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LM

General-relativistic Kinematics ??

Recall that the standard flat-space kinematics of a relativistic object (in the sense of its energy-momentum-angular-momentum structure, the term "dynamics" applying only when a specific Hamiltonian — or equivalent — has been provided governing the equations of motion of the structure of the object) may be described by the symmetric kinematic twistor $A_{\alpha\beta}$ $(= A_{\beta\alpha})$ subject to

$$A_{\alpha\delta} I^{\beta\delta} = \overline{A_{\beta\delta} I^{\alpha\delta}} \qquad (1)$$

The Hermiticity condition (1) is needed to reduce the number of independent components of $A_{\alpha\beta}$ from 10 complex ones (i.e. 20 real ones) to 10 real ones. The obtaining of an appropriate analogue of (1) has proved to be a major stumbling block in the interpretation of the "$A_{\alpha\beta}(\mathcal{S})$" that arises in the general-relativistic theory of 2-surface ("superficial") twistors, this $A_{\alpha\beta}(\mathcal{S})$ being associated with a spacelike topological 2-sphere $\mathcal{S}$ in (curved) space-time $\mathcal{M}$ and being intended to describe the "kinematics" (in the above sense) of the (total) object surrounded by $\mathcal{S}$ (cf. R.P. in TN 13 and Proc. R. Soc A 381 (1982) 53).

I have always regarded (1) (or equivalent) as a rather unpleasant-looking relation, from the point of view of twistor theory, for such a basic system of physical quantities as that described by $A_{\alpha\beta}$. Perhaps there is some deep-seated connection between this awkwardness and the difficulties in obtaining an appropriate analogue of (1) in general relativity, for a finite 2-surface $\mathcal{S}$. What I have in mind is the possibility that the 2-surface twistor space $\mathbb{T}^\alpha(\mathcal{S})$

may have a structure slightly different from that of the standard $\mathbb{T}^\alpha$ associated with Minkowski space $\mathbb{M}$, and that the $A_{\alpha\beta}$ in $\mathbb{M}$ may be describing an infinitesimal change away from this standard Minkowskian twistor space structure. The idea is that the $A_{\alpha\beta}(\mathcal{S})$, for a finite $\mathcal{S}$ in a curved $\mathcal{M}$ would have a somewhat different structure from the $A_{\alpha\beta}$ in $\mathbb{M}$ and would perhaps describe a finite change away from the standard Minkowskian twistor space structure.

How can we associate the Minkowskian $A_{\alpha\beta}$ with an infinitesimal change in twistor space structure? A clue may be found in the "kinematic sequence" (cf. Penrose & Rindler, Spinors and Space-time, Vol. 2 1983 (or 84?) and L.P.H & T.R.H. in 'T' N 13, p. 18) where it is found that the space of $A_{\alpha\beta}$'s plays a dual role, either as a subspace of $\mathbb{T}_{(\alpha\beta)}$ (subject to (1)), or as a factor space of the trace-free Hermitian part $\widehat{\mathbb{H}}^\alpha_\beta$ of $\mathbb{T}^\alpha_\beta$. We can regard $\widehat{\mathbb{H}}^\alpha_\beta$ as the space of real conformal Killing vectors k^a in $\mathbb{M}$, these providing infinitesimal motions of $\mathbb{T}^\alpha$ preserving the twistor norm $Z^\alpha \bar{Z}_\alpha$. The relation between k^a and the corresponding angular momentum structure, given by M^{ab} regarded as a field (a function of the origin point), may be expressed as

$$M^{ab} = \nabla^{[a} k^{b]} \qquad (2)$$

(cf. R.P. & M.A.H. MacC. Phys Rep & R.P. & W.R., S&S-T, Vol 2). The required dependence on position of M^{ab} is a consequence of (2) (where $\nabla^{(a} k^{b)} = \frac{1}{4} g^{ab} \nabla_c k^c$) and all M^{ab}'s arise in this way. The translations and dilations (and their compositions)

are the k^a's which are killed by (2), so we may regard the space of M^{ab}-fields (i.e. the space of $A_{\alpha\beta}$'s) as the space of conformal Killing vectors for $\mathcal{M}$ factored out by the translations and dilations. Thus to describe this space (or its non-linear "exponentiation") we seek some structure in $\mathcal{M}$ which is invariant precisely under these translations and dilations and look for the family of conformal transforms (infinitesimal or finite) of this structure.

One such structure is the "projective π-projection" of $\mathbb{T}$

$$\mathbb{T}^\alpha \longrightarrow \{\$_{A'}\} \qquad (3)$$

where $\{\$_{A'}\}$ stands for the projective space of $\pi_{A'}$'s, taken as a given space. Thus a proposal for a "non-linear $A_{\alpha\beta}$" would be a map (3) which is a general $SU(2,2)$-transform of the standard one. Geometrically, (2) is described by a line $\hat{I}$ in $\mathbb{PN}$, where the family of planes through $\hat{I}$ is given a parameterization as a standard Riemann sphere $\mathbb{CP}^1$ (namely $\{\$_{A'}\}$), the fibres of the projection being the individual planes. It is perhaps a little easier to think in dual terms, where the injection

$$\{\$^A\} \longrightarrow \{\mathbb{T}^\alpha\} \qquad (4)$$

plants a standard Riemann sphere $\mathbb{CP}^1$ as a parameterized projective line $\hat{I}$ in $\mathbb{PN}$, and the different ways of doing this give the required space. Note that there is a 10-parameter (real) freedom in doing this (4 for the line $\hat{I}$; 6 for the different choices of parameterization on $\hat{I}$). When $\hat{I}$ lies infinitesimally

separated from I, with an infinitesimal shift in parameterization, we recover the original space of "linear" $A_{\alpha\beta}$'s.

I am trying to put forward a minor modification of this tentative suggestion. In place of (3) and (4), consider the possible exact sequences

$$0 \to \$^A \to \mathbb{T}^\alpha \to \$_{A'} \to 0 \qquad (5)$$

where $\$^A$ is a given 2-dimensional complex vector space and $\$_{A'}$ is its complex conjugate dual. $\mathbb{T}^\alpha$ is also taken to be a given space, namely a particular 4-dimensional complex vector space with standard (++--) twistor norm $Z^\alpha \bar{Z}_\alpha$, but the maps can vary. (5) is to be compatible with this structure of $\mathbb{T}^\alpha$ in the (standard) sense that the dual sequence

$$0 \leftarrow \$_A \leftarrow \mathbb{T}_\alpha \leftarrow \$^{A'} \leftarrow 0 \qquad (6)$$

to (5) is also the complex conjugate of (5) (in the reverse order). In terms of indexed quantities, the sequences (5), (6) can be expressed by use of objects

$$e_{A'\alpha}\ ,\quad e_A{}^\alpha \qquad (7)$$

where

$$\overline{e_{A'\alpha}} = e_A{}^\alpha \qquad (8)$$

and

$$e_{A'\alpha}\, e_B{}^\alpha = 0\ , \qquad (9)$$

$\$^A \to \mathbb{T}^\alpha$ being achieved by $\omega^A e_A{}^\alpha = Z^\alpha$ and $\mathbb{T}^\alpha \to \$_{A'}$ by $Z^\alpha e_{A'\alpha} = \pi_{A'}$, etc.

