_\beta^\alpha$  part of (15), so twistors whose primary parts are killing vectors constitute the image of this map. The kernel provides those  $\gamma^{\alpha\beta}$ 's which give zero in (6) (i.e. in (10)). This kernel is the image of  $\mathbb{H}_\beta^\alpha \rightarrow \mathbb{T}^{(\alpha\beta)}$ , which is the set of twistors of the form  $iH_\gamma^{\alpha\beta} I^\gamma$ . (16)

The  $\eta$ 's of (4), with (13) substituted, arise as real parts of expressions like

$$\zeta = \frac{1}{16\pi G} \oint K_{AA'BB'} dx^A dx^B \gamma^{AB} \in A^{AB} (17)$$

which we can write

$$\zeta = A_{\alpha\beta} G^{\alpha\beta} \quad (18)$$

where  $A_{\alpha\beta}$  is the total kinematic twistor of the sources surrounded by  $S$  and satisfies

$$A_{\alpha\beta} I^{\beta} = \bar{A}^{\beta} I^{\alpha} \quad (19)$$

$A_{\alpha\beta}$  is dual (over the reals) to  $G^{\alpha\beta}$ , so it is located in the dual sequence to (5), which is another sequence of just the same form as (15),  $A_{\alpha\beta}$  lying in the image of the corresponding map  $\mathbb{H}_\beta^\alpha \rightarrow \mathbb{T}^{(\alpha\beta)}$ , which is what (19) entails.

All this refers just to  $M^I$ ; we wish to do a corresponding thing in a general curved space  $M$ . We cannot expect to solve (1) (or (9)) in general, but we need its appropriate analogue only on the surface  $S$  itself. However, if we try to restrict (1) to  $S$ , we find only two equations which are entirely tangential to  $S$ , which is insufficient to determine the three complex components of  $\gamma^{\alpha\beta}$  on  $S$ . The key idea is not to try to define the  $\gamma$ 's directly but to consider them as tensor products of 1-index objects. Any solution

of (1) is, indeed, of the form

$$\gamma^{AB} = \sum_i w_i^{(A} w_i^{B)} \quad (20)$$

in  $M^I$ , where each  $w_i^A$  satisfies the 1-index twistor equation

$$\nabla_{\alpha}^{\beta} \omega^{\alpha} = 0 \quad (21)$$

and it is (21) rather than (1) that we restrict to  $\mathcal{S}$ . We take  $\mathcal{S}$  to be spacelike and (normally) to have the topology of a sphere  $S^2$ . Using the GHP formalism (Geroch, Held & R.P., J. Math. Phys. 14 (1973) 874), we find, for the two components tangential to  $\mathcal{S}$  (in a spin-frame family whose flagpoles are orthogonal to  $\mathcal{S}$ ) make up

$$\delta' \omega^0 = \sigma' \omega^1, \quad \delta \omega^1 = \sigma \omega^0. \quad (22)$$

It follows from Atiyah-Singer that (22) always have at least 4 (complex-) independent solutions & that generically (22) have precisely 4 independent solutions. (In the "canonical" case,  $\sigma = \sigma' = 0$ , and the adjoint eqns. have no solutions.) The solution space  $\mathbb{T}^\alpha(\mathcal{S})$  is the 2-surface twistor space of  $\mathcal{S}$ , which will be a complex 4-dim. vector space, assuming that we are in this normal situation. Then the symmetric tensor product  $\mathbb{T}^{(\alpha\beta)}(\mathcal{S})$  will be 10-complex-dimensional. Each element of  $\mathbb{T}^{(\alpha\beta)}(\mathcal{S})$  can be interpreted as a spinor field  $\psi^{\alpha\beta}$  over  $\mathcal{S}$  (the tensor product of the  $\omega^\alpha$ 's being taken pointwise on  $\mathcal{S}$ ). The quantities (17) can now be defined in a general  $M$ , as before but with Riemann's  $R_{abcd}$  replacing Kabsch.

For a 1-term sum (20) we get

$$S = \frac{-i}{4\pi G} \oint \{ (\Psi_1 - \Phi_{10}) \omega_1^0 \omega_2^0 + (\Psi_2 - \Phi_{11}) (\omega_1^0 \omega_2^1 + \omega_1^1 \omega_2^0) + (\Psi_3 - \Phi_{21}) \omega_1^1 \omega_2^1 \} \mathcal{L} \quad (23)$$

( $\mathcal{L}$  being the surface-area 2-form on  $\mathcal{S}$ ). This defines  $A_{\alpha\beta} \in \mathbb{T}_{[\alpha\beta]}(\mathcal{S})$  via (18), which has 10 complex components. To define mass-momentum & angular momentum quasi-locally we need an analogue of (19) so as to have only 10 real pts. So far this is lacking; we don't have a convincing  $I_{\alpha\beta} \in \mathbb{T}_{[\alpha\beta]}(\mathcal{S})$ , nor a convincing  $Z^\alpha \mapsto \bar{Z}_\alpha$  (though tentative definitions exist for the latter — not conformally invariant whereas  $\mathbb{T}^\alpha(\mathcal{S})$  is). Things are no better at  $\mathcal{P}^+$  where  $\sigma' \neq 0$ .

$I_{\alpha\beta}$  exists via  $I_{\alpha\beta} Z^\alpha \bar{Z}^\beta = i \omega_1^0 \pi_{11} - i \omega_2^0 \pi_{11} = \omega_1^0 \delta \omega_2^0 - \omega_2^0 \delta \omega_1^0$  (24)

being constant over  $\mathcal{S} \subset \mathcal{P}^+$ , where  $\pi_{10} = i \delta \omega_1^1 - i p \omega^0$ ,  $\pi_{11} = i \delta \omega_2^0 - i p' \omega^1$ . (25)

Still  $Z^\alpha \mapsto \bar{Z}_\alpha$  is not totally convincing, but the Bondi-Sachs 4-momentum is reproduced exactly and a new improved angular momentum at  $\mathcal{P}^+$  is obtained. (Note K.P.T.'s tantalizing formula for (23), using (25):  $S = \frac{-i}{4\pi G} \oint (\pi_{10} \pi_{21} + \pi_{11} \pi_{20}) \mathcal{L}$ .)

Paper submitted to Proc. Roy. Soc.; more work in progress.

Thanks especially to I.M. Singer and K.P. Tod.

~ R.P.

Codeformations?

My aim in this note is to interpret the usual graviton construction in a way which appears to lend itself naturally to dualization, and to use this technique to construct a googly graviton. The eventual construction is related to RP's work in TN11, and seems to have most of the nice properties he developed there. (For example, it gives the right answer for Eguchi-Hansen.) The motivation seems to much clearer, however, and the final result appears considerably more tractable than the integro-differential equations of the earlier article.

The basic idea is to treat a deformation not as a deformation of the underlying manifold, but instead as a deformation of the sheaf  $\mathcal{O}$  of holomorphic functions on that manifold. (GAJS has also had this idea, some time ago.) Specifically, let  $\mathcal{T}'$  be a deformation of a region  $T$  in twistor space.  $\mathcal{T}'$  and  $T$  are diffeomorphic; suppose  $\phi: \mathcal{T}' \rightarrow T$  is a diffemorphism between them. Now instead of considering  $\mathcal{T}'$  and the sheaf  $\mathcal{O}$  of germs of holomorphic functions on it, we consider the sheaf  $\tilde{\mathcal{O}}$  on  $T$  given by

$$\tilde{\Gamma}(U; \tilde{\mathcal{O}}) = \Gamma(\phi^{-1}(U); \mathcal{O})$$

for any  $U$  open in  $T$ . It is fairly clear that the sheaf  $\tilde{\mathcal{O}}$  over  $T$  contains all of the "information" in the deformation; indeed, the ringed spaces  $(\mathcal{T}, \mathcal{O})$  and  $(T, \tilde{\mathcal{O}})$  are isomorphic.

We can now interpret the non-linear construction using the sheaf  $\tilde{\mathcal{O}}$ , without ever mentioning the deformed space  $\mathcal{T}'$ . Further, since  $\mathcal{T}'$  no longer appears, it is fairly clear how to dualize the construction of the metric (for example) from the sheaf  $\tilde{\mathcal{O}}$ , if we could only find a way to dualize the construction of the sheaf  $\tilde{\mathcal{O}}$  itself.

I want to suggest that this dual, which I will write  $\mathcal{Q}$ , is also a sheaf, and is also a deformation of the sheaf  $\mathcal{O}$ . It appears, however, that there may not be a manifold  $\mathcal{T}$  such that the ringed spaces  $(\mathcal{T}, \mathcal{O})$  and  $(T, \mathcal{Q})$  are isomorphic - there is no obvious reason why a "codeformation" of a manifold should also be a manifold.

To construct the sheaf  $\mathcal{Q}$ , we first examine the sheaf  $\tilde{\mathcal{O}}$ . Suppose the transition functions for  $\mathcal{T}$  are given by

$$\hat{z}^\alpha = z^\alpha + I^{\alpha\beta} \frac{\partial g}{\partial z^\beta} \quad (1)$$

where  $g$  is some (not necessarily infinitesimal) twistor function, homogeneous of degree 2. (This is the case, for example, in RSW's "A Class of Self-dual Solutions to Einstein's Equations, Proc. Roy. Soc.") A holomorphic function on  $\mathcal{T}$  is given by a pair of functions  $f(z^\lambda), \hat{f}(\hat{z}^\lambda)$  satisfying

$$f(z^\lambda) = \hat{f}(\hat{z}^\lambda) = \hat{f}(z^\lambda + I^{\lambda\sigma} \frac{\partial g}{\partial z^\sigma}). \quad (2)$$

The "googlization" of (1) is

$$\frac{\partial}{\partial z^\alpha} = \frac{\partial}{\partial \hat{z}^\alpha} - I_{\alpha\beta} [\hat{z}^\beta, \tilde{g}(\frac{\partial}{\partial \hat{z}^\lambda})]$$

where  $\tilde{g}$  is still homogeneous of degree 2 but is now defined on dual twistor space. The appearance of the commutator is explained in RP's article in TN11. This suggests a dual to (2):

$$\frac{\partial \hat{f}}{\partial \hat{z}^\alpha} - I_{\alpha\beta} [\hat{z}^\beta, \tilde{g}(\frac{\partial}{\partial \hat{z}^\lambda})] \hat{f} = \frac{\partial f}{\partial z^\alpha}. \quad (3)$$

There are several problems with this definition. For one thing, the set of solutions (i.e., sections of  $\tilde{\mathcal{O}}$ ) do not form a ring! Another difficulty is that  $f = \text{constant}$ ,  $\hat{f} = \text{different constant}$  satisfies (3) in the flat case  $\tilde{g} = 0$ . Let me ignore these difficulties for the moment, however.

RP shows that (3) can be rewritten

$$\frac{\partial \hat{f}}{\partial z^\alpha} + \frac{I_{\alpha\beta}}{(2\pi i)^2} \int R^\beta \tilde{g}(R^\lambda) \hat{f}(R^\lambda + z^\lambda) d^4 R = \frac{\partial f}{\partial z^\alpha}. \quad (4)$$

If I assume that  $f$  is homogeneous of degree  $n$ , it follows (since  $I$  is preserved under a deformation and is self-dual, and therefore should be preserved in the new construction) that  $\hat{f}$  is homogeneous, and since  $I$  must have  $\hat{f} = f$  when  $g = 0$ ,  $\hat{f}$  is also homogeneous of degree  $n$ . Contracting (4) with  $\tilde{z}_\alpha$  now gives (for  $n \neq 0$ )

$$\hat{f}(\frac{1}{z}) + \frac{1}{n(2\pi i)^2} \int \tilde{z}^\alpha \tilde{R}^\beta \tilde{g}(\tilde{R}^\lambda) \hat{f}(\tilde{R}^\lambda + \frac{1}{z}) d^4 R = f(\frac{1}{z}). \quad (5)$$

There are new problems with this definition. The most serious is that it is non-local; there are convincing physical arguments to the effect that the googly construction (whatever it turns out to be) should depend only on the behavior of  $g(R)$  in a neighborhood of the line  $I$ . We can recover the locality by rewriting (5) as

$$\hat{f}(\vec{z}) + \frac{1}{n(2\pi i)^2} \left\{ \int \overline{R} g(\vec{z}) \sum_{m=0}^{\infty} \frac{1}{m!} \frac{\partial^{(m)} \hat{f}}{\partial z^p \cdots \partial z^r} R^p \cdots R^r d^4 R = f(\vec{z}) \right.$$

The overall  $Z$ -homogeneity of the  $m$ th term in the sum is 1 (from the  $\overline{Z}^R$ ) plus  $n-m$  (from the derivative term). This suggests that the only contributing term is  $m=1$ , and

$$\hat{f}(\vec{z}) + \frac{1}{n(2\pi i)^2} \left\{ \int \overline{Z}^R \frac{\partial \hat{f}}{\partial z^p} R^p g(\vec{z}) d^4 R = f(\vec{z}). \right. \quad (6)$$

The set of solutions to (6) still does not form a ring, because of the presence of the  $n$  in the denominator preceding the integral. However, the functions of primary interest to us are the googly maps, which are homogeneous of degree 1. If we set  $n=1$  in (6) and require the new relation to hold for all homogeneities, we get

$$\hat{f}(\vec{z}) + \frac{1}{(2\pi i)^2} \frac{\partial \hat{f}}{\partial z^p} \left\{ \int \overline{Z}^R g(\vec{z}) R^p d^4 R = f(\vec{z}), \right. \quad (7)$$

the solutions to which do form a ring.

This equation has a variety of attractive features. For example, the  $Z$ -homogeneity is preserved. Also the integrand is homogeneous of degree -4 in  $R$ , as it should be. In addition, if  $Z$  is on  $I$ , so that  $\overline{Z}^R = 0$ , we get

$$\hat{f}(\vec{z}) = f(\vec{z}),$$

implying that the line  $I$  is itself not deformed, although no neighborhood of it need also be preserved.