Suppose we have two such pairs of objects, namely (7), subject to (8), (9), and

$f_{A'\alpha}$, with $f_A{}^\alpha = \overline{f_{A'\alpha}}$, $f_{A'\alpha} f_B{}^\alpha = 0$. (10)

Let us define

$$A_{\alpha\beta} = -i C \epsilon_{A'(\alpha} f^{A'}{}_{\beta)}, \quad (11)$$

for some positive constant C, and

$$B_{\alpha\beta} = \epsilon_{A'[\alpha} f^{A'}{}_{\beta]} \quad (12)$$

(indices raised with the "standard" $\epsilon^{A'B'}$ of $\mathbb{S}_{A'}$). When $f_{A'\alpha} = e_{A'\alpha}$ = the canonical $\pi_{A'}$-projector for $\mathcal{M}$, then $A_{\alpha\beta} = 0$ and $B_{\alpha\beta} = I_{\alpha\beta}$. When $f_{A'\alpha}$ differs from the canonical $e_{A'\alpha}$ by a small quantity of order ε, then $A_{\alpha\beta}$ is of order ε and is, to 1st order in ε, a quantity having the structure of a kinematic twistor (satisfying (1) to 1st order in ε), where $I_{\alpha\beta} = B_{\alpha\beta} + O(\varepsilon)$. For a general $A_{\alpha\beta}$, $B_{\alpha\beta}$ given by (11), (12) we have the Hermiticity property

$$A_{\alpha\delta} \bar{B}^{\beta\delta} = \bar{A}^{\beta\delta} B_{\alpha\delta}, \quad (13)$$

like (1), but $B_{\alpha\beta}$ is not normally "simple". (Certain restrictions may be put on $e_{A'\alpha}$, $f_{A'\alpha}$ if $\epsilon_{\alpha\beta\gamma\delta}$ is regarded as part of the structure of $\mathbb{T}^\alpha$ e.g.

$$e_{A'[\alpha} e^{A'}{}_{\beta]} = \tfrac{1}{2} \epsilon_{\alpha\beta\gamma\delta} e_A{}^\gamma e^{A\delta} \quad (14)$$

and the corresponding relation for the f's, but the status of these is unclear.)

Recall, now, the Tod form of the expression

for $A_{\alpha\beta}(\mathcal{S})$:

$$A_{\alpha\beta}(\mathcal{S}) = -\frac{i}{4\pi G}\oint\left(\pi_{\alpha}^{0'}\pi_{\beta}^{1'} + \pi_{\alpha}^{1'}\pi_{\beta}^{0'}\right)\underset{\sim}{\mathcal{S}} \qquad (15)$$

($\underset{\sim}{\mathcal{S}}$ = surface-area 2-form on $\mathcal{S}$) (cf. R.P., PRS A381(1982)53 and TWN 13), and also the local expression of

$$\pi_{\alpha}^{0'}\pi_{\beta}^{1'} - \pi_{\alpha}^{1'}\pi_{\beta}^{0'} \qquad (16)$$

for $I_{\alpha\beta}$. This tantalizingly suggests a relation between (15) and (11), and between (16) and (12), with something like an integral of $\pi_{\alpha}^{0'}\pi_{\beta}^{1'}$ over $\mathcal{S}$ producing the quantity

$$S\,e_{A'\alpha}f^{A'}{}_{\beta} = SB_{\alpha\beta} + \frac{iS}{C}A_{\alpha\beta} \qquad (17)$$

(S being the total surface area of $\mathcal{S}$, and $C = \frac{S}{4\pi G}$). For this to make sense, the integral of $\pi_{\alpha}^{0'}\pi_{\beta}^{1'}$ would have to have matrix rank 2 (as an element of $\mathbb{T}_{\alpha\beta}$). I have not been able to prove (or disprove) this, and the whole scheme remains highly conjectural. It is possible that something along these lines may work, however. Roger Penrose

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Solution to sequence problem in TWN 10, p.22:

..., 7, 9, 12, (16.6355323...), 24, 36, 56, 90, ...

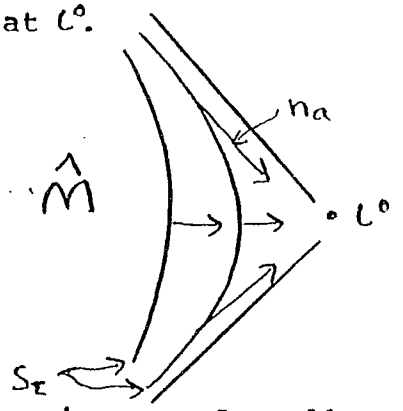
The formula for the nth term is $\frac{24(2^n-1)}{n}$, where (?) is the 0th term — for which a limit needs to be taken. L'Hopital's rule gives $24\log 2$.

Solution to puzzle in TWN 14 Direct method: $a(bcd)e = (add)(bcd)e =$
$= ad(d(bcd)e) = ad((ccd)(bcd)e) = ad(((cbb)cd)(bcd)e) = ad((cb(bcd))(bcd)e) =$
$= ad(cb((bcd)(bcd)e)) = ad(cbe)$. What's it all about? The axioms are those of a "group action" or "torsor" (or fibre of a principal fibre bundle!), which is a group but with no specified identity ("affine space"). Select any element. Call it 1. Def. $a\cdot b$ = "a1b" and a^{-1} = "1a1", then "abc" = $a\cdot b^{-1}\cdot c$, and result follows.

SPINORS, ZRM FIELDS AND TWISTORS AT SPACELIKE INFINITY.

To describe spinor fields at spacelike infinity ($\mathcal{I}^0$), it is necessary to construct an SU(1,1) spin structure on K, the hyperboloid of spacelike directions at $\mathcal{I}^0$. Then, given spinor differential equations on the space-time, one can compute the asymptotic spinor d.e.s induced on the hyperbolic structure.

One way of doing this is to reduce SL(2,$\mathbb{C}$) spinors to SU(1,1) spinors on a foliation of timelike hypersurfaces contracting to a point at $\mathcal{I}^0$.



$$M \to \hat{M}$$
$$g_{ab} \to \Omega^2 \overset{*}{g}_{ab}$$

If one is working in a conformally rescaled space-time a natural foliation is the family:

$$S_\Sigma = \text{surfaces } \{\Sigma \equiv \Omega^{1/2} = \text{constant}\}$$

with outward spacelike normal form $\eta_a = \hat{\nabla}_a \Sigma$, with normalised form $n^a = \eta^a / \|\eta^b \eta_b\|^{1/2}$

SL(2,$\mathbb{C}$) may be reduced to SU(1,1) by the standard procedure: let

$$\lambda^{\dagger A} = \sqrt{2}\, n^{AB'} \overline{\lambda}_{B'}$$

then

$$(\lambda^\dagger)^{\dagger A} = +\lambda^A$$

and an inner product defined by $\langle \alpha, \beta \rangle = \alpha^{\dagger A} \beta_A$.

Define (Witten operator)

$$\mathbb{D}_{AB} = \sqrt{2}\, n^{B'}_{(A} \hat{\nabla}_{B)B'}$$

$\mathbb{D}_{AB}$ is related to the intrinsic spinor covariant derivative D_{AB} preserving

$$h_{ab} = \hat{g}_{ab} + n_a n_b = h_{(AC)(BD)} = -\frac{1}{2}\left(\epsilon_{AB}\epsilon_{CD} + \epsilon_{CB}\epsilon_{AD}\right)$$

by

$$\mathbb{D}_{AB}\lambda_C = D_{AB}\lambda_C + \frac{1}{\sqrt{2}}\pi_{ABCD}\lambda^D \qquad (*)$$

where $\pi_{ab} = \pi_{(AC)(BD)}$ is the second fundamental form, which may be decomposed into shear and expansion parts as

$$\pi_{ABCD} = \sigma_{(ABCD)} - \frac{\theta}{6}\left(\epsilon_{AC}\epsilon_{BD} + \epsilon_{BC}\epsilon_{AD}\right)$$

One then computes the spinor d.e.s induced on S_Σ, imposes a regularity condition on the spinors, and defines

$$D^K_{AB} = \lim_{\Sigma \to \mathcal{I}^0} D_{AB}$$

Then the curvature of D^K is that of a unit timelike hyperboloid, and the limit $\Sigma \to 0$ can be taken.

Firstly this is done for ZRM fields, and secondly for twistors.

1. ZERO REST MASS FIELDS

Let $\phi_{AB\ldots CD}$ be a field with 2n indices and helicity s=-n satisfying

$$\hat{\nabla}^{A}_{A'}\phi_{AB\ldots CD} = 0$$

in $\hat{\mathcal{M}}$-(image of sources in $\mathcal{M}$).

and suppose that

$$\tilde{\phi}_{AB\ldots LM} = \Sigma^{2r}\phi_{AB\ldots LM}$$

is finite but direction dependent at i^0. Then applying the above and computing Π_{ab} to $O(\Sigma^{-1})+O(1)$, one finds that the limit $\overset{\circ}{\tilde{\phi}}$ of $\tilde{\phi}$ satisfies

$$\mathcal{D}^{K}_{DA}\overset{\circ}{\tilde{\phi}}^{A}{}_{B\ldots LM} = \frac{1}{\sqrt{2}}[n+1-2r]\overset{\circ}{\tilde{\phi}}_{DB\ldots LM}$$

For positive helicity fields $\psi_{A'B'\ldots C'D'}$, let

$$\overline{\psi}_{AB\ldots CD} = 2^{n}n_{A}{}^{A'}n_{B}{}^{B'}\ldots n_{D}{}^{D'}\psi_{A'B'\ldots C'D'}$$

and a similar computation yields,

$$\mathcal{D}^{K}_{DA}\overset{\circ}{\tilde{\overline{\psi}}}^{A}{}_{B\ldots LM} = -\frac{1}{\sqrt{2}}[n+1-2r]\overset{\circ}{\tilde{\overline{\psi}}}_{DB\ldots LM}$$

Reconstituting these (for $n\in\mathbb{Z}$) into vector equations and then recombining as electric and magnetic parts the leading order equations of Geroch[1], Sommers[2], and Ashtekar-Hansen[3] can be recovered.

2. TWISTORS.

The same procedure can be applied to the twistor eqn:

$$\nabla^{(A}_{A'}\omega^{B)} = 0$$

provided that the leading order magnetic part of the Weyl curvature vanishes, i.e.

$$0 = B_{ab} = \lim_{\Sigma\to 0}\left(\Sigma\hat{C}_{acbd}n^{c}n^{d}\right).$$

(This will be demonstrated shortly). The most straightforward way of doing the calculation is to firstly work it out in $\mathcal{M}$, and then ask when the resulting equations generalise to a conformally curved space.

Writing, in $\hat{\mathcal{M}}$, $\hat{\omega}^{A} = \omega^{0A} + \Sigma\omega^{1A}$

then one finds a pair of differential eqns. on $\mathcal{K}$, viz

$$\mathcal{D}^{K}_{CA}\omega^{0}_{B} = \frac{1}{\sqrt{2}}\omega^{0}_{(A}\epsilon_{C)B} \qquad (1)$$

$$\mathcal{D}^{K}_{CA}\omega^{1}_{B} = -\frac{1}{\sqrt{2}}\omega^{1}_{(A}\epsilon_{C)D} - \frac{\mu}{\sqrt{2}}K_{CABD}\omega^{0D} \qquad (2)$$

[μ is coefficient $O(1)$]

where K_{ab} is the $O(1)$ part of Π_{ab}. In a conformally curved space K_{ab} acts as a tensor potential for B_{ab} via

$$B_{ab} \propto \epsilon_{a}{}^{mn}\mathcal{D}^{K}_{m}K_{bn}$$

10. 1 and 2 are the asymptotic twistor eqns for $\mathbb{M}$. When do they generalise to a curved space? Now the Ricci identity for K is

$$2\overset{K}{D}_{E(F}\overset{K}{D}_{A)}{}^{E}\lambda_{C} = \lambda_{(F}\epsilon_{A)C}$$

and applying this to 2, gives immediately

$$\overset{K}{D}_{(A}{}^{B}K_{C)BEF} = 0$$

so $B_{ab}=0$.