The googly maps are functions from twistor space to the line  $I$ , and can therefore be thought of as functions from twistor space to itself. In RP's notation, they are given in the flat case by

$$\underline{z} \rightarrow \underline{z} = \underline{\int z} (= I^{\alpha\beta} \times_{\beta\gamma} z^\gamma).$$

Suppose we consider maps from twistor space to its dual, using as a prototype

$$\hat{f}(\underline{z}) = \underline{A}_z.$$

This gives  $\underline{\partial f} = \underline{A}$ , so that (7) becomes

$$\underline{f}(\underline{z}) = \underline{A}_z + \frac{1}{(2\pi i)^2} \underline{\int z^R} \underline{\int_R g(k)} d^4 R,$$

at variance with RP's earlier result. The quantity of interest, however, is

$$\underline{\int f(z)} = \underline{z} + \frac{1}{(2\pi i)^2} \underline{\int z^R} \underline{\int_R g(k)} d^4 R, \quad (8)$$

and this expression is identical to the one evaluated in TN11!

Penrose proceeds from (8) to obtain the googly maps in the linearized Eguchi-Hansen twistor space, where  $g(k) = 1/(A \cdot R)^3 (B \cdot R)^3$ . He has arrived at (8) by methods different from ours, and takes for the boundary in the integration the four planes characterized by the dual twistors  $\underline{x}, \underline{A}, \underline{z}$  and  $\underline{\bar{z}}$ , where  $\underline{x}$  is an additional twistor appearing in his construction.

RP continues by splitting the quadrilateral generated by these four planes (when intersected with the plane B) into two triangles, one given by the points  $I \cap B$ ,  $\underline{z} \cap \underline{\bar{z}} \cap B$ , and  $L_x \cap B$ , where  $L_x$  is the line in question and is given by  $\underline{z} = 0$ . Integrating (8) over this triangle produces an expression for the googly maps in agreement with the one derived by Law in TN11.

A possible (moral) objection to this is that the line  $L_x$  may be some distance from the line I, and this is at variance with the idea that the whole construction should be local to the line I. This suggests that we replace  $L_x = \{ \underline{\int f(z)} = 0 \}$  with the inhomogeneous boundary

10.

$$\{ \underline{f(z)} = \underline{f(r)} \}. \quad (7) \text{ now becomes:}$$

$$f_x(z) = \hat{f}_x(z) + \frac{1}{(2\pi i)^2} \frac{\partial \hat{f}_x}{\partial z^1} \int_{L_1} \sum_R g(R) R^P d^4 R \quad (9)$$

$$L_1 = \mathbb{I}$$

$$L_2 = \{ \underline{f(z)} = \underline{f(r)} \}$$

$$L_3 = \left\{ \begin{array}{l} [f(z) - f(r)] \\ z \end{array} = 0 \right\} \cap \{ Rz = 0 \}.$$

In the case where  $g$  is an elementary state, it is possible to interpret the three line boundary by restricting the analysis to a plane, and this again produces the correct result for linearized Eguchi-Hansen. In the general case, it is less clear how this boundary is to be interpreted; it seems that some sort of limit may be involved. It may also be relevant that the three lines intersect pairwise (non-projectively, of course) if and only if  $z$  is not on the line  $L_x$ . This is suggestive, since there are other reasons to believe that the googly maps should be well-behaved only for spacetime points  $y$  where  $L_y$  is outside the region of "codeformation."

Note also that because of the nature of the boundary, we are only able to define functions from twistor space to its dual (is this a good or a bad thing?). It is also interesting to note that (9) is already non-linear (as opposed to the material in TN11). Specifically, if  $f$ ,  $\hat{f}$  satisfy (9) for a particular  $g$ , and  $f'$  and  $\hat{f}'$  satisfy it for  $g'$ , it is not necessarily the case that  $f+f'$  and  $\hat{f}+\hat{f}'$  satisfy it for  $g+g'$ . The reason for this is that the boundaries for the integration in (9), which depend on  $\hat{f}$ , will not be the same in the three cases.

Matt Guberg

Many thanks due to R.P., P.L., and A.P.H.

Let  $M$  be a complex 4-manifold with conformal structure which can, say, be represented by some global complex-Riemannian metric with respect to which  $M$  is geodesically convex, and let  $N$  be the associated complex 5-manifold of (complex torsion-free) null geodesics. If, by chance,  $M$  happened to be a region in Minkowski space, then  $N$  would be a neighborhood of a quadric in  $\Lambda := \{(z, w) \in \mathbb{P}T \times \mathbb{P}T^* \mid z \cdot w = 0\}$ , and so come equipped with the sheaf  $\mathcal{O}^{(1)}$  defined by the exact sequence

$$0 \rightarrow \mathcal{I}^2 \rightarrow \mathcal{O} \rightarrow \mathcal{O}^{(1)} \rightarrow 0$$

of sheaves on  $\mathbb{P}T \times \mathbb{P}T^*$ , where  $\mathcal{I}$  is the ideal of functions which vanish on  $\Lambda$ ;  $N^{(1)} := (N, \mathcal{O}^{(1)})$  is the so-called first neighborhood of  $\Lambda$  in  $\mathbb{P}T \times \mathbb{P}T^*$ . I will give a quick sketch here of a generalization of the object  $N^{(1)}$  to the case where  $M$  isn't flat and then state some results relating cohomology groups of  $N^{(1)}$  to the solution spaces of the Klein-Gordon equations for various masses and conformal scalings on  $M$ ; these results seem to be new even in the flat case.

To start, let  $V \rightarrow N$  be the rank 6 vector bundle with fibre at  $\gamma$  defined to be the solution space of the following system:

$$D^2 \begin{bmatrix} \omega^A \\ \xi^A \end{bmatrix} = - \begin{bmatrix} \phi_B \lambda^A & \psi_B \lambda^A \\ \dots & \dots \\ \psi_B \pi^A & \phi_B \pi^A \end{bmatrix} \begin{bmatrix} \omega^B \\ \xi^B \end{bmatrix} \quad (1)$$

and

$$D \begin{bmatrix} \lambda_A \omega^A \\ \pi_A \xi^A \end{bmatrix} = 0 \quad (2)$$

WHERE  $D$  is covariant derivation along  $\gamma$  with respect to  $X^{AA'} = \lambda^A \pi^{A'}$   
 $\omega$  has homogeneity  $(0,1)$  in  $(\lambda, \pi)$   
 $\xi$  has homogeneity  $(1,0)$  in  $(\lambda, \pi)$

and

$$\phi_A := \phi_{ABA'B'} \lambda^B \pi^{A' B'} , \quad \phi_{A'} := \phi_{ABA'B'} \lambda^A \lambda^B \pi^{A' B'} ,$$

$$\psi_A := \psi_{ABCD} \lambda^B \lambda^C \lambda^D , \quad \tilde{\psi}_{A'} := \tilde{\psi}_{A'B'C'D'} \pi^{B' C' D'} ,$$

Then there is a natural inclusion  $j: T^*N \hookrightarrow V$  given by

$$j(J^a) := \begin{bmatrix} J^{AA'} \pi_{A'} \\ J^{AA'} \lambda_A \end{bmatrix}$$

where a tangent vector on  $N$  is represented by the Jacobi field  $J^a$  satisfying the constraint  $X_a DJ^a = 0$ . (Thus (1) is just a super up form of Jacobi's equation.)  $V$  is, incidentally, a conformally invariant object.

Now, as a sheaf of abelian groups, let

$$\mathcal{O}^{(1)} := \{(f, \varphi) \in \mathcal{O} \oplus \mathcal{O}(V^*) \mid j^* \varphi = df\}$$

and then give  $\mathcal{O}^{(1)}$  a ring structure by the multiplication

$$(f_1, \varphi_1)(f_2, \varphi_2) := (f_1 f_2, f_1 \varphi_2 + f_2 \varphi_1)$$

Then this gives the usual sheaf  $\mathcal{O}^{(1)}$  if  $N = A$ , and quite generally has  $V$  as its set of derivations (linear maps  $\delta: \mathcal{O}_Y^{(1)} \longrightarrow \mathbb{C}$ , for some  $\gamma \in N$ , such that  $\delta(g_1 g_2) = g_1(\gamma) \delta(g_2) + g_2(\gamma) \delta(g_1)$ ).

This completes our construction of  $N^{(1)}$ .

This done, we can state a result concerning the Klein-Gordon equation. Let us call a locally free sheaf of rank 1 over  $\mathcal{O}^{(1)}$  a line-bundle on  $N^{(1)}$ , and notice that it makes sense to talk about the restriction of such an object to  $N$  to give a line-bundle in the usual sense.

THEOREM. Let  $K$  be a function on  $M$  of conformal weight -2. Then there exists a line-bundle  $\mathcal{O}_K^{(1)}(0, -2)$  extending  $\mathcal{O}(0, -2)$  to  $N^{(1)}$  for which

$$H^1(\mathcal{O}_K^{(1)}(0, -2)) = \{\text{Solutions of } (\square + K)f = 0\} .$$

(Proof in IHES preprint M/81/54.)

Claude LeBrun

NOTE ON THE WAVE EQUATION

N.P. Buchdahl

Let  $\mathcal{E}$ ,  $\mathcal{A}$  be generic for the sheaves of germs of smooth and real-analytic (real- or complex-valued) functions respectively. For a constant coefficient linear partial differential operator  $P$  on  $\mathbb{R}^n$ , it is well-known that  $P\mathcal{E}(\Omega) = \mathcal{E}(\Omega)$  for any open convex subset  $\Omega \subset \mathbb{R}^n$ . However, it is not necessarily the case that  $P\mathcal{A}(\Omega) = \mathcal{A}(\Omega)$ ; for example, if  $\Delta_n$  is the  $n$ -dimensional Laplacian, then both  $P = \Delta_n$  and  $P = \Delta_n + \partial/\partial t$  when  $n \geq 2$  are operators for which  $P\mathcal{A}(\mathbb{R}^{n+1}) \neq \mathcal{A}(\mathbb{R}^{n+1})$  (!), (result of DeGiorgi and Piccinini). Nevertheless, it turns out that for the wave operator  $\square = \frac{\partial^2}{\partial t^2} - \Delta_n$  it is the case that  $\square\mathcal{A}(\mathbb{R}^{n+1}) = \mathcal{A}(\mathbb{R}^{n+1})$ :

For  $\Omega \subset \mathbb{R}^m$  open, a function  $g \in \mathcal{E}(\Omega)$  is real-analytic iff for each compact subset  $K \subset \Omega$  there is a constant  $C_K$  such that  $\sup_{x \in K} |D^\alpha g| \leq C_K^{|\alpha|+1} |\alpha|!$ , (where  $|\alpha| := \alpha_1 + \dots + \alpha_m$  and  $D^\alpha := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \dots \frac{\partial^{\alpha_m}}{\partial x_m^{\alpha_m}}$  for  $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m$ ). If  $f(x, t)$  is an element of  $\mathcal{A}(\mathbb{R}^{n+1})$ , then

$$\sum_{\alpha} \frac{\partial^\alpha f}{\partial x_1^{\alpha_1} \dots \partial x_m^{\alpha_m}}(0,0) \cdot \frac{\partial^\alpha u}{\partial x_1^{\alpha_1} \dots \partial x_m^{\alpha_m}}(x,t) := \frac{1}{a_n} \int_0^t \int_{|y|^2=1} (t-s)f((t-s)y+x, s) dS ds$$

satisfies  $\square u = f$ , (where  $a_n :=$  area of unit ball in  $\mathbb{R}^n$ ), and by using the above characterisation of real-analyticity it is easy to see that  $u$  belongs to  $\mathcal{A}(\mathbb{R}^{n+1})$ .

For twistor theory, the surjectivity of  $\square$  on  $\mathcal{A}(M^I)$  implies for example, the surjectivity of  $\nabla^{AA'} : \Gamma(M^I, \mathcal{A}_A^{(B' \dots D')}) \rightarrow \Gamma(M^I, \mathcal{A}^{(A'B' \dots D')})$ . Using the fact that  $M^I$  has a fundamental system of contractible Stein neighbourhoods in  $\mathbb{C}M^I$  together with the exactness of the generalised holomorphic DeRham sequence on  $\mathbb{C}M$  (N.P.B. TM.10), this enables one to deduce the exactness of

$$\Gamma(M^I, \mathcal{A}_{(ABC \dots E)}) \xrightarrow{\nabla^{AA'}} \Gamma(M^I, \mathcal{A}_{(BC \dots E)}^{(A)}) \xrightarrow{\nabla^{AB}} \Gamma(M^I, \mathcal{A}_{(C \dots E)}),$$

a fact useful for hyperfunctional aspects of twistor theory.

### Coupled Propagators

In TN11, I made some remarks suggesting that it might be possible to generalize the flat-space propagator construction to produce propagators coupled to self-dual fields. The basic idea was to use the then new description of the propagators in terms of constructions on non-projective twistor space.

This can be summarized by defining non-projective ambitwistor space  $\tilde{\Omega}^-$  to be that subspace of  $\mathbb{W}^* \times \mathbb{W}^{*-}$  where  $w \cdot z = 0$ .  $\tilde{\Omega}^-$  is a fiber bundle with base  $\mathbb{P}^*$  (a deformation retract of  $S^3$ ) and fiber a deformation retract of  $C^*$ . It can be shown that  $H^*(\tilde{\Omega}^-; \mathbb{Z}) \cong \mathbb{Z}$ , and if we take  $\tilde{k}$  to be the generator of this group, then the natural injection  $i: \mathbb{Z} \rightarrow \mathcal{O}$  allows us to treat  $\tilde{k}$  as an element of  $H(\tilde{\Omega}^-; \mathcal{O})$ . This element is homogeneous of degree zero in its various arguments and corresponds to an element  $k \in H(\Omega^-; \mathcal{O}(0,0))$  where  $\Omega^-$  is the projective version of  $\tilde{\Omega}^-$ . The twistor propagators are now given by

$$\phi_n = (w, z)_{-n-1} \cdot k \in H^2(\mathbb{P}^* \times \mathbb{P}^{*-}; \mathcal{O}(-n-2, -n-2)).$$

These propagators have been described in a variety of places by MGE and myself.