These eqns. have many interesting properties. For example, asymptotic twistors act as helicity-raising operators on asymptotic ZRM fields. One can do this twice, in which case acting on a ZRM field, one is essentially contracting the field with a vector

$$W^{a} = W^{AB} = \omega^{(A}\tilde{\omega}^{B)}$$

for a pair $\omega^{A}, \tilde{\omega}^{A}$ of solutions to 1 and 2. These vectors are essentially the conformal Killing vectors on $I^{<}$. To see this, one can first perform a SPI supertranslation[3], $\Sigma \to \Sigma(1+\alpha\Sigma)$, and set $K_{ab}=0$. Let $W^{AB}=\omega^{(A}\tilde{\omega}^{B)}$ where $\lambda, \tilde{\lambda} = \pm 1$

$$\overset{K}{D}_{CA}\omega_{B} = \frac{\lambda}{\sqrt{2}}\omega_{(A}\epsilon_{C)B} \quad ; \quad \overset{K}{D}_{CA}\tilde{\omega}_{B} = \frac{\tilde{\lambda}}{\sqrt{2}}\tilde{\omega}_{(A}\epsilon_{C)B}$$

There are two possibilities $\lambda = -\tilde{\lambda}$ or $\lambda = +\tilde{\lambda}$.

a) $\lambda = -\tilde{\lambda}$. Then W^{a} satisfies

$$D_{a}W_{b} + \left(\frac{\lambda}{\sqrt{2}}\omega_{c}\tilde{\omega}^{c}\right)h_{ab} = 0$$

b) $\lambda = \bar{\lambda}$. Then

$$D_{a}W_{b} = \lambda i\epsilon_{abc}W^{c}$$

It is well known that the ten <u>real</u> conformal Killing fields on K are precisely

a) 4 curl-free conformal Killing fields.

b) 6 Killing fields with rotation. So, in fact, when $K_{ab}=0$,

$$\mathbb{T}^{\mathbb{M}}(I^{0}) \otimes_{S} \mathbb{T}^{\mathbb{M}}(I^{0}) \cong \mathbb{C}\{\text{conformal Killing fields}\}$$

When $K_{ab}\neq 0$ the translational (type a) fields acquire additional terms unless one of ω^{0A} or $\tilde{\omega}^{0A}$ vanish.

These ideas have applications to the theory of conserved quantities. Certainly in linearised theory one can recover the ADM mass and a supertranslation invariant notion of angular momentum. For the curved asymptotically flat space the ADM mass is again recovered an a new notion of angular momentum obtained. Indeed, it is conjectured, that (possibly with some modifications) fields of the type described above represent the limits of 2-surface twistors[4] at I^{0}.

William Shaw

Thanks to R. Penrose & K. P. Tod.

<u>Refs</u>

① Geroch in "Asymptotic Structure of Spacetime"

② Sommers: J. Math Phys 19(3), 549

③ Ashtekar-Hansen J. Math Phys 19(7) 1542

④ Penrose T.N. 13

The Manifold of Pure Spinors is a Homogeneous Space

In TN 14, I described an inductive approach to defining spinor representations, whereby an $SO(2k)$-spinor could be regarded as a direct sum of two $SO(2k-2)$-spinors — a generalization of the correspondence between a twistor Z^α and a pair of $SU(4)$ (or Lorentz) spinors $(\omega^A, \pi_{A'})$.

Further, it was seen that in higher dimensions, not all spinors correspond to null α- or β- planes. Rather, the spinors that do so correspond must lie on the intersection of various quadrics in (projective) spinor space. Thus for example, in 8 dimensions, an arbitrary spinor is a pair (Z^α, W_β) of two $SO(6)$-spinors (twistors!), while a pure spinor (ie one which corresponds to a null 4-plane in $\mathbb{C}^8$) has the special form $Z^\alpha W_\alpha = 0$. The problem is that the dimension of the spinor space goes up as 2^k, while the number of quadrics goes up as approximately 2^{2k}. The resulting variety of pure spinors is thus rather hard to understand. In fact, Hodge & Pedoe calculated directly the (real) dimension of this variety ( ) and found the simple answer $k(k-1)$ — a rather easier size than either of the powers of 2 we dealt with earlier. Their calculations are very long & unilluminating, so the obvious question is to understand this manifold in some more concise, 'spinorial' fashion.

In fact, the answer is as obvious as the question. It is easy to see that the rotation group $SO(2k)$ acts transitively on all totally null k-planes in $\mathbb{C}^{2k}$. Thus, the manifold of totally null k-planes is simply the group manifold $SO(2k)$ modulo the stabilizer of a single null plane. The stabilizer is not hard to compute (thanks to K. Hannabuss) — and it is $U(k)$. Thus this horrible intersection of 2^{2k} quadrics inside $\mathbb{CP}^{2k-1}$ has a simple form:

Prop: The variety of pure $SO(2k)$-spinors, which is the sub-manifold of projective $SO(2k)$-spinor space that corresponds to the null k planes in $\mathbb{C}^{2k}$ is precisely the homogeneous space $SO(2k)/U(k)$.

As a trivial corrollary, we see that the (real) dimension of the manifold is $k(2k-1) - k^2 = k(k-1)$, which agrees with Hodge & Pedoe.

Rmk: LPH suggested an extended connection between solutions to the wave equation in $\mathbb{C}^n$ and contour integrals over the pure spinor space (TN 14). Integration over a homogeneous space

(cont'd. on p. 17)

Curved Space Twistors and GHP

by B Jeffryes

As is well known, if $\omega^{A\ldots B}$ is a solution to the n index twistor equation,

$$\nabla^{A'(A}\omega^{B\ldots C)} = 0 \qquad 1.$$

then an additional consistency condition may be found by applying $\nabla_{A'}{}^{D}$ and symmetrizing over unprimed indices (Penrose 1975); namely

$$\Psi_F{}^{(ABC}\omega^{D\ldots E)F} = 0 \qquad 2.$$

It is clear that if ω is null then Ψ_{ABCD} is also null and they share the same principal null direction (henceforth abbreviated to pnd). It is not immediately clear what the restriction on Ψ_{ABCD} is if ω is of a more general type. The conditions may be simply found by taking components of 1 in the Geroch-Held-Penrose formalism (GHP 1973). Those not familiar with the formalism should read what follows in conjunction with the above reference.

Theorem

If $\omega^{A\ldots B}$ is a solution to the n index twistor equation then Ψ_{ABCD} is proportional to $(\omega_{A\ldots B})^{4/n}$.

Proof.

If we choose $\{o_A, \iota^A\}$ as a spinor dyad, normalized by $\iota^A o_A = 1$ then if $\omega^{AB\ldots CD}$ is a solution to 1. we write

$$\begin{aligned}
\omega_0 &= o^A \ldots o^D \omega_{A\ldots D} && \text{type } \{n,0\}\\
\omega_1 &= \iota^A o^B \ldots o^D \omega_{A\ldots D} && \text{type } \{n-2,0\}\\
&\vdots \qquad\qquad \vdots\\
\omega_{n-1} &= \iota^A \ldots \iota^C o^D \omega_{A\ldots D} && \text{type } \{2-n,0\}\\
\omega_n &= \iota^A \ldots \iota^D \omega_{A\ldots D} && \text{type } \{-n,0\}
\end{aligned}$$

First we consider the case when ω is null. We set o_A as its pnd. ω_n is then the only non-zero component and 1 becomes.

$$\begin{aligned}
&K\omega_n = 0 && \sigma\omega_n = 0 && 3/.\\
&(\text{Þ} + n\rho)\omega_n = 0 && (\eth + n\tau)\omega_n = 0 && 4/.\\
&\eth'\omega_n = 0 && \text{Þ}'\omega_n = 0 && 5/.
\end{aligned}$$

3/. implies $\sigma = K = 0$. By GHP 2.23 therefore $\Psi_0 = 0$. Applying the commutator $[\text{Þ}', \eth]$ to ω_n, using GHP 2.31 gives.

$$0 = n\Psi_3.$$

applying $[\text{Þ}, \eth]$ and using 2.21 & 2.24 we have

$$0 = 3n\Psi_1$$

$[\text{Þ}', \text{Þ}]$ and $[\eth', \eth]$ give us.

$$\text{Þ}'\rho = -\tau\bar{\tau} + \Psi_2 + \Phi_{11} - \Lambda$$

$$\eth'\tau = -\rho\bar{\rho}' - \Psi_2 + \Phi_{11} - \Lambda.$$

substituting into 2.26 gives $\Psi_2 = 0$, thus Ψ_4 is the only non-vanishing component.

14. If ω is not null then it has at least 2 prds. We chose 2 of these as our dyad. Thus $\omega_0 = \omega_n = 0$.

1/ becomes

$$
\begin{array}{llll}
K\omega_1 & & & = 0 \\
(\text{Þ} + \rho)\omega_1 & - n K \omega_2 & & = 0 \\
(\text{ð}' + (n-1)\tau')\omega_1 & - n(\text{Þ} + 2\rho)\omega_2 & + \ldots & = 0 \quad 6/ \\
\sigma'\omega_1 & - n(\text{ð}' + (n-2)\tau')\omega_2 & + \ldots & = 0 \\
 & - n\sigma'\omega_2 & + \ldots \quad + \ldots & = 0.
\end{array}
$$

Similarly for $\sigma, \text{ð}, \text{Þ}', K'$ substituting for $K, \text{Þ}, \text{ð}', \sigma'$ respectively

Firstly we have $\sigma = K = \sigma' = K' = 0$. Thus $\Psi_0 = \Psi_4 = 0$.

Applying $[\text{Þ}, \text{ð}]$ to the non-vanishing ω_m with the lowest value of m gives

$$(n - 4m)\Psi_1 = 0 \qquad 7/$$

Similarly applying $[\text{Þ}', \text{ð}']$ to the non-vanishing ω_{n-k} with the lowest value of k implies.

$$(n - 4k)\Psi_3 = 0 \qquad 8/$$

First we consider the case where n is not a multiple of 4. $\Psi_1 = \Psi_3 = 0$, thus Ψ_{NSCO} has only 2 prds. Since all prds of ω are also prds of Ψ_{NSCO}, ω has only one non vanishing component, whose derivatives are given by.

$$
\begin{aligned}
(\text{Þ} + m\rho)\omega_m &= (\text{ð} + m\tau)\omega_m = 0 \\
(\text{Þ}' + (n-m)\rho')\omega_m &= (\text{ð}' + (n-m)\tau')\omega_m = 0.
\end{aligned}
\qquad 9/
$$

Applying $[3', 3]$ and $[\eth', \eth]$ to ω_m and using 2.26 we find

$$(n-2m)\Psi_2 = 0 \qquad 10/.$$

To recap; if n is not a multiple of 4 then ω is either a power of $\omega_0 o_A$ or of $\omega_1 o_{(A} \iota_{B)}$ and the Weyl spinor is given respectively by $\Psi_4 o_A o_B o_C o_D$ and $6\Psi_2 o_{(A} o_B \iota_C \iota_{D)}$.

If $n = 4m$ then we seek to show that ω is a power of a 4-twistor which is proportional to Ψ_{ABCD}. ω can have a maximum of 4 pods since every pod of ω is one of Ψ_{ABCD}. We examine the cases according to the type of Ψ_{ABCD}.

$\underline{\Psi_{ABCD} = o_{(A} \iota_B \alpha_C \beta_{D)}.}$

Since for no choice of pod of ω do either Ψ_1 or Ψ_3 vanish each pod must by 7/. & 8/. have multiplicity $n/4$.

$\underline{\Psi_{ABCD} = o_{(A} o_B \iota_C \alpha_{D)}.}$

If our dyad is $\{o_A, \iota^A\}$ then 8/. tells us ι_A has multiplicity $n/4$, similarly if $\{o_A, \alpha^A\}$ is the dyad α^A_A has multiplicity $n/4$, hence o_A has multiplicity $n/2$.

$\underline{\Psi_{ABCD} = o_{(A} o_B \iota_C \iota_{D)}.}$

ω has only 1 term, by 10/. we see o_A & ι_A have multiplicity $n/2$.

$\underline{\Psi_{ABCD} = o_{(A} o_B o_C \iota_{D)}}$

By 8/. ι_A has multiplicity $n/4$, hence o_A has multiplicity $3n/4$.

(References, p. 17.)

□ B Jeffreys 1983.

Integral Formulae for $\Box\phi = 0$

I want to make some (mostly trivial) remarks about various integral solution formulae for the wave equation in n-dimensional flat space, and about whether these integrals generalize to non-linear equations.

The first formula is as follows. Let x^a denote the coordinates of complexified flat n-space C^n (so the index a runs from 1 to n). Let $v^a = v^a(\zeta_1,\ldots,\zeta_{n-2})$ parametrize the null directions; the ζ's are complex variables and are allowed to take on the value ∞. Then

$$\phi(x) = \oint f(v_a(\zeta)x^a,\zeta)d\zeta \qquad (W)$$

solves the wave equation. The integral is (n-2)-dimensional.

E.T. Whittaker [Math. Ann. 57 (1903) 333-355] in effect gave this formula for n = 3 and 4. He said nothing about higher n, but H. Bateman [Proc. Lond. Math. Soc. (2) 1 (1904) 451-458] clearly knew that it worked for all n (although he didn't write it down).