What about the coupled case? For definiteness, I will consider propagators coupled to an electromagnetic field given by a line bundle  $L$  over  $\mathbb{P}^-$ . (Other self-dual fields are essentially the same.) The coupled propagators will be elements

$$\phi_n \in H^2(\mathbb{P}^- \times \mathbb{P}^{*-}; \mathcal{O}(L \times \bar{L}; -n-2, -n-2)),$$

where I have used the fact that  $\bar{L}$  is a line bundle over  $\mathbb{P}^{*-}$  and have written  $\mathcal{O}(L \times \bar{L}; -n-2, -n-2)$  for the sheaf of germs of homogeneous sections of the product bundle  $L \times \bar{L}$ , homogeneous of degree  $-n-2$  in each argument. If I can find such  $\phi_n$  which vary holomorphically with the line bundle  $L$  and which restrict to the uncoupled propagators if  $L=0$ , I will have an explicit construction of the fully coupled propagators. The result will be explicit

only on twistor space; to construct the spacetime form of the coupled propagators involves integrations which, although straightforward, are extremely messy, and I have not as yet been able to bring myself to do them.

The construction of the coupled propagators is very straightforward.

Let  $\mathcal{T}^-$  and  $\mathcal{T}^{*-}$  be the deformed versions of  $\mathbb{P}^-$  and  $\mathbb{P}^{*-}$  corresponding to  $L$  and  $\bar{L}$  respectively, and  $\tilde{\mathcal{M}}_L$  the subvariety of  $\mathcal{T}^- \times \mathcal{T}^{*-}$  where  $w \cdot z = 0$ . (This is a well-defined notion, since the projective versions of  $\mathcal{T}^-$  and  $\mathcal{T}^{*-}$  are still dual.) It now follows (as MGE has pointed out) from the fact that  $L$  (and  $\bar{L}$ ) is trivial when restricted to any line in  $\mathbb{P}^-$  that

$$H^1(\tilde{\mathcal{M}}_L; \mathbb{Z}) \cong H^1(\tilde{\mathcal{M}}^-; \mathbb{Z}). \quad (1)$$

Letting  $\tilde{k}_L$  generate the left-hand group, we have  $\tilde{k}_L \in H^1(\tilde{\mathcal{M}}^-; \theta)$  and, since  $\tilde{k}_L$  is homogeneous of degree zero and satisfies (by definition) the transition relations for  $\tilde{\mathcal{M}}_L$ , we get a projective

$$k_L \in H^1(\mathcal{M}^-; \mathcal{O}(L \times \bar{L}; 0, 0)).$$

Defining

$$\phi_n \equiv (w \cdot z)^{-n-1} \cdot k_L \in H^2(\mathbb{P} \times \mathbb{P}^{*-}; \mathcal{O}(L \times \bar{L}; -n-2, -n-2))$$

provides us with the coupled propagators.

If  $\mathcal{T}^-$  is replaced by any other vector bundle (such as that in the Ward construction of Yang-Mills fields), the argument is unchanged. The gravitational case is different, since the parenthetical argument preceding (1) may no longer hold. Here, the propagators are constructed using homological results on the deformed projective spaces; this work has been described elsewhere.

Matt Gubenberg

### Twistors versus Antitwistors

I will list here some more properties of the antitwistor functions introduced by Richard Jozsa in his thesis and in TN12. These properties are somewhat surprising and a general appeal to aesthetics gives the impression that a hidden truth is lurking. However, I do not know what this truth is and, in particular, I cannot as yet give good answer to the question: "what are these things good for?" They have a certain air of "googliness" which can be seen, for example, in the relationship between antitwistors and dualtwistors described later but, again, this is rather too vague to suggest a clear direction to follow.

First some motivation, albeit slight. In the integral formula  $\int \phi_{A_1 \dots A_k} = \int \pi_{A_1} \dots \pi_{A_k} f \Delta x$  it is clear that  $\phi_{A_1 \dots A_k}$  satisfies  $\nabla^{AA'} \phi_{A_1 \dots A_k} = 0$  because  $\pi_{A_1} \nabla^{AA'} f = 0$  and this leads to thinking of twistor functions as functions  $f = f(x, \pi)$  defined on the primed spin-bundle (i.e.  $x$  is a plane in  $\mathbb{P}$  and  $\pi$  is a point on  $x$ ) and satisfying  $\pi_{A_1} \nabla^{AA'} f = 0$ . More precisely, if we restrict to homogeneous functions, then the sheaf of twistor functions homogeneous of degree  $k$  is the kernel of  $\pi_{A_1} \nabla^{AA'} : \mathcal{O}(k) \xrightarrow{\sim} (\mathcal{O}^k(k+1)[-1])'$  on  $\mathbb{F}$ . Denote this sheaf by  $\mathcal{O}(k)$ . It is the same as the topological inverse image sheaf of  $\mathcal{O}(k)$  on  $\mathbb{P}$  via the natural projection  $\mu : \mathbb{F} \rightarrow \mathbb{P}$ . In particular it is a sheaf of rings. Now what about the integral formulae for massless fields of the other helicity?:  $\int \partial_{\partial w^A} \partial_{\partial w^B} f \Delta x$ . Again, the field equations are a consequence of an equation on  $f$ :  $\partial_{\partial w^A} \nabla^{AA'} f = 0$  but this equation is harder to interpret since  $\partial_{\partial w^A}$  is an operation on twistor functions (before pulling back to  $\mathbb{F}$ ). Jozsa defined antitwistor functions as solutions of  $\partial_{\partial \pi_A} \nabla^{AA'} f = 0$  for  $f = f(x, \pi)$ . The similarity to  $\partial_{\partial w^A} \nabla^{AA'} f = 0$  is clear. A different but more precise connection is that  $\partial_{\partial \pi_A} \nabla^{AA'}$  coincides with  $\partial_{\partial w^A}$  (up to scale if the homogeneity is not -2) when acting on twistor functions. So much for motivation.

Jozsa's motivation came from intuitionistic sheaf models for massless fields. He defined  $A(k)$ , the sheaf of antitwistor functions homogeneous of degree  $k$ , as the kernel of  $\partial_{\partial \pi_A} \nabla^{AA'} : \mathcal{O}(k)[-k-1] \xrightarrow{\sim} \mathcal{O}_A[-k-1]$  on  $\mathbb{F}$ . The conformal weight  $[-k-1]'$  is included to make  $\partial_{\partial \pi_A} \nabla^{AA'}$  conformally invariant. In a particular frame it can be ignored as usual. Actually, it is better to arrange that the weight is independent of the homogeneity. This can be achieved as follows. Let  $\mathbb{F}^*$  denote the projective dual primed spin-bundle i.e.  $\mathbb{F}^* = \{(x, \bar{s}) \text{ st. } x \text{ is a plane in } \mathbb{P} \text{ and } \bar{s} \text{ is a line in } x^* \text{ (the vector-space dual of } x\}$ . As a complex manifold,  $\mathbb{F}^*$  is canonically isomorphic to  $\mathbb{F}$  since every line  $\bar{s}$  in  $x^*$  determines (as the kernel of any non-zero element of  $\bar{s}$ ) a line in  $x$  and vice versa. However,  $\mathbb{F}^*$  has a different "universal bundle" i.e. if we denote by  $\mathcal{O}^*(-1)$  the line-bundle which assigns to every pair  $(x, \bar{s})$  the line  $\bar{s}$  then  $\mathcal{O}^*(-1)$  is different from  $\mathcal{O}(-1)$  (when we identify  $\mathbb{F}^*$  and  $\mathbb{F}$ ). In fact, there is a determinant (a skew form,  $\epsilon$ ) involved so that, in general,  $\mathcal{O}^*(k)[l]' = \mathcal{O}(k)[l-k]'$ . In particular,  $A(k)$  is the kernel of  $\partial_{\partial \bar{s}^A} \nabla^{AA'} : \mathcal{O}^*(k)[-1] \xrightarrow{\sim} \mathcal{O}^*(k-1)[-1][2]'$ . This makes antitwistor functions look much like twistor functions - the only difference is that  $\partial_{\partial \bar{s}^A}$  replaces  $\pi_A$ . This calls to mind the "spinor transform," a spinor version of the twistor transform. Namely, for  $f \in \mathcal{O}(k)$   $k \geq -1$ , define the spinor transform  $Sf \in \mathcal{O}^*(-k-2)[-1]'$  by  $\int (Sf)(x, \bar{s}) = \frac{(k+1)!}{2\pi i} \int f(x, \pi) \frac{dx}{(\pi, \pi)^{k+1}} \Delta x$  and for  $g \in \mathcal{O}^*(k)[l]$ ,  $Sg \in \mathcal{O}(l-k-2)$  by  $\int Sg(x, \bar{s}) = \frac{(-1)^k (k+1)!}{2\pi i} \int g(x, \pi) \frac{dx}{(\pi, \pi)^{k+1}} \Delta x$ . It is immediate from these integral formulae that  $\partial_{\partial \bar{s}^A} (Sf) = -S(\pi_A f)$ ,  $S^m(Sf) = S(\partial_f / \partial \pi_A)$ ,  $\partial_{\partial \pi_A} (Sg) = S(S^m g)$ , and  $\pi_A (Sg) = -S(\partial_g / \partial \bar{s}^A)$ . It is clear by pure thought that  $S$  is really some sort of

differential operator (of order  $k$  on  $\Omega(k)$ ) and a calculation shows it is just  $\delta^{k+1}$  in disguise (recall  $\delta^{k+1}: \Omega(k) \rightarrow \Omega(-k-2)[k+1]'$  is defined by  $\pi^{\alpha_1} \dots \pi^{\alpha_k} \delta^{k+1} f = \partial^{\alpha_1} f / \partial \pi_{\alpha_1} \dots \partial \pi_{\alpha_k}$  (edit: is usually taken to lower conformal weight by 1 but this is just a matter of sign convention which has been switched here);  $\Omega(k) \xrightarrow{\delta^{k+1}} \Omega(-k-2)[k+1]' = \Omega^+(-k-2)[-1]'$  is  $S$ ). The properties just claimed for  $S$  become  $\pi^{\alpha_1} \delta^{k+1} f = \delta^k \partial f / \partial \pi_{\alpha_1}$  and  $\partial / \partial \pi_{\alpha_1} \delta^{k+1} f = \delta^{k+2} \pi_{\alpha_1} f$ . Of these, the first is obvious and the second is a very messy calculation if the integral formula or some other trick is not employed. In these formulae it is clear that  $\pi$  just goes along for the ride. We can deduce that ①  $S$  and  $\nabla^{\alpha_1 \dots \alpha_k}$  commute, ② there are exact sequences  $0 \rightarrow \mathcal{M}_{(A_1 \dots A_k)}[-1] \xrightarrow{\delta^{k+1}} \Omega(k) \rightarrow (\Omega^+(-k-1)[-1])' \rightarrow 0$ ,  $0 \rightarrow \mathcal{M}_{(A_1 \dots A_k)}[-1] \xrightarrow{\delta^{k+1}} \Omega^+(k)[-1] \rightarrow \Omega(-k-2) \rightarrow 0$ , where  $\mathcal{M}$  denotes the sheaf of functions of  $\pi$  alone (i.e. functions on Minkowski space regarded as up on  $\mathbb{F}$ ). Putting all this together we get the following commutative diagrams on  $\mathbb{F}$ :

I.  $\forall k \geq 1$ :  
 All columns are exact

Hence: If  $Z_{(A_1 \dots A_k)}$  denotes the kernel of  $\nabla^{\alpha_1 \dots \alpha_k}$  (i.e. massless fields) we obtain (as did ROJ in TN12)

$$0 \rightarrow Z_{(A_1 \dots A_k)} \xrightarrow{\delta^{k+1}} A(k) \xrightarrow{S} J(-k-2) \rightarrow 0$$

an exact sequence. We also deduce

that the middle row gives a resolution of  $A(k)$ . I find this very surprising. This allows us to calculate  $H^j(A(k)) = T^j(Z_{(A_1 \dots A_k)})$ ,  $H^j(A(k)) = 0$  for  $j \neq 0$ . (Note:  $H^j(J(-k-2)) \cong H^j(Z_{(A_1 \dots A_k)})$  Poincaré Duality).

These cohomology groups are calculated over  $\nu^{-1}(U)$  for  $U^{\text{open}} \subset \mathbb{M}$ ,  $\nu: \mathbb{F} \rightarrow \mathbb{M}$ .

II.  $\forall k \geq -1$ :  
 All columns are exact

Hence: If  $Z_{(A_1 \dots A_k)}$  denotes massless fields then we obtain the exact sequence

$$0 \rightarrow Z_{(A_1 \dots A_k)} \xrightarrow{\delta^{k+1}} J(k) \xrightarrow{S} A(-k-2) \xrightarrow{J(-k-2)} Z_{(A_1 \dots A_k)} \rightarrow 0$$

Sols of dual k-twistor eqns  $5^{alpha_1} 5^{alpha_2} 5^{alpha_k} \nabla^{\alpha_1 \dots \alpha_k} f = 0$

We also deduce that the bottom row

provides a resolution of  $A(-k-2)$ . Thus  $H^j(A(-k-2)) \cong T^j(Z_{(A_1 \dots A_k)})$  and  $H^j(A(-k-2)) = T^j(Z_{(A_1 \dots A_k)})$ . Others vanish.

It is a good exercise to check that all these statements are consistent, i.e. check cohomology etc..

$A(0)$  is omitted from the above discussion. In this case things go slightly wrong and it seems that  $A(0)$  should be redefined so as to impose  $\square f = 0$  (which is automatic for other homogeneities).