Example: dimension 3. For n = 3, (W) can be rewritten as

$$\phi(x) = \oint f(x^{AB}\,\pi_A\,\pi_B,\,\pi_C)d\pi, \text{ contour } S^1. \qquad (W3)$$

Here π is a 2-spinor and $x^{AB} = x^{BA}$ are the coordinates on C^3. This is exactly the formula that one gets from mini-twistor theory (see, e.g., PEJ in TN 14).

Example: dimension 4. Here the formula looks like

$$\phi(x) = \oint f(x^{AA'}\,\eta_A\,\pi_{A'},\,\eta_B,\,\pi_{B'})d\eta d\pi, \text{ contour } S^2. \qquad (W4)$$

This is, I think, the Kirchoff integral which gives ϕ in terms of its null-datum on $\mathscr{I}$. It is what one gets from a "twistor theory" in which the "twistor space" is the set of all null hyperplanes in C^4.

Bateman (see ref. cited above) realized that in even dimensions there is an alternative formula. (He only dealt with dimensions 4 and 6 explicitly). Again we're in dimension n, but now n is restricted to be even. A null cone is "made up" out of totally null $\frac{1}{2}$n-planes. We have to choose a$(\frac{1}{2}n-1)$-parameter family of these $\frac{1}{2}$n-planes which fill up the null cone of the origin. Thus, choose $v_a^1(\zeta),\ldots,v_a^{\frac{1}{2}n}(\zeta)$ such that

(i) ζ denotes a $(\frac{1}{2}n-1)$-tuple of complex parameters;

(ii) for each ζ, $v_a^j(\zeta)x^a = 0$ for $j = 1,\ldots,\frac{1}{2}n$ defines a totally null $\frac{1}{2}$n-plane;

(iii) as ζ varies, these sweep out the null cone of the origin in C^n.

$$\phi(x) = \oint f(v_a^j(\zeta)x^a,\zeta)d\zeta \qquad (B)$$

then gives the solution of $\Box\phi = 0$ in C^n. One way of implementing this construction was invented by LPH (see TN 9). In his notation, (Z^α, W_α), $\alpha = 1,2,\ldots,\frac{1}{2}n$, are the coordinates on C^n, the metric is $dZ^\alpha dW_\alpha$, and the formula reads

$$\phi(Z,W) = \oint f(W_\alpha\pi^\alpha,\, Z^{[\beta}\,\pi^{\gamma]},\,\pi^\delta)d\pi.$$

Example: dimension 4. Here the formula reduces to the standard twistor formula

$$\phi(x) = \oint f(ix^{AA'}\,\pi_{A'},\,\pi_{B'})d\pi, \text{ contour } S^1. \qquad (B4)$$

Remarks/questions

(i) The "twistor space" on which the integrand f of (B) is defined, is a space whose points correspond to some (but not, for $n \geq 6$, all) of the totally null $\frac{1}{2}n$-planes in C^n. There are also formulae which use <u>all</u> the α-planes, as was pointed out by LPH in TN 14. For example, in dimension six, this formula takes a function of 6 variables and does a 3-dimensional integral, whereas B6 takes a function of 5 variables and does a 2-dimensional integral. Presumably one can get from the first formula to the second by doing one dimension's worth of integration.

(ii) The main interest for me of all this is the question of whether any of these formulae generalize to non-linear equations. We know that W3 and B4 do: W3 generalizes to solve the Bogomolny equations, and B4 to solve the self-dual Yang-Mills and self-dual Einstein equations. In each case, an "ordinary" H^1 generalizes to a "non-linear" H^1. Now in dimension ≥ 5, it seems to me that a solution formula for a hyperbolic equation could never be of H^1 type, because there are too many null directions. The formula might be of H^p type for $p \geq 2$ (can the formulae W and B above be interpreted cohomologically?). But I don't know of any "non-linear" H^p for $p \geq 2$. So I would conjecture that in dim ≥ 5 there are no non-linear hyperbolic equations which are "nice" (in the sense that the self-duality equations in dim 4 are nice; for example, one can write down large classes of solutions of them). [Exercise for the reader: define "nice", and then prove or disprove this conjecture.] Perhaps this is another reason why space-time is four-dimensional — equations in higher dimensions are too hard to solve!

Richard Ward.

(cont'd from p. 11)

is surely someone's favourite pastime. Is it possible that there are deeper connections between this homogeneous space & the complex wave-equation?

-Scott Petrack

(cont'd from p. 15)

References.

Gerach, Held and Penrose 1973 J Math Phys 14,7 p874

Penrose 1975, in Quantum Gravity ed Isham, Penrose, Sciama, OUP p 370.

The "'Normal' Situation" for Superficial Twistors

In Proc. Roy. Soc. A381 (1982) 53-63, or TWN13, R.P. refers to the "'normal' situation" for superficial twistors as that in which the solution space of the superficial twistor equations, $\eth'\omega^0 = \sigma'\omega'$ and $\eth\omega' = \sigma\omega^0$, has $\mathbb{C}$-dimension four. Since $\mathcal{D} \equiv \begin{bmatrix} \eth' & -\sigma' \\ \sigma & \eth \end{bmatrix}$ is an elliptic operator on the sphere with adjoint $\mathcal{D}^* = \begin{bmatrix} \eth & \bar{\sigma}' \\ \bar{\sigma} & \eth' \end{bmatrix}$, its index, $\dim\ker\mathcal{D} - \dim\ker\mathcal{D}^*$, is independent of σ, σ'. In this case the spin weights involved ($-\frac{1}{2}$ & $\frac{1}{2}$ for ω^0 & ω', $-\frac{3}{2}$ & $\frac{3}{2}$ for the adjoint) imply that, in case $\sigma = 0 = \sigma'$ $\dim\ker\mathcal{D} = 4$, $\dim\ker\mathcal{D}^* = 0$. Thus in general (as observed in Proc. loc. cit.) $\dim\ker\mathcal{D} \geq 4$ with equality generic and occuring exactly when $\ker\mathcal{D}^* = 0$. The aim of this note is to give an explicit argument with explicit bounds to show that the 'normal' situation occurs for small shears σ & σ'.

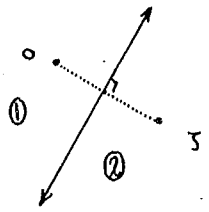
If the sphere under consideration is regarded as the usual Riemann sphere then the adjoint equations become $\begin{pmatrix} \bar{\eth} f \\ \eth g \end{pmatrix} = \chi \begin{pmatrix} f \\ g \end{pmatrix}$ for $f \in T'(H^3)$, $g \in T'(\bar{H}^3)$, and $\chi \in \mathrm{Hom}\left(\begin{matrix} H^3 \\ \oplus \\ \bar{H}^3 \end{matrix}, \begin{matrix} \Lambda^{0,1}(H^3) \\ \oplus \\ \Lambda^{1,0}(\bar{H}^3) \end{matrix} \right)$ where H is the Hopf bundle and $\Lambda^{p,q}$ denotes forms of type (p,q). Here, χ is the matrix $\begin{pmatrix} 0 & -\bar{\sigma} \\ -\bar{\sigma}' & 0 \end{pmatrix}$ although we may as well consider the seemingly more general case of χ arbitrary. [That this greater generality is illusory may be seen by using integrating factors (change of gauge) $f \mapsto e^u f$, $g \mapsto e^v g$ for arbitrary smooth functions u & v whence X is changed according to $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \mapsto \begin{bmatrix} \alpha + \bar{\eth}u & e^{u-v}\beta \\ e^{v-u}\gamma & \delta + \eth v \end{bmatrix}$ which, since $H'(\text{Sphere}, \mathcal{O}) = 0$, may be used to eliminate the diagonal entries. This may be rephrased as the complex structure on H^3 being unique - note that the sphere is special in this respect, an added difficulty in defining superficial twistors with respect to other 2-surfaces.] If we choose an affine coordinate patch on the sphere then these equations reduce to equations on genuine functions on the $\mathbb{C}$-plane:

$$\left.\begin{aligned} \partial F/\partial\bar{z} &= AF + BG \\ \partial G/\partial z &= CF + DG \end{aligned}\right\}$$

where the twisted nature at infinity is precisely reflected in the existence of the limits of z^3F, $\bar{z}^3G$, z^2A, $z^5\bar{z}^{-3}B$, $\bar{z}^5 z^{-3}C$, and $\bar{z}^2 D$ as $z \to \infty$. Note that, in particular, F and G are $O(1/|z|^3)$ and $X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is $O(1/|z|^2)$ as $z \to \infty$. Recall the Cauchy integral formula $u(\zeta) = \frac{1}{2\pi i}\left[\int_{\partial\Omega} u \frac{d\bar{z}}{z-\zeta} + \int_\Omega \frac{\partial u}{\partial \bar{z}} \frac{d\bar{z}\wedge dz}{z-\zeta}\right]$ for any u smooth on $\bar{\Omega}$. If $u(z) \to 0$ as $z \to \infty$ then we may take $\Omega = \mathbb{C}$ and obtain $u(\zeta) = \frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial u}{\partial \bar{z}} / z-\zeta$. Applying this to F for (F, G) a solution of the adjoint equations, we obtain $F(\zeta) = \frac{1}{\pi}\int_{\mathbb{C}} \frac{AF + BG}{z-\zeta}$. Thus, if we set $a = \sup_{\zeta\in\mathbb{C}} \frac{1}{\pi}\int_{z\in\mathbb{C}} \left|\frac{A(z)}{z-\zeta}\right|$ etc.; assuming for the moment that these exist, we conclude that $\sup|F| \leq a\sup|F| + b\sup|G|$. A similar bound holds

for $\sup|G|$ and so $\begin{pmatrix}\sup|F|\\ \sup|G|\end{pmatrix} \leq \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\sup|F|\\ \sup|G|\end{pmatrix}$ which is a load of dingo's kidneys if $\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ is small (for example distance decreasing), unless $F \equiv 0 \equiv G$. Note that in this argument we have used only that $|F|, |G|$ vanish at infinity whereas we really know more rapid decay. Perhaps this can be used to improve the bounds. Indeed, it is quite conceivable that D^* never has non-trivial kernel, i.e. the normal situation always occurs.

It remains to shew that $\int_{z\in\mathbb{C}} \left|\frac{A(z)}{z-\zeta}\right|$ is bounded uniformly in ζ. Without loss of generality we may assume that A is circularly symmetric and a monotone decreasing function of distance from the origin. Dividing the integration into two pieces according to region of integration as indicated in the following diagram:



, it is easy to see that each is bounded by $\int_{\mathbb{C}} \left|\frac{A}{z}\right|$, which exists because the integrand decays like constant$/|z|^3$ at infinity. Michael Eastwood.

Superstructure versus Formal Neighbourhoods

In the ambitwistor description of (not necessarily $\pm$self-dual) Yang-Mills of Green-Isenberg-Yasskin or Witten, formal neighbourhoods of $\mathbb{A}$ in $\mathbb{P}\times\mathbb{P}^*$ appear. Formal neighbourhoods of lines in $\mathbb{P}$ occur in the Penrose transform for left-handed fields. There are other examples. A supermanifold is rather similar except that instead of adding commuting extra variables (a truncated polynomial algebra), anticommuting variables are added (an exterior algebra). The purpose of this note is to compare the relevant obstruction theory in the two cases. Suppose X is a manifold (smooth or holomorphic) with structure sheaf $\mathcal{O}$, and E is a vector bundle on X.

Formal Neighbourhoods: Suppose we wish to extend E to formal neighbourhoods of X. Then there is an exact sequence of sheaves (see MGE in TN12 for notation)

$$0 \longrightarrow (\mathcal{N}^*)^n \otimes \mathcal{O}^{m\times m} \longrightarrow GL(m, \mathcal{O}_{(n)}) \longrightarrow GL(m, \mathcal{O}_{(n-1)}) \longrightarrow 0 \qquad *$$

and the obstruction theory follows (as in TN12).

Supermanifolds: Suppose we wish to extend X to be a supermanifold with underlying vector bundle E. Let $\mathcal{A}$ = sheaf of $\mathbb{Z}_2$-graded automorphisms of $\wedge\mathcal{O}^m$ and let $\mathcal{A}_j$ = those automorphisms which are the identity on $\wedge^i\mathcal{O}^m$, $i<j$. Then we have a filtration $\mathcal{A}=\mathcal{A}_1 \supset \mathcal{A}_2 \cdots \supset \mathcal{A}_{n+1} = 1$ and it is easy to identify the corresponding graded object: $\mathcal{A}_1/\mathcal{A}_2 = GL(m,\mathcal{O})$, $\mathcal{A}_j/\mathcal{A}_{j+1} = Der_{\mathcal{O}}(\mathcal{O}, \wedge^j\mathcal{O}^m)$ for j even, $= Hom_{\mathcal{O}}(\mathcal{O}^m, \wedge^j\mathcal{O}^m)$ for j odd and $\neq 1$ (see M. Batchelor, The structure of supermanifolds, Trans AMS 253 (1979) or P. Green, On holomorphic graded manifolds, Univ. of Maryland preprint). The problem is to see whether E is in the image of $H^1(X,\mathcal{A}) \longrightarrow H^1(X, \mathcal{A}_1/\mathcal{A}_2) = H^1(X, GL(m,\mathcal{O}))$. The analogue of $*$ is

$$0 \longrightarrow \mathcal{A}_j/\mathcal{A}_{j+1} \longrightarrow \mathcal{A}/\mathcal{A}_{j+1} \longrightarrow \mathcal{A}/\mathcal{A}_j \longrightarrow 0$$

and the obstruction theory follows in the same way. Michael Eastwood.