Integral formula for solving  $\partial / \partial \pi^{\alpha_1} \nabla^{\alpha_1 \dots \alpha_k} f = 0$ . Paul Tod suggested the following contour integral formula:  $f(\pi, \xi) = \oint F(\pi^{\alpha_1} \pi_{\alpha_1}, \xi^{\alpha_1} \pi_{\alpha_1}, \pi^{\alpha_2} \pi_{\alpha_2}) \Delta \pi$ , for  $F$  a function of 5 complex variables  $F(w^{\alpha_1}, w, \pi_{\alpha_1})$  homogeneous of degree -2. This gives the general solution of  $\partial / \partial \pi^{\alpha_1} \nabla^{\alpha_1 \dots \alpha_k} f = 0$  locally (when interpreted cohomologically). This is because if  $\pi$  in  $\pi^{\alpha_1 \dots \alpha_k}$  is allowed to take on 3 (or more) values then the usual arguments of the Penrose transform are unchanged and the wave operator in this "hypertwistor" theory is the antitwistor operator. In this way we obtain a hypertwistor correspondence lying over the usual twistor correspondence and there is some interesting geometry to be explored. The  $f$  constructed above is not homogeneous but if we split it into its homogeneous parts then  $H^j(A(k)) = T^j(Z_{(A_1 \dots A_k)})$  above recovers the usual contour integral formulae for massless fields (right-handed anyway).

Antitwistors v. Dualtwistors:  $\forall k \geq 1$  we have  $H^j(A(k)) \cong H^j(J(-k-2))$ , and  $\forall k \geq -1$  we have  $H^j(J(k)) \cong H^j(A(-k-2))$ ,  $H^j(J(k)) \cong H^j(A(-k-2))$ . This suggests connecting homomorphisms. Roger Penrose also came up with this suggestion based on general dual/googly speculations. I can't see quite how to make this work but the following is close: Let  $\mathcal{D}$  = dualtwistor functions ( $\partial / \partial \pi^{\alpha_1} \nabla^{\alpha_1 \dots \alpha_k} f = 0$ ) and  $\mathcal{B}$  = dualantitwistor functions ( $\partial / \partial w^{\alpha_1} \nabla^{\alpha_1 \dots \alpha_k} f = 0$ ). Then there is an exact sequence (for example)  $0 \rightarrow \mathcal{D}(-1) \rightarrow \mathcal{B}(-1) \xrightarrow{\partial / \partial w^{\alpha_1} \nabla^{\alpha_1 \dots \alpha_k}} A(1) \xrightarrow{S} J(-3) \rightarrow 0$ . ( $\Rightarrow H^j(\mathcal{D}(-1)) \cong H^j(A(1))$ ,  $H^j(\mathcal{B}(-1)) \cong H^j(J(-3))$ ). Michael Eastwood.

### The Kinematic Sequence (Revisited)

$$0 \longrightarrow C \xrightarrow{A^i} C^i \xrightarrow{A^j} [ij] \xrightarrow{A^k} [ijk] \xrightarrow{A^\ell} [ijkl] \xrightarrow{A^m} [ijklm] \xrightarrow{A^n} [ijklmn] \longrightarrow 0 \quad (*)$$

with  $A^i$   $i=0,1,\dots,5$  a fixed point of  $C^6$ , is an exact sequence of vector space maps; the vector spaces  $C, C^i$  etc being irreducible representations of  $GL(6, \mathbb{C})$ .

If we restrict attention to the subgroup  $SO(6, \mathbb{C})$  preserving a non-degenerate quadratic form  $\Omega_{ij}$ , then some of these vector spaces will decompose in an interesting fashion. To see this, rewrite things in "skew bitwistor" indices. Then, for instance,

$$\begin{aligned} [ij] &\longleftrightarrow C^{[\alpha\beta][\gamma\delta]} \xrightarrow{\cong} C_p^{\alpha\text{TF}} \quad (\text{trace-free}) \\ b^{\alpha\beta\gamma\delta} &\mapsto b_p^\alpha = b^{\alpha\beta\gamma\delta} \Omega_{\beta\gamma\delta\alpha} \\ \text{and } [ijk] &\longleftrightarrow C^{[\alpha\beta\gamma\delta\epsilon\eta\zeta]} \xrightarrow{\cong} C^{(\alpha\beta)} \oplus C_{(\eta\zeta)} \\ b^{\alpha\beta\gamma\delta\epsilon\eta\zeta} &\mapsto (b^{\alpha\beta\gamma\delta} \Omega_{\beta\gamma\delta\alpha}, b^{\alpha\beta\gamma\delta\epsilon} \Omega_{\gamma\eta\delta\alpha} \Omega_{\eta\zeta\beta}) \end{aligned}$$

where  $\Omega_{\beta\gamma\delta\alpha} = \Omega_{[\beta\gamma\delta\alpha]} \leftrightarrow \Omega_{ij}$ . The exact sequence  $(*)$  now takes the form

$$\begin{aligned} 0 \longrightarrow C &\longrightarrow C^{[\alpha\beta]} \longrightarrow C_Y^{\alpha\text{TF}} \longrightarrow C^{(\alpha\beta)} \oplus C_{(\eta\zeta)} \\ b &\mapsto b^{\alpha\beta} \\ b^\alpha &\mapsto b^{\alpha\beta} A_{\beta\gamma} - \frac{1}{4} (b^{\beta\beta} A_{\beta\gamma}) \delta_\gamma^\alpha \\ b_\gamma^\alpha &\mapsto (b_Y^\alpha A_\beta)^\gamma, b_\gamma^\alpha A_{\beta\gamma}) \\ &\longrightarrow C_Y^{\alpha\text{TF}} \longrightarrow C_{(\eta\zeta)} \longrightarrow C \longrightarrow 0 \\ (b^{\alpha\beta}, d_{\gamma\beta}) &\mapsto b^{\alpha\beta} A_{\beta\gamma} - d_{\gamma\beta} A^{\alpha\beta} \\ b_\gamma^\alpha &\mapsto b_{[\alpha}^\beta A_{\beta]\gamma} \quad (***) \\ b_{\gamma\beta} &\mapsto b_{\gamma\beta} A^{\alpha\beta} \end{aligned}$$

This we shall call the kinematic sequence

R.P.'s well-known cyclic sequence looks rather similar

$$\dots \longrightarrow C_Y^\alpha \xrightarrow{I} C^{[\alpha\beta]} \xrightarrow{I} C_Y^\alpha \xrightarrow{I} C^{(\alpha\beta)} \oplus C_{(\eta\zeta)} \xrightarrow{I} C_Y^\alpha \xrightarrow{I} C^{[\alpha\beta]} \longrightarrow \dots$$

III  
 $C_Y^{\alpha\text{TF}} \oplus C$  (\*\*\*)

where  $I^{\alpha\beta}$  is the vertex of  $\phi$ . In fact, this can be cut into an infinite number of finite chunks, each of which is like (\*): (\*\*\*) and (\*\*) contain the same information. In particular, the generators of the conformal group are parametrized by  $C_Y^{\alpha\beta}$ ; the Poincaré generators are given by the kernel of  $I : C_Y^{\alpha\beta} \rightarrow C_{[\alpha\beta]}$ .

Remarks 1. Nothing so far has depended on whether or not  $A^{\alpha\beta}$  is simple ( $\Lambda^{\alpha\beta}\Lambda^{\gamma\delta}J_{\alpha\beta\gamma\delta} = 0$ ). However, in the case when  $A^{\alpha\beta}$  is not simple, the sequence (\*) splits into two pieces:

$$0 \longrightarrow C \xrightarrow{A} C_{[\alpha\beta]} \xrightarrow{A} C_Y^{\alpha\beta} \xrightarrow{A} C_{[\alpha\beta]} \longrightarrow 0$$

and its dual

$$0 \longrightarrow C_{[\alpha\beta]} \xrightarrow{A} C_d^{\alpha\beta} \xrightarrow{A} C_{[\alpha\beta]} \xrightarrow{A} C \longrightarrow 0$$

2. The Koszul complex is an exact sheaf sequence on  $\mathbb{C}^6$  which is analogous to (\*\*):

$$0 \longrightarrow \mathcal{O} \xrightarrow{X} \mathcal{O}^{[\alpha\beta]} \xrightarrow{X} \mathcal{O}_Y^{\alpha\beta} \xrightarrow{X} \mathcal{O}^{[\alpha\beta]} \oplus \mathcal{O}_{[\alpha\beta]} \xrightarrow{X} \mathcal{O}_Y^{\alpha\beta} \xrightarrow{X} \mathcal{O}_{[\alpha\beta]} \xrightarrow{X} \mathcal{O} \longrightarrow 0$$

where the  $X^{\alpha\beta}$  are  $\mathbb{C}^6$ -coordinates. This sequence and a similar one on  $\mathbb{CP}^5$  are useful in the study of conformally invariant space-time fields.

Lars Hughston & Tom Hurd

20.

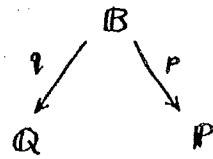
COHOMOLOGY OF BUNDLES ON KLEIN QUADRIC

Ross Moore

$\mathbb{P}^3 = \mathbb{P}\mathbb{T}$ .

$Q$  = Grassmann variety  $G(1,3)$  embedded as  
quadric hypersurface in  $\mathbb{P}^5 = \mathbb{P}\tilde{\mathbb{T}}$ ,  $\tilde{\mathbb{T}} = \Lambda^2 \mathbb{T}$ .

$B$  = flag manifold, projective spin bundle.



On  $\mathbb{P}$ , the following short exact sequences define the tangent bundle  $T$ , and the cotangent bundle  $\Omega'$ :

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathcal{O}(-1) & \rightarrow & T^* \otimes \mathcal{O}_{\mathbb{P}} & \rightarrow & T(-1) \rightarrow 0 \\ 0 & \rightarrow & \Omega' & \rightarrow & T^* \otimes \mathcal{O}(-1) & \rightarrow & \mathcal{O}_{\mathbb{P}} \rightarrow 0 \end{array} \quad \left. \begin{array}{c} \textcircled{1} \\ \textcircled{2} \end{array} \right\}$$

Write  $V^{(k)} = k^{\text{th}}$  symmetric power of a vector space, or bundle  $V$ .

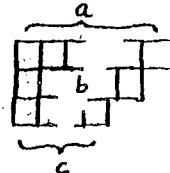
Taking exterior & symmetric powers of  $\textcircled{1}$  —  $T \cong \Omega^2(4)$ ,  $\Lambda^2 T \cong \Omega^1(4)$  — one calculates the following cohomology groups on  $\mathbb{P}$ :

$$\textcircled{2} \quad \left\{ \begin{array}{lcl} H^0(\Omega'^{(2)}(-j)(k)) & = & \mathfrak{W}^{j+k} \langle j+k^2, k \rangle & j, k \geq 0 \\ H^1(\quad " \quad) & = & \mathfrak{W}^{j-1} \langle j-1^2, -k-2 \rangle & -j-1 \leq k \leq -2 \\ H^3(\quad " \quad) & = & \mathfrak{W}^{j+1} \langle -k-4, j^2 \rangle & k \leq -j-4 \end{array} \right.$$

$$\textcircled{3} \quad \left\{ \begin{array}{lcl} H^0(T(-2)^{(j)}(k)) & = & \mathfrak{W}^{k-j} \langle k, k-j^2 \rangle & k \geq j \geq 0 \\ H^2(\quad " \quad) & = & \mathfrak{W} \langle j-1, j-k-3 \rangle & -2 \leq k \leq j-3 \\ H^3(\quad " \quad) & = & \mathfrak{W} \langle j-k-4, j \rangle & k \leq -4 \end{array} \right.$$

all others vanish. Note that  $\textcircled{2}, \textcircled{3}$  are mutually Serre dual. These cohomology groups are given as irreducible representations  $\langle a, b, c \rangle$  of  $GL(4, \mathbb{C})$ , corresponding to the Young tableaux :

with  $a \geq b \geq c$ . ( $\langle a^2, b \rangle \equiv \langle a, a, b \rangle$  and  $\langle a, b \rangle \equiv \langle a, b, 0 \rangle$ .)  $\mathfrak{W} = \langle 1^4 \rangle$  is the determinant character, corresponding to the natural action of  $GL(4)$  in  $\Lambda^4 \mathbb{T} \cong \mathbb{C}$ .



The sequences on  $\mathbb{Q}$  analogous to ① define the universal bundle  $S$ , and the universal quotient bundle  $Q$ , each of rank 2, and their duals  $S^* \cong S(1)$ ,  $Q^* \cong Q(-1)$ . (Line bundles  $\mathcal{O}_{\mathbb{Q}}(j)$  on  $\mathbb{Q}$  are induced by restriction of  $\mathcal{O}(j)$  on  $\mathbb{P}^5$ .)

$$\begin{aligned} 0 \rightarrow S \rightarrow T \otimes \mathcal{O}_{\mathbb{Q}} &\rightarrow Q \rightarrow 0 \\ 0 \rightarrow Q^* \rightarrow T^* \otimes \mathcal{O}_{\mathbb{Q}} &\rightarrow S^* \rightarrow 0 \end{aligned} \quad \left. \right\} \textcircled{4}$$

The following direct images, of bundles on  $B$ , can be identified:

$$\textcircled{5} \quad \left\{ \begin{array}{lcl} q_* [ p^* \mathcal{O}_{\mathbb{P}}(k) \otimes q^* \mathcal{O}_{\mathbb{Q}}(j) ] & = & S^{*(k)}(j) \\ q'_* [ p^* \mathcal{O}_{\mathbb{P}}(-k-2) \otimes q^* \mathcal{O}_{\mathbb{Q}}(j+1) ] & = & S^{(k)}(j) \\ p_* [ p^* \mathcal{O}_{\mathbb{P}}(k) \otimes q^* \mathcal{O}_{\mathbb{Q}}(j) ] & = & \Omega'^{(2)}(j)(k) \\ p'_* [ p^* \mathcal{O}_{\mathbb{P}}(k+2) \otimes q^* \mathcal{O}_{\mathbb{Q}}(-j-3) ] & = & \Omega'(-2)(j)(k) \end{array} \right\} \begin{array}{l} k \geq 0, j \in \mathbb{Z} \\ j \geq 0, k \in \mathbb{Z} \end{array}$$

Using the Leray spectral sequences of the (holomorphic) fibrations p.q of  $B$ , the following cohomology groups on  $\mathbb{Q}$  are identified from ②, ③.

$$\textcircled{6} \quad \left\{ \begin{array}{lcl} H^0(S^{(k)}(j)) & \cong & \langle j^2, k \rangle & \text{for } j \geq k \geq 0 \\ H^2(S^{(k+2)}(j-1)) & \cong & \langle k, j^2 \rangle & " \quad k \geq j \geq 0 \\ H^4(S^{(k)}(-j-4)) & \cong & \langle j+k, j \rangle & " \quad j, k \geq 0 \\ H^i(Q^{(k)}(j)) & \cong & H^{4-i}(S^{(k)}(-j-4)) & \text{for } k \geq 0, j \in \mathbb{Z} \end{array} \right.$$

Here powers of the determinant  $\det$  have been ignored, so that these groups may be considered as representation spaces of  $SL(4)$ , or of  $SO(6) \cong SL(4)/\mathbb{Z}_2$  which fixes the defining equation of  $\mathbb{Q} \subseteq \mathbb{P}^5$ .

The tangent & cotangent bundles on  $\mathbb{Q}$  are  $T_{\mathbb{Q}} = S^* \otimes Q$ ,  $\Omega'_{\mathbb{Q}} = Q^* \otimes S$ . The following sequence is exact

$$0 \rightarrow \mathcal{O}_{\mathbb{Q}}(-2) \rightarrow \Omega'_{\mathbb{P}^5}|_{\mathbb{Q}} \rightarrow \Omega'_{\mathbb{Q}} \rightarrow 0$$

giving

$$\textcircled{7} \quad \left\{ \begin{array}{lcl} H^0(\Omega'_{\mathbb{P}^5}|_{\mathbb{Q}} \otimes \mathcal{O}_{\mathbb{Q}}(j)) & = & \langle j-2^2 \rangle \oplus \langle j, j-1, 1 \rangle, & j \geq 2 \\ H^1(\Omega'_{\mathbb{P}^5}|_{\mathbb{Q}}) & = & \emptyset \\ H^4(\Omega'_{\mathbb{P}^5}|_{\mathbb{Q}} \otimes \mathcal{O}_{\mathbb{Q}}(-j-4)) & = & \langle j+2^2 \rangle \oplus \langle j+2, j+1, 1 \rangle, & j \geq -1 \end{array} \right.$$

## The conformal group action on $\mathbb{P}^1(\mathbb{CH})$ .

The multiplication in the space  $\mathbb{CH}$  of complex quaternions seems to be well suited to the expression of Lorentz and conformal group action on Minkowski space. The basic twistor diagram is a natural part of the picture.

Notation.  $\mathbb{CH}$  will stand for the space of complex quaternions  $q = q_0 + i_1 q_1 + i_2 q_2 + i_3 q_3$ , where  $i_1, i_2, i_3$  are three quaternionic units with the usual multiplicative properties and  $q_0, \dots, q_3$  are complex numbers. The complex imaginary unit  $i$  is supposed to commute with  $i_1, i_2, i_3$ . Basic conjugations in  $\mathbb{CH}$  are

$$q \mapsto q^t = q_0 - i_1 q_1 - i_2 q_2 - i_3 q_3; \quad q \mapsto q^* = q_0^* + i_1 q_1^* + i_2 q_2^* + i_3 q_3^*; \quad q \mapsto q^{t*} = q_0^* - i_1 q_1^* - i_2 q_2^* - i_3 q_3^*,$$

where  $*$  means the complex conjugation. The squared norm  $|q|^2$  is defined by

$$|q|^2 = q_0^2 + q_1^2 + q_2^2 + q_3^2 \in \mathbb{C}. \text{ Note that } q^t \text{ exists iff } |q|^2 \neq 0 \text{ and } q^t = \frac{q^*}{|q|^2}.$$

### I. Lorentz group action ([1]).

The complex Minkowski space  $\mathbb{M}^I$  will be identified with the space  $\mathbb{CH}$  as well as with its representation by  $2 \times 2$  complex matrices through mappings

$$z = [z_0, \dots, z_3] \in \mathbb{M}^I \longleftrightarrow q = z_0 + i_1 z_1 + i_2 z_2 + i_3 z_3 \in \mathbb{CH} \longleftrightarrow z^{AA'} = \begin{bmatrix} z_0 + z_3 & z_1 - iz_2 \\ z_1 + iz_2 & z_0 - z_3 \end{bmatrix}.$$

We have  $z^\mu z_\mu = |q|^2 = \det(z^{AA'})$ .

The real Minkowski space  $\mathbb{M}^I$  is fixed by the reality condition

$$M = \{q \in \mathbb{CH} \mid q^{t*} = q\} = \{x = x_0 + i \vec{x}^T \mid x_0, x_j \in \mathbb{R}\}$$

Denoting  $G = \{\alpha \in \mathbb{CH} \mid |\alpha|^2 = 1\}$  we have:

a)  $G \times G$  is the usual covering group for the (restricted) complex Lorentz group; the action is given by  $(\alpha, \beta) \in G \times G : q \in \mathbb{CH} \mapsto \alpha q \beta^t \in \mathbb{CH}$ .

b) the subgroup of  $G \times G$ , which leaves  $M$  invariant, has  $l$  connected pieces; the identity component is just the usual  $2-l$  cover of (restricted) Lorentz group:

$$\times_{G \times G} : \quad q \in M \mapsto \alpha q \alpha^{t*} \in M.$$

Examples: The rotations correspond to  $\alpha = e^{i\vec{x}^T}$ , the boosts to  $\alpha = e^{i\vec{x}^T}$  ( $x_1, \dots, x_3 \in \mathbb{R}$ ).

### II. Conformal group action.

There are simple and useful descriptions of the conformal groups on  $S^2$  and  $S^4$  using the projective groups over  $\mathbb{C}$  and  $\mathbb{H}$ . So the analogy could be:

$$(\mathbb{C} \cong \mathbb{R}_2 \xrightarrow{\text{conf. compactification}} (\mathbb{C}) \cong S^2; \text{ conf. group} \cong \text{PGL}(2, \mathbb{C}) \quad [\text{i.e. } z \mapsto \frac{az+b}{cz+d}])$$

$$(\mathbb{H} \cong \mathbb{R}_4 \xrightarrow{\text{conf. comp.}} \mathbb{P}^1(\mathbb{H}) \cong S^4; \text{ conf. group} \cong \text{PGL}(2, \mathbb{H}))$$

$$(\mathbb{CH} \cong \mathbb{M}^I \xrightarrow{\text{conf. comp.}} (?)) \quad \mathbb{P}^1(\mathbb{CH}) \cong G_{2,4}; \text{ conf. group} \cong \text{PGL}(2, \mathbb{CH}).$$

We shall see that the question mark is necessary and that some unexpected changes have to be made, which lead them directly to the basic twistor diagram. Some care is necessary in the definition of  $\mathbb{P}^1(\mathbb{CH})$  and  $\text{PGL}(2, \mathbb{CH})$ .

Def.  $\mathbb{P}^1(\mathbb{CH}) = \mathbb{CH} \times \mathbb{CH} \setminus \{0, 0\} / \sim$ , where  $\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \sim \begin{bmatrix} q_1 \lambda \\ q_2 \lambda \end{bmatrix}$ , if  $|\lambda|^2 \neq 0$ !

Def.  $\text{GL}(2, \mathbb{CH})$  stands for the group of invertible  $2 \times 2$  matrices with entries in  $\mathbb{CH}$ .

The group  $\text{PGL}(2, \mathbb{CH})$  is defined to be the factor-group of the group  $\text{GL}(2, \mathbb{CH})$  by the subgroup of matrices, which act like the identity on  $\mathbb{P}^1(\mathbb{CH})$ .

Remark. The subgroup of matrices, acting like the identity on  $\mathbb{P}^1(\mathbb{C}\text{IH})$ , consists of matrices  $\begin{bmatrix} z & 0 \\ 0 & \bar{z} \end{bmatrix}$ ,  $z \in \mathbb{C}$ . Hence  $\text{PGL}(2, \mathbb{C}\text{IH})$  is the 15 (complex) parameter group (in fact the (identity component of) complexification of the Minkowski conformal group).

Reality condition.

Denote by  $N$  the subset  $N = \left\{ \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \mid q_1^{+*}q_2 + q_2^{+*}q_1 = 0 \right\} \subset \mathbb{P}^1(\mathbb{C}\text{IH})$ .

Def. Denote by  $C$  the subgroup of  $\text{PGL}(2, \mathbb{C}\text{IH})$ , which leaves  $N$  invariant.

We shall show that the action of  $C$  on  $N$  is at the same time the action of the conformal group on the compactified Minkowski space  $M$  and on the projective twistor space  $\mathbb{P}_3$ .

Remark. The group  $C$  has two components - the one containing the identity and another one, containing the matrix  $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$  (representing PT-transformation). They are just two pieces of the full Minkowski conformal group. The space or time reflections are not included. The action of the group  $C$  will be shown by examples, but first we should collect some informations on  $\mathbb{P}^1(\mathbb{C}\text{IH})$ .

Some facts on  $\mathbb{P}^1(\mathbb{C}\text{IH})$ .

- i) A point in  $\mathbb{C}\text{IH} \times \mathbb{C}\text{IH}$  can be represented by a  $2 \times 4$  matrix:  $\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \sim \boxed{\quad}^2 \sim [U_x V_x]$ , where  $U_x, V_x \in \mathbb{C}_4$  can be considered to be a basis for the 2-dim. (or 1-dim.) subspace in  $\mathbb{C}_4$ .
- ii) The equivalence relation means simply non-singular lin. combination of  $U_x, V_x$ .
- iii) There are 2 types of points in  $\mathbb{P}^1(\mathbb{C}\text{IH})$  /  $U_x, V_x$  indep.  $\Rightarrow$  eq. class  $\sim$  2-dim. subspace  $T_2 \subset \mathbb{C}_4$   
 $U_x, V_x$  dep.  $\Rightarrow$  eq. class  $\sim$  1-dim. subspace  $T_1 \subset \mathbb{C}_4$
- iv) Hence  $\boxed{\mathbb{P}^1(\mathbb{C}\text{IH}) \cong G_{2,4} \cup \mathbb{P}_3}$ .

- v)  $\mathbb{C}M^I$  can be imbedded into  $\mathbb{P}^1(\mathbb{C}\text{IH})$  in many ways. One possibility is

$$\mathbb{C}M^I \simeq U = \left\{ \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \mid |q_2|^2 \neq 0 \right\} \subset \mathbb{P}^1(\mathbb{C}\text{IH})$$

$$z \in \mathbb{C}M^I \simeq \mathbb{C}\text{IH} \longleftrightarrow \begin{bmatrix} iz \\ 1 \end{bmatrix} \sim \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}, q_1 q_2^{-1} = iz \quad \begin{array}{l} \text{the choice of } i \text{ in } q_1 q_2^{-1} = iz \\ \text{is done to fit in with usual} \\ \text{twistor conventions.} \end{array}$$

with reality condition

$$x \in M^I \iff \begin{bmatrix} ix \\ 1 \end{bmatrix} \in U \cap N = \left\{ \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \in U \mid q_1^{+*}q_2 + q_2^{+*}q_1 = 0 \right\} = \left\{ \begin{bmatrix} ix \\ 1 \end{bmatrix} \mid x = x^{+*} \right\}.$$

- vi) If  $\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \sim [U_x V_x]$ ;  $U_x, V_x$  dependent generators of  $T_1 \subset \mathbb{C}_4$ , then the notation

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \sim \begin{bmatrix} w^A \\ \pi_A \end{bmatrix} \otimes \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix} := \begin{bmatrix} w^0 \lambda_0, w^1 \lambda_1 \\ w^1 \lambda_0, w^0 \lambda_1 \\ \pi_0 \lambda_0, \pi_1 \lambda_1 \\ \pi_1 \lambda_0, \pi_0 \lambda_1 \end{bmatrix} \quad \text{is convenient for the comparison with twistor theory.}$$

In this representation we have

$$q_1^{+*}q_2 + q_2^{+*}q_1 = [\bar{\lambda}_0] \otimes [\lambda_0 \lambda_1], \quad (\overline{\pi_A} w^A + \overline{w^A} \pi_A) \quad \begin{array}{l} [\bar{\lambda}_0 \text{ stands for complex conjugation}] \\ = \bar{z} \cdot z^* \end{array}$$

hence  $N \cap \mathbb{P}_3$  is just the space of real twistors ( $\bar{z} \cdot z^* = 0$ ).

- vii) By the action of any element from  $\text{GL}(2, \mathbb{C}\text{IH})$  the space  $\mathbb{C}M^I$  can be embedded into  $\mathbb{P}^1(\mathbb{C}\text{IH})$  in another way. The conformal action then will be simply the same action as before changed only by conjugation in  $\text{PGL}(2, \mathbb{C}\text{IH})$  by the chosen element.

## 24. Examples:

|                         | Real.<br>coord.   | Homog.<br>coord.   | Mink.<br>space  | Twistor<br>space  |  |
|-------------------------|---|--|---|---|--|
| 1) <u>translation</u> : | $\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$<br>$b^{t*} = -b$<br>$b = ia$<br>$a \in \mathbb{M}^+$ | $\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \mapsto \begin{bmatrix} q_1 + bq_2 \\ q_2 \end{bmatrix}$ | $x \mapsto x + a$   | $\begin{bmatrix} w^A \\ \bar{w}_N \end{bmatrix} \otimes \begin{bmatrix} \lambda_2 \\ \bar{\lambda}_2 \end{bmatrix} \mapsto \begin{bmatrix} w^A + ia^{t*} \bar{w}_N \\ \bar{\lambda}_2 \end{bmatrix} \otimes \begin{bmatrix} \lambda_2 \\ \bar{\lambda}_2 \end{bmatrix}$ |  |
| 2) <u>Lorentz</u> :     | $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$<br>$ ad  =  d  = 1$                                  | $a^{t*} \cdot d = 1$<br>$a^{t*} = d$   | $\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \mapsto \begin{bmatrix} aq_1 \\ a^*q_2 \end{bmatrix}$         | $x \mapsto axa^{t*}$<br>$= axa^{t*}$  | $\begin{bmatrix} w^A \\ \bar{w}_N \end{bmatrix} \otimes \begin{bmatrix} \lambda_2 \\ \bar{\lambda}_2 \end{bmatrix} \mapsto \begin{bmatrix} L_A^B w^B \\ L_N^{B'} \bar{w}_{B'} \end{bmatrix} \otimes \begin{bmatrix} \lambda_2 \\ \bar{\lambda}_2 \end{bmatrix}$<br>where $a \in \mathbb{CH} \cup L_A^B \in SL(2, \mathbb{C})$<br>$a^* \in \mathbb{CH} \cup L_N^{B'} \in SL(2, \mathbb{C})$ |
| <u>dilation</u> :       | $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$<br>$a, d \in \mathbb{R}$                             | $ad = 1$<br>$d = \frac{1}{a}$  | $\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \mapsto \begin{bmatrix} aq_1 \\ \frac{1}{a}q_2 \end{bmatrix}$ | $x \mapsto a^2 x$   | $\begin{bmatrix} w^A \\ \bar{w}_N \end{bmatrix} \otimes \begin{bmatrix} \lambda_2 \\ \bar{\lambda}_2 \end{bmatrix} \mapsto \begin{bmatrix} aw^A \\ \frac{1}{a}\bar{w}_N \end{bmatrix} \otimes \begin{bmatrix} \lambda_2 \\ \bar{\lambda}_2 \end{bmatrix}$  |
| 3) <u>involution</u> :  | $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$  |  | $\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \mapsto \begin{bmatrix} q_2 \\ q_1 \end{bmatrix}$             | $x \mapsto -x - \frac{1}{ x ^2} x$  | $\begin{bmatrix} w^A \\ \bar{w}_N \end{bmatrix} \otimes \begin{bmatrix} \lambda_2 \\ \bar{\lambda}_2 \end{bmatrix} \mapsto \begin{bmatrix} \bar{w}_N \\ w_N^A \end{bmatrix} \otimes \begin{bmatrix} \lambda_2 \\ \bar{\lambda}_2 \end{bmatrix}$  |

Remarks. The action of  $GL(2, \mathbb{CH})$  on  $\mathbb{P}^1(\mathbb{CH})$  gives the conformal group action on both compactified Minkowski space and the projective twistor space in an unified way. If elements  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2, \mathbb{CH})$  are represented by  $4 \times 4$  complex matrices, then the  $SU(2, 2)$  subgroup leaves the 'norm'  $q_1 q_1^{t*} + q_2 q_2^{t*}$  invariant. In fact, the (identity component of) group  $C$  is just the factor group of  $SU(2, 2)$  by the finite subgroup  $\{\pm 1, \pm i\} \cdot 11$ , which leaves the 'norm' invariant, but acts on the projective space  $\mathbb{P}^1(\mathbb{CH})$  like the identity.

## III. The topology on $\mathbb{P}^1(\mathbb{CH})$ .

There is a surprise at the end of the story. The space  $\mathbb{P}^1(\mathbb{CH})$  carries the natural factor-topology. It is easy to see that the topology is not quite usual. Consider 2-space  $T_2 \in G_{2,4} \subset \mathbb{P}^1(\mathbb{CH})$ . Then the corresponding equivalence class is not a closed subset of  $\mathbb{CH} \times \mathbb{CH}$ . Possible limits of the representatives  $[V_1 V_2] \in T_2$  clearly contain representatives of another classes, namely those of all 1-spaces  $T_1, T_1 \subset T_2$ . Hence we have:

$\forall T_2 \in G_{2,4} \subset \mathbb{P}^1(\mathbb{CH})$  the closure of  $\{T_2\} = \{T_2\} \cup \{T_1 \mid T_1 \subset T_2\} \subset \mathbb{P}^1(\mathbb{CH})$   
Similarly it can be shown that

$$\forall T_1 \in \mathbb{P}_3 \subset \mathbb{P}^1(\mathbb{CH}) \quad \text{Open}_{T_1 \in \mathcal{O}} \quad \mathcal{O} = \{T_1\} \cup \{T_2 \mid T_2 \supset T_1\} \subset \mathbb{P}^1(\mathbb{CH}).$$

Hence the basic twistor correspondence between  $G_{2,4}$  and  $\mathbb{P}_3$  is encoded in the non-Hausdorff behaviour of the factor-topology. The pathology of the topology sits only in this relation between  $G_{2,4}$  and  $\mathbb{P}_3$ .

The restrictions of the topology to either  $G_{2,4}$  or  $\mathbb{P}_3$  give back the standard topology.

Rewards (for those, who like the space  $\mathbb{P}^0(\mathbb{C})$ ).

The projective space of complex numbers  $\mathbb{P}^0(\mathbb{C})$  became popular recently on Friday meetings. So let me add another strange being to the collection:

Projective group  $\mathrm{PGL}(1, \mathbb{CH})$  acting on the projective space  $\mathbb{P}^0(\mathbb{CH})$ .

The standard definitions (left action, right projective space) are:

$\mathrm{GL}(1, \mathbb{CH}) := \{x \in \mathbb{CH} \mid |x|^2 \neq 0\}$  acts projectively (from the left) on  $\mathbb{P}^0(\mathbb{CH}) := \mathbb{CH}/\nu$ , where  $q \sim q \cdot \lambda$ , if  $|\lambda|^2 \neq 0$ . The subgroup of  $\mathrm{GL}(1, \mathbb{CH})$ , acting like the identity on  $\mathbb{P}^0(\mathbb{CH})$  are just complex multiples of 1, hence  $\mathrm{PGL}(1, \mathbb{CH}) \cong \mathrm{SL}(2, \mathbb{C})/\{\pm 1\} \cong \text{Lorentz}$ .

We have  $\mathbb{P}^0(\mathbb{CH}) \cong \{1\} \cup \mathbb{P}^1(\mathbb{C})$  {the same analysis as before - either  $|q|^2 \neq 0$  or  $q \cong [w^1] \otimes [\lambda_2]$ , the topology is again non-Hausdorff (between two pieces)!}.

So this funny example can be considered to be the Lorentz group action on the projective spinor space  $\mathbb{P}(\mathbb{S}^4)$ .

Remarks. i) The whole discussion can be presented in purely quaternionic notation. It's just suitable for clear explanation to relate things through  $2 \times 2$  matrices to the well-known spinor notation. So only abstract quaternionic properties are at the base and the whole thing doesn't depend on special properties of  $2 \times 2$  matrices representing them.

ii) The use of  $\mathbb{CH} \times \mathbb{CH}$  for the definition of  $\mathbb{P}^1(\mathbb{CH})$  is a 'coordinate' description. To have an analogy of projective spaces of complex vector spaces, the modules over  $\mathbb{CH}$  can be used instead. Only the mapping  $q \mapsto q^\#$ ,  $q \mapsto |q|^2$  are needed on such modules. Such description is then 'coordinate-free', but more abstract and not so suitable for the first presentation.

iii) There is a mirror symmetric version of  $\mathbb{P}^1(\mathbb{CH})$  - left and right multiplication reversed - which correspond to the dual projective twistor space  $\mathbb{P}_3(\mathbb{T}^4)$  together with the 'dual' version of  $G_{2,4}$ . Another larger scheme is possible, if wanted, where both twistor and dual twistor space appear together.

iv) The projection (on the second component)  $\mathbb{P}^1(\mathbb{CH}) \cdot \{\text{something}\} \longrightarrow \mathbb{P}^0(\mathbb{CH})$  is the  $\mathbb{CH}$ -version of the standard nonlinear graviton projection  $\mathbb{P}^3 - \mathbb{P}^1 \rightarrow \mathbb{P}^1$ . An alternative 'topological' formulation of the nonlinear graviton construction is possible, where the usual deformation of the complex structure is substituted by a 'deformation of non-Hausdorff topology' on  $\mathbb{P}^1(\mathbb{CH})$ . But whether this reformulation can give something new is far from clear at the moment.

#### Reference:

- [1] J. Ehlers, W. Rindler, I. Robinson: Quaternions, bivector and the Lorentz group, in Perspectives in Geometry and Relativity, 1966

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Vladimir Souček

## Maxwell's equations from topology

Consider a point positron in an electromagnetic field. The equations of motion are:

$$f^{ab}{}_{;b}(\xi) = -4\pi e \int ds \frac{dx^a}{ds} \delta^4(\xi - x(s)) \quad (\text{Me})$$

$$\bar{f}^{ab}{}_{;b}(\xi) = 0 \quad (\text{Mg})$$

$$m \frac{d^2 x^a}{ds^2} = -e f^{ab} \frac{dx_b}{ds}, \quad (\text{Le})$$

where  $x(s)$  is the worldline of the positron, of charge  $e$  and mass  $m$ .

The equation (Mg) is Bianchi's identity, and hence entirely kinematical.

It is well known that the remaining Maxwell and Lorentz equations can be derived from an action integral by varying with respect to the field  $A_a$  and the worldline. This action has in addition to the free particle and free fields parts an interaction term which comes from assuming a "minimal coupling", thus:

$$S_L(x, A) = -m \left[ \int ds [c]_{\text{positon}} - \frac{1}{16\pi} \int d^4 s F_{ab}(\xi) F^{ab}(\xi) + e \int A_a(x) \frac{dx^a}{ds} ds \right].$$

From the exact duality between electric and magnetic charges, we can regard the positron as an electric monopole and make use of the topological results known in magnetic monopoles. We claim that by thus regarding the electric charge as a topological charge we can do away with assuming any explicit form for the interaction. In other words, the interaction of a topological charge with the gauge field is intrinsically given by the theory.

Electric-magnetic duality means that given any electric potential  $A_a$  there exists, at least locally, a magnetic potential  $A_m$  such that their respective fields are dual to each other as 2-forms. However, in the dual description of a positron,  $A_a$  can no longer be given by a single function but must be patched. This condition is exactly equivalent to (Me). To see this, we observe that  $F^{ab}{}_{;b} = 0$  except possibly on the positron worldline, because it is the Bianchi identity for the connection  $A_a$ ; hence we must have

$$F^{ab}{}_{;b}(\xi) = \int ds \lambda^a(s) \delta^4(\xi - x(s)),$$

or some four-vector  $\lambda^a(s)$  defined on the positron worldline. By integrating over a sphere

surrounding the position we get on the one hand the total flux and on the other (via Stokes' theorem) a line integral involving the patching function. It is straightforward to see that

$$\Delta^2(s) = -4\pi e \frac{dx^a}{ds}.$$

Hence  $(Me)$  is given by the topology.

Now we can use  $(Me)$  as a kinematic constraint, and vary the free action

$$S_{\text{L}}(x, A) = -m \left[ \int ds \right]_{\text{perifrom}} - \frac{1}{16\pi} \int d^4\xi F_{ab}(\xi) F^{ab}(\xi)$$

under such a constraint, with respect to  $\delta A_a$  and  $\delta x_a$ . By the method of Lagrange multipliers we easily obtain  $(Le)$ .

#### Remarks

1) It would be nice to use the same constraint method to treat a position-magnetic photon system. However, the presence of both  $A_a$ - and  $A_a$ -topological charges seems to require treating both  $A_a$  and  $A_a$  as independent variables. NMJN suggests using the Palatini method. This is under investigation.

2) The motivation for this exercise was to treat the case of non-abelian topological charges whose interaction is not known; whereas some form of kinematic constraint must exist by virtue of the topology. The problem is to write this down in a convenient form. So far I can only write it down in an extremely exotic form — using the second loop space.

Discussions with David Matravers are gratefully acknowledged.

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Tim Sheung Tsun

## The Status of Affine Bundles in Twistor Theory.

The 'twisted photon' construction (Ward [1]) provides a direct geometric interpretation of a right handed Maxwell field on some region  $U \subset CM$ , namely as a holomorphic line bundle over the corresponding region  $U'' \subset PT$ . In fact, ZRM fields of any helicity can be represented as holomorphic fibrations (Green [2]). The purpose of this note is to make explicit the ideas involved and to point out a deficiency in the scheme.

Briefly the affine bundle description (for a right handed Maxwell field  $\phi_{\alpha\beta}$  for definiteness) is as follows: Suppose  $\phi_{\alpha\beta}$  on  $U$  corresponds (Eastward, Penrose, Wells [3]) to a cohomology class  $[f] \in H^1(U'', \mathcal{O}(-4))$  which can be represented by a cocycle  $\{f_{ij}\}$  with respect to a Stein cover  $\{U_i\}$  of  $U'' \subset PT$ . Let  $T_{ij}$  be the transition functions for flat twistor space (considered as a line bundle over the projective space) with respect to  $\{U_i\}$ . Let  $J_i$  be the fibre coordinate over  $U_i$  in the locally free sheaf  $\mathcal{O}(-4)$ . Thus we have  $J_i = (T_{ii})^4 J_i$ . If  $s$  is some local section of  $\mathcal{O}(-4)$ , write  $J_i(s)$  for the local function describing it over  $U_i$ . Construct the affine line bundle corresponding to  $\phi_{\alpha\beta}$  by patching fibre coordinates  $E_i$  on  $U_i$  according to

$$\left( \begin{matrix} E_j \\ 1 \end{matrix} \right) = g_{ij} \left( \begin{matrix} E_i \\ 1 \end{matrix} \right) \quad \text{where} \quad g_{ij} = \left( \begin{matrix} (T_{ij})^4 & J_i(s) \\ 0 & 1 \end{matrix} \right) \quad (1)$$

Now suppose the affine line bundle  $A$  has transition functions  $\left( \begin{matrix} a_{ij} & b_{ij} \\ 0 & 1 \end{matrix} \right)$  with respect to the Stein cover  $\{U_i\}$ ,

then its associated vector bundle of translations  $V$  is defined by the transition functions  $a_{ij}$ . Any other affine line bundle with translation bundle  $V$  has transition functions of the form  $\left( \begin{matrix} a_{ij} & b_{ij}' \\ 0 & 1 \end{matrix} \right)$  with respect to  $\{U_i\}$  and

two such bundles are regarded as equivalent if their transition functions satisfy  $\left( \begin{matrix} a_{ij} & b_{ij}' \\ 0 & 1 \end{matrix} \right) = \left( \begin{matrix} \phi_i & r_i \\ 0 & 1 \end{matrix} \right) \left( \begin{matrix} a_{ij} & b_{ij} \\ 0 & 1 \end{matrix} \right) \left( \begin{matrix} \frac{1}{\phi_i} & -\frac{r_i}{\phi_i} \\ 0 & 1 \end{matrix} \right) \quad (2)$

where  $\phi_i$  and  $\gamma_i$  are holomorphic on  $U_i$  and  $\phi_i$  is nowhere zero on  $U_i$  (Atiyah [4]).

Regarding the transition functions ① as local sections of the (non abelian) sheaf  $\Omega^* \circ \Omega$  (the semi-direct product of  $\Omega^*$  and  $\Omega$ ), the patching relations become cocycle conditions and the bundle equivalence ② becomes coboundary freedom. Affine line bundles on  $U'' \subset \mathbb{P}\mathbb{T}$  are thus classified by elements of the cohomology set  $H^1(U'', \Omega^* \circ \Omega)$ .

The construction of the affine line bundle from  $\phi_{\alpha' \alpha}$  can thus be interpreted as a map

$$H^1(U'', \Omega(-4)) \rightarrow H^1(U'', \Omega^* \circ \Omega).$$

Unfortunately this is not injective as the following argument shows.

③ gives  $a_{ij} \phi_j = \phi_i a_{ij}$  and, cancelling the  $a_{ij}$ 's since they're nowhere zero, we get a global holomorphic function  $\phi$ . This must be constant on lines and hence must be globally constant, equal to  $k$  say. The bundle equivalence freedom is thus  $(a_{ij}, b_{ij}) \mapsto (a_{ij}, \gamma_i - a_{ij} \gamma_j + k b_{ij})$ .

In the Maxwell case where  $A$  is obtained from the element  $[f] \in H^1(U'', \Omega(-4))$ ,  $b_{ij}$  is  $S_i(f_{ij})$  so the bundle equivalence freedom is, with  $a_{ij} = T_{ij}^{-1}$ ,

$$S_i(f_{ij}) \mapsto S_i(f_i) + a_{ij} S_j(f_i) + k S_i(f_{ij})$$

(where  $f_i$  is the section of  $\Omega(-4)$  over  $U_i$  represented by the function  $\gamma_i$ )

$$\text{i.e. } S_i(f_{ij}) \mapsto S_i(f_i) - S_i(f_j) + k S_i(f_{ij})$$

$$\text{i.e. } f_{ij} \mapsto f_i - f_j + k f_{ij}.$$

The first two terms on the RHS are just a coboundary. The freedom to choose an equivalent affine bundle thus corresponds to the freedom of multiplying the cohomology class  $[f]$  and hence  $\phi_{\alpha' \alpha}$  by a complex number  $k$ . The affine bundle only encodes the field up to a constant scaling.

In fact the field value at a point  $x_0$  (up to scale) is encoded in the restricted bundle  $A|_{L_{x_0}}$ . This is in contradistinction to the twisted photon where any such restriction is trivial.

A classification theorem of Frenkel [5] has the consequence that affine line bundles over  $L_\infty$  ( $\cong \mathbb{C}P^1$ ) are in one to one correspondance with points of the space  $\mathbb{C}P^2 \cup \{\text{pt}\}$ . The  $\mathbb{C}P^2$  is the  $\mathbb{C}P^2$ 's worth of non zero  $\phi_{n'0}$  up to scale and the point is the zero field value.

As pointed out by K.P.Tod, evaluation of the field at  $\infty$  is a realization of Serre duality

$$H^1(L_\infty, \mathcal{O}(4)) \cong H^0(L_\infty, \mathcal{O}(2))^* = \text{space of symmetric 2 index spinors } \phi_{n'0}.$$

As  $A|_{L_\infty}$  only gives the element of  $H^1(L_\infty, \mathcal{O}(4))$  up to scale, the field  $\phi_{n'0}(\infty)$  at  $\infty$  is only given up to scale.

To obtain a scheme which encoded the field  $\phi_{n'0}$  properly, one could restrict to bundle equivalence transformations of the form  $(a_{ij}, b_{ij}) \mapsto \begin{pmatrix} 1 & t_i \\ 0 & 1 \end{pmatrix} (a_{ij}, b_{ij}) \begin{pmatrix} 1 & -t_j \\ 0 & 1 \end{pmatrix}$  ③

but then one is not dealing with affine bundles and ③ is not the full coboundary freedom in  $H^1(U'', \mathcal{O}^4, \mathcal{O})$ .

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Phil Jones

## A NEW APPROACH TO BOSE AND FERMI STATISTICS

by L.P. Hughston & T.R. Hurd

Perhaps too much attention is devoted to the spin and statistics 'theorem' of quantum field theory (Pauli 1940, Schwinger 1951, Streater & Wightman 1964) which takes the form that 'when such and such conditions are imposed on the quantum fields an inconsistency emerges if the "incorrect" statistics for the fields are assumed.'

It would seem much more natural to look for a geometrical construction which produces the correct spin and statistics relations in a neat and tidy way. We present such a construction here for the special case of non-interacting zero-rest-mass fields.

Although the case is very special, nevertheless the result is sufficiently pretty that we would expect some analog of the idea to go through even when interactions are incorporated.

Wave Functions. For non-interacting fields it pays to forget the quantum field operators and look directly at the  $n$ -particle wave functions. For two scalar particles, for example, one wants a two-point function  $\psi(x, y)$  which (for zero rest mass in each variable) satisfies

$$\overset{x}{\square} \psi(x, y) = 0 , \quad \overset{y}{\square} \psi(x, y) = 0 .$$

The condition of Bose statistics is that the wave function be symmetric, i.e.:

$$\psi(x, y) = \psi(y, x).$$

In the case of two zero rest mass particles each with helicity  $s$  ( $s$  negative, say) we have the field equations:

$$\overset{x}{\nabla}^{A'A} \psi_{A \dots B, P \dots Q}(x, y) = 0 , \quad \overset{y}{\nabla}^{P'P} \psi_{A \dots B, P \dots Q}(x, y) = 0 .$$

The condition of Bose (or Fermi) statistics is then:

$$(*) \quad \psi_{A \dots B, P \dots Q}(x, y) = \pm \psi_{P \dots Q, A \dots B}(y, x) ,$$

the plus sign being taken when  $s$  is integral (Bose statistics), and the minus sign when  $s$  is  $\frac{1}{2}$  odd-integral (Fermi statistics). The relation (\*) above is equivalent to the statement one frequently sees in quantum theory books requiring the wave function for two-particle states to be symmetric (resp., anti-symmetric) under 'simultaneous interchange of the space and spin coordinates' for Bose (resp., Fermi) statistics.

Main Result for Bose Statistics. We assume the reader is familiar with the standard twistor isomorphism (Penrose 1979,

§§ 2.2 & 2.6 ; see also Eastwood et al 1981) :

$$H^1(P_+, \mathcal{O}(-2s-2)) \cong \{ \text{future analytic z.r.m. fields of helicity } s \},$$

where 'future-analytic' means analytic on  $CM^+$  (open future tube), the term 'positive frequency' being reserved for fields analytic on  $CM^+$  (terminology suggested by Tolby Bailey).

Consider the space of symmetrized pairs of twistors  $X^{(\alpha} Y^{\beta)}$ . This space is a 6-dimensional algebraic variety  $V^6$  sitting in the 9-dimensional projective space of symmetric twistors  $A^{\alpha\beta}$ . The equation for  $V^6$  is  $A^{\alpha\beta} A^{\rho\sigma} A^{\delta\rho} = 0$ . By requiring both  $X^\alpha$  and  $Y^\alpha$  to lie in  $P_+^3$  (top half of twistor space) we get a region  $V_{++}^6 \subset V^6$ . Let  $\mathcal{O}_V(m)$  denote the standard sheaf of twisted holomorphic functions on  $V^6$ , obtained from  $\mathcal{O}(m)$  on  $P^9$  by restriction. Then we have the following result:

$$H^2(V_{++}^6, \mathcal{O}_V(-2n-2)) \cong \{ \text{two-particle states of future-analytic z.r.m. fields of helicity } n \text{ (n integral) satisfying Bose statistics} \}.$$

The corresponding result for fermions is somewhat more complicated (it is given below), but what is remarkable is that the same underlying 'symmetrized twistor space'  $V_{++}^6$  can be used for both fermions and bosons, the only difference being a different choice of sheaf. This is nice, because it would be very awkward if systems of fermions and systems of bosons had to live on different 'kinds' of twistor spaces (supersymmetry, for example, would be very difficult to cope with under such circumstances).

Symmetric Functions. To get a picture of why the isomorphism described above works one must consider properties of symmetric functions of two twistors. Suppose  $f(X^\alpha, Y^\alpha)$  is homogeneous of the same degree ( $m$ , say) in each variable and is symmetric, i.e.  $f(X^\alpha, Y^\alpha) = f(Y^\alpha, X^\alpha)$ . Then there exists a function  $F(A^{\alpha\beta})$  homogeneous of degree  $m$  such that  $f(X^\alpha, Y^\alpha) = F(X^{(\alpha} Y^{\beta)})$ . This is perhaps not entirely obvious, since one might imagine other symmetric combinations of  $X^\alpha$  and  $Y^\alpha$  could be manufactured which were in some sense 'independent' of  $X^{(\alpha} Y^{\beta)}$ . A plausible candidate might be  $X^{[\alpha} Y^{\beta]} X^{[\gamma} Y^{\delta]}$ , for example, which is certainly symmetric, but not manifestly related to  $X^{(\alpha} Y^{\beta)}$ . However  $X^{[\alpha} Y^{\beta]} X^{[\gamma} Y^{\delta]}$  can in fact be constructed by applying a Riemann tensor type Young tableau symmetry operator to  $X^{(\alpha} Y^{\beta)} X^{(\gamma} Y^{\delta)}$ ; following this line of reasoning one concludes that any symmetric homogeneous function of two twistors can be expressed as a function  $F(X^{(\alpha} Y^{\beta)})$  i.e. a function on a region of  $V^6$ , and it is not difficult to check (using twistor contour integral formulae) that symmetric two-twistor functions

do indeed give rise to two-particle states satisfying Bose statistics.

Geometrically the situation is summarized in figure A. The space of projective rank-2 twistors  $T^{\alpha\beta}$  is  $P^{15}$ . This space contains 3 important subspaces:  $P^9$ , the space of projective symmetric twistors  $A^{\alpha\beta}$ ;  $P^5$ , the space of projective skew-symmetric twistors; and  $Q^6$ , a 6-dimensional variety which is the Segre embedding of  $P^3 \times P^3$  in  $P^{15}$  given by  $T^{\alpha\beta} = X^\alpha Y^\beta$ . The equation for  $Q^6$  is  $T^{\alpha\beta} T^{\delta\gamma} = T^{\alpha\delta} T^{\beta\gamma}$ . The variety  $V^6$  is, as noted earlier, a subvariety of  $P^9$ . Note that  $P^9 \cap Q^6 = V^6 \cap Q^6 = (P^3)^2$ , where  $P^3$  is projective twistor space, appearing with multiplicity 2, i.e.  $T^{\alpha\beta} = Z^\alpha Z^\beta = A^{\alpha\beta}$ .

For any point in  $V^6 = V^6 - (P^3)^2$  there exists a unique line in  $P^{15}$  through it that intersects  $Q^6$  precisely twice. If  $A^{\alpha\beta} = X^\alpha Y^\beta$  is the point in  $V^6$ , then  $X^\alpha Y^\beta$  and  $Y^\alpha X^\beta$  are the two points in  $Q^6$  that the special line intersects. The line then goes on to intersect  $P^5$  at the point  $X^\alpha Y^\beta$ , which lies on the Klein quadric  $\Omega^4$  (spacetime\*) in  $P^5$ .

Now consider a set  $\mathcal{V}$  in  $V^6$ . For each point in  $\mathcal{V}$  we construct the corresponding 'special line', and by intersecting this set of lines with  $Q^6$  we obtain a pair of regions  $\mathcal{D}_1$  and  $\mathcal{D}_2$  in  $Q^6$ . If  $F(X^\alpha Y^\beta)$  is a function defined on  $\mathcal{V}$ , then by mapping along the special lines a symmetric function is defined on  $\mathcal{D}_1 \cup \mathcal{D}_2 \subset P^3 \times P^3$ .

Main Result for Fermi Statistics. Here we require antisymmetric functions. Such a function necessarily satisfies  $f(X, X) = 0$ , and so is of the form  $X^{[\alpha} Y^{\beta]} G_{\alpha\beta}(X^\alpha Y^\beta)$  for some  $G_{\alpha\beta}(X^\alpha Y^\beta)$  on  $V^6$ . For a fermion state we want  $f(X, Y)$  to have odd homogeneity in each variable, therefore  $G$  must have even homogeneity. Since we are interested in  $G_{\alpha\beta}$  only modulo terms annihilated by  $X^{[\alpha} Y^{\beta]}$  the relevant sheaf  $\mathcal{I}_1(m)$  for fermions is given by the following exact sequence:

$$(V^6) \quad \mathcal{O}_{\mathcal{D}}^{-1}(m-1) \xrightarrow{\varphi} \mathcal{O}_{\mathcal{D}}^{-1}(m) \longrightarrow \mathcal{I}_1(m) \longrightarrow 0$$

where  $\varphi$  is  $\mathcal{O}_{\mathcal{D}}^{-1} \mapsto \mathcal{O}_{\mathcal{D}}^{-1} X^{[\alpha} Y^{\beta]} \epsilon_{\alpha\beta\gamma\delta}$ ; the cohomology result is:

$H^2(V_{++}^6, \mathcal{I}_1(-2n-2)) \cong \{ \text{two-particle states of future-analytic E.R.M. fields of helicity } n - \frac{1}{2} \text{ (n integral) satisfying Fermi statistics} \}$ .

\*Compactified complex spacetime  $\Omega^4$  is the quadric  $\sum_{\alpha\beta} T^{\alpha\beta} T^{\delta\gamma} = 0$ , with  $T^{\alpha\beta} = T^{[\alpha\beta]}$ . It is interesting to note in our picture that both twistor space and spacetime appear. This leads to a very direct geometrical 'construction' of the Klein representation for lines in  $P^3$ .

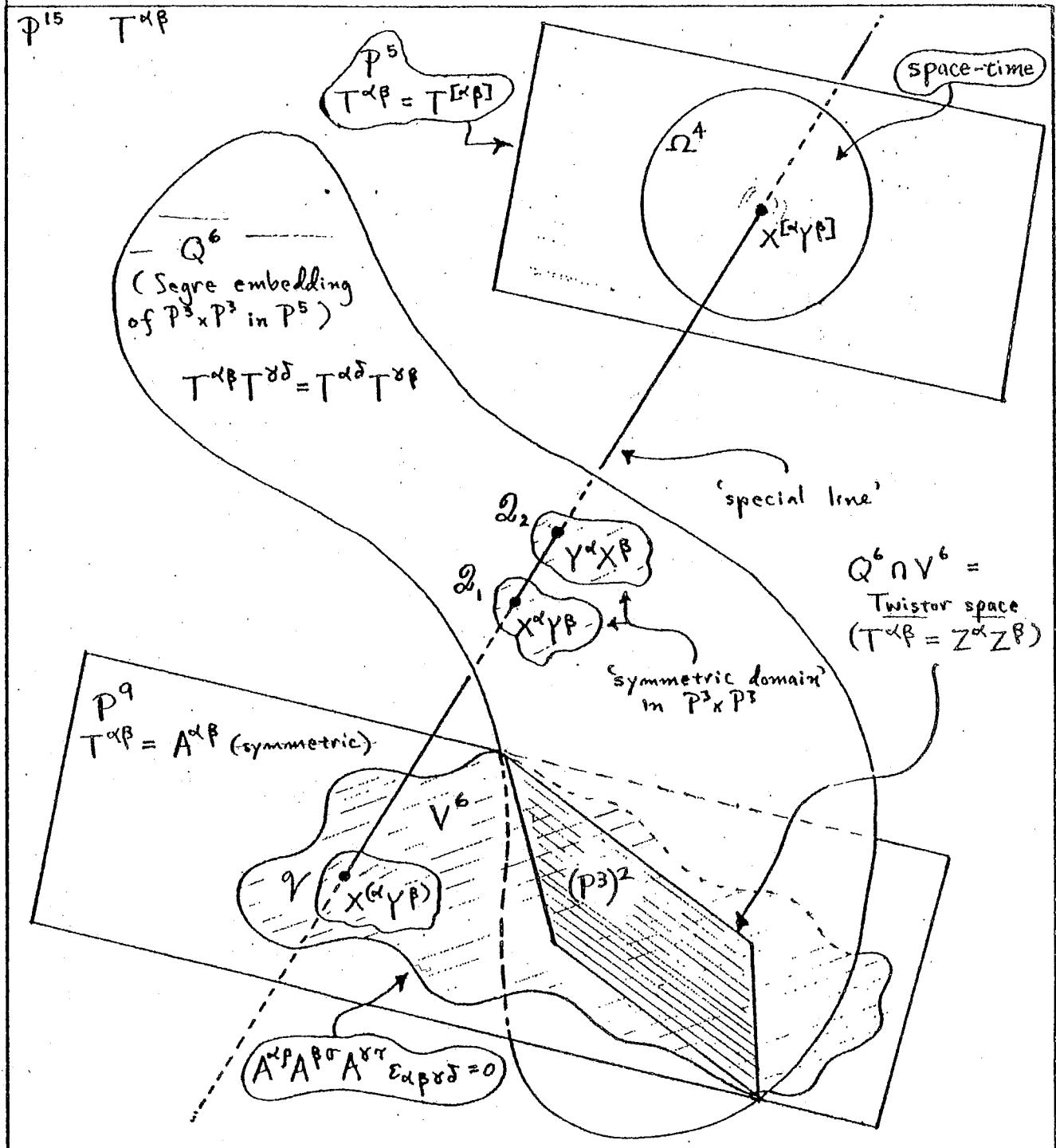


Figure A

Gratitude is expressed to A.D.Helfer, R.Penrose, and M.C.Sheppard for useful discussions.

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## The Geometry of The Ward Construction

Richard Ward (1977) has described an elegant way of coding the information of an asd Yang-Mills field into a deformed twistor space  $\mathcal{I}$ . In fact, the bundle structure of  $\mathcal{I}$  reflects the gauge structure of the YM field in a natural and explicit way, analogous to the way the leg-break graviton's twistor space encodes the connection of an asd spacetime. This note is devoted to a description of this correspondence.

We will assume we are given a connection  $-ig\mathbb{E}_a(x)$  (group indices suppressed) on a region of  $\mathbb{C}M$  corresponding to a region  $X$  of  $\mathbb{R}^{10}$ . The vector space over a point in spacetime on which the group representation acts will be called the YM space of the point.

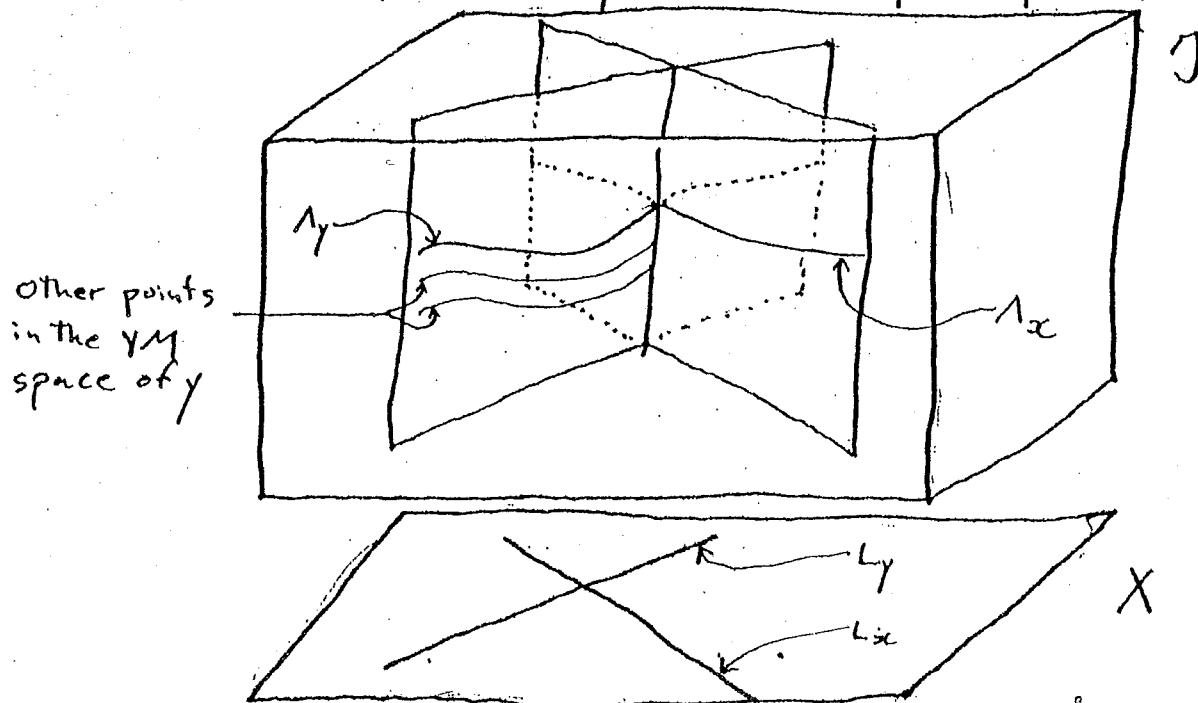
Now recall the Ward picture. The YM field is represented by a vector bundle  $\mathcal{I}$  over  $X$ . The restriction of  $\mathcal{I}$  to any projective line is required to be trivial.

So let  $L_x$  be a projective line in  $X$  corresponding to a point  $x \in \mathbb{C}M$ . Then

$$\mathcal{I}|_{L_x} \cong L_x \times \mathbb{C}^n$$

where  $n$  is the dimension of the YM space at  $x$ . The compactness of  $L_x$  implies that  $\mathcal{I}|_{L_x}$  is ruled in a unique way by  $\mathbb{C}^n$  projective lines, no two of them intersecting. We will regard this  $\mathbb{C}^n$  as the YM space of  $x$ . That is, each point in the YM space of  $x$  corresponds to a line  $L_x \times \{p\}$  in  $\mathcal{I}|_{L_x}$ .

To complete this picture, we need a description of the connection. This is equivalent to a prescription for



3.6.

parallel propagation of YM-space vectors along curves in  $\mathbb{C}M$ . By linearity, it is enough to describe parallel propagation in null directions.

Let  $x$  and  $y$  be null-separated, then, and let  $Lx$  correspond to a point in the YM space at  $x$ .  $Lx$  and  $Ly$  intersect in a point, so let  $Ny$  be the point in  $y$ 's YM space meeting  $Lx$ . (See diagrams.)

This is all very natural and tempting; it remains to be shown that it is correct. We must verify that the connection is linear and that it reproduces the curvature of the field Ward uses to construct  $J$ .

Linearity is immediate from the bundle structure. I will outline the proof of the second desideratum.

Recall the definition of the patching for  $J$ . Pick a line in  $X$  and fix distinct twistors  $P^a, Q^a$  on the line.

For any  $Z^a \in X$ , let  $p^{aa'}(Z^a)$  be the line common to  $P^a$  and  $Z^a$

$$q^{aa'}(Z^a) \quad Q^a \text{ and } Z^a$$

Let

$$U = \{ z^a \in X \mid z^a p \neq 0 \text{ and } L_p(z) \subset X \}$$

$$O = \{ z^a \in X \mid z^a q \neq 0 \text{ and } L_q(z) \subset X \}$$

Let  $Z^a = (\omega^a, \pi_a) \in U \cap O$ , and  $\tilde{z}^a$  be an n-tuple of elements in the proportionality class  $Z^a$ . Now solve

$$\pi^{a'} (\nabla_{AA'} - i g \bar{\Phi}_{AA'}) \tilde{z}^b = 0 \quad (*)$$

on the  $\alpha$ -plane  $Z^a$ . Then the patching identifies

$$\tilde{z}^a = \tilde{z}^a(p) \quad \text{and} \quad \tilde{z}^a = \tilde{z}^a(q)$$

In fact, we can solve  $(*)$  explicitly. Define

$$f_{xp}(p^{aa'}, q^{aa'}, Z^a) = \sum_{n=0}^{\infty} (ig)^n \int_1^n dx_1 \int_2^n dx_2 \dots \int_{n+1}^n dx_n \bar{\Phi}_c(x_n) \dots \bar{\Phi}_1(x_2) \bar{\Phi}_0(x_1)$$

where the path of integration is the same for all terms.  
Then

$$\tilde{z}^a(q) = f_{xp}(p, q, Z^a) \tilde{z}^a(p)$$

If we restrict ourselves to a line  $Lx$ , then in fact we find

$$\tilde{z}^a = f_{xp}(p, x, Z) f_{xp}(x, q, Z) \tilde{z}^a$$

This is in fact a 'splitting' over the line  $Lx$ , which

enables us to write down sections of  $\mathcal{J}/L_\alpha$  immediately. This tells us what the YM space points' equations are, and from there it is straightforward if tedious to compute the curvature from the parallel-propagation prescription.

### Note

The function  $f_{\text{exp}}$  defined here is the factor-ordered analog of the exponential

$$\exp -ig \int_p^q \mathbb{E}_a(x) dx^\alpha$$

to which it reduces if the  $\mathbb{E}_a$ 's commute.  $f_{\text{exp}}$  is not a sheaf map and its cohomological interpretation is unclear.

- Adam Helfer

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## Abstracts

### THE FIRST FORMAL NEIGHBOURHOOD OF AMBITWISTOR SPACE

#### FOR CURVED SPACE-TIME

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Abstract : An analogue of the first neighbourhood of ambitwistor space is constructed for curved space-time, and certain analytic cohomology groups of this neighbourhood are shown to correspond to the solution spaces of the Wave equation and Klein-Gordon equation.

## SCATTERING THEORY AND THE GEOMETRY OF MULTI-TWISTOR SPACES

Matthew L. Ginsberg

**Abstract.** Existing results which show the zero rest mass field equations to be encoded in the geometry of projective twistor space are extended, and it is shown that the geometries of spaces of more than one twistor contain information concerning the scattering of such fields. Some general constructions which describe spacetime interactions in terms of cohomology groups on subvarieties in twistor space are obtained and are used to construct a purely twistorial description of spacetime propagators and of first order  $\theta^4$  scattering. Spacetime expressions concerning these processes are derived from their twistor counterparts, and a physical interpretation is given for the twistor constructions.