Minitwistors & "Anti Self Dual" Monopoles

This note will explain the minitwistor interpretation of an electrically and magnetically charged monopole situated at the origin of $\mathbb{R}^3$, and will show in minitwistor terms how the description of the magnetic potential as a connection on a non trivial bundle over $\mathbb{R}^3 - \{0\}$ arises.

Denote by MT the complex manifold whose points represent oriented geodesics in $\mathbb{R}^3$. MT is the total space of the $\mathcal{O}(2)$ bundle over $\mathbb{P}_1$ [1]. A point x in $\mathbb{R}^3$ is specified by the sphere's worth of oriented geodesics through it, and so corresponds to a holomorphic section, L_x, of MT. The origin of $\mathbb{R}^3$ corresponds to the zero section. Suppose now, we remove the origin, then each geodesic which contained it splits into two parts, thus the minitwistor space MT_0 relevant to $\mathbb{R}^3 - \{0\}$ has 2 copies of the zero section, one corresponding to the outgoing geodesics and one to the incoming ones. In fact MT_0 is a non-Hausdorff complex manifold, obtained by taking two copies, MT_α and MT_β of MT and identifying them everywhere except along their zero section. (cf. [2]).

On MT_α introduce the open cover $\{U_{\alpha 0}, U_{\alpha 1}\}$:

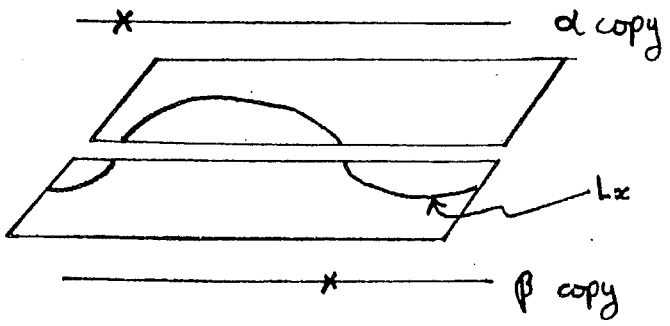
$$U_{\alpha 0} = \{(\zeta, Z^0)\} \qquad \zeta = \frac{1}{\zeta}, \quad Z' = \zeta^{-2} Z^0$$
$$U_{\alpha 1} = \{(\tilde{\zeta}, Z')\}$$

and a similar one $\{U_{\beta 0}, U_{\beta 1}\}$ on MT_β. The four sets $\{U_{\alpha 0}, U_{\alpha 1}, U_{\beta 0}, U_{\beta 1}\}$ give a Stein cover of MT_0. One of the zero sections is taken care of by $\{U_{\alpha 0}, U_{\alpha 1}\}$ and the other by $\{U_{\beta 0}, U_{\beta 1}\}$. Now introduce a line bundle $\mathcal{B}$ on MT_0, with the following transition functions:

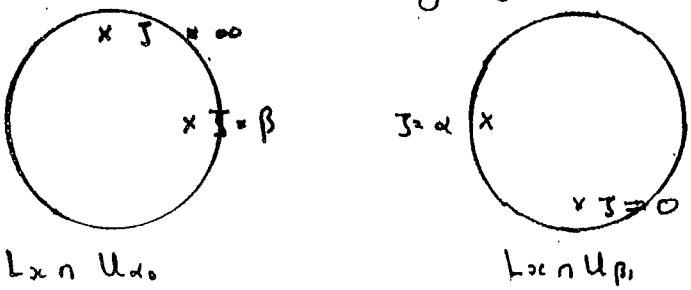
$U_{\alpha 0} \cap U_{\alpha 1}$:	$g_{\alpha 0 \alpha 1} = 1$	$U_{\alpha 1} \cap U_{\beta 1}$:	$g_{\alpha 1 \beta 1} = Z'$
$U_{\beta 0} \cap U_{\beta 1}$:	$g_{\beta 0 \beta 1} = \zeta^2$	$U_{\alpha 0} \cap U_{\beta 1}$:	$g_{\alpha 0 \beta 1} = Z'$
$U_{\alpha 0} \cap U_{\beta 0}$:	$g_{\alpha 0 \beta 0} = Z^0$	$U_{\alpha 1} \cap U_{\beta 0}$:	$g_{\alpha 1 \beta 0} = Z^0$

$\mathcal{B}$ is thus constructed from the bundle $\mathcal{O}$ on MT_α and $\mathcal{O}(-2)$ on MT_β.

Now take a point $x \neq 0$, in $\mathbb{R}^3$. The section L_x, being a quadratic polynomial, intersects the zero section in two points. Take one from the α-copy and one from the β-copy.



The open sets $\{U_{\alpha_0}, U_{\beta_1}\}$ restrict to give a two set Stein cover of L_x. Denote by α (resp. β) the value of ζ for which L_x intersects the α (resp. β) copy of the zero section.



L_x is the section represented by the quadratic polynomial $z^\circ = i\{-(x+iy) + 2z\zeta + (x-iy)\zeta^2\}$, and so the transition function for the restricted bundle $B|_{L_x}$ is, on $(L_x \cap U_{\alpha_0}) \cap (L_x \cap U_{\beta_1})$:

$$\frac{\zeta^2}{a(\zeta - A)(\zeta - B)} \qquad \text{where} \quad a = i(x-iy)$$

$$A = \frac{(x+iy)}{(z-r)} \quad ; \quad B = \frac{(x+iy)}{(z+r)}$$

$$r^2 = x^2+y^2+z^2 \cdot (x+iy)(x-iy) = (r+z)(r-z)$$

On multiplying by $\zeta - \beta$ (holomorphic and non zero on $L_x \cap U_{\alpha_0}$) and then by $a(1-\zeta)/\zeta$ (holomorphic and non zero on $L_x \cap U_{\beta_1}$), we obtain the transition function ζ, so the bundle $B|_{L_x}$ is isomorphic to $\mathcal{O}(-1)$. In order to extract a potential, however, we require a bundle which is trivial on lines, so we consider $G := B \otimes \mathcal{O}(1)$. This has transition function

$$\frac{\zeta}{a(\zeta - A)(\zeta - B)}$$

ie.

$$\frac{\zeta}{i(x-iy)\left(\zeta - \frac{(x+iy)}{(z-r)}\right)\left(\zeta - \frac{(x+iy)}{(z+r)}\right)}$$

Now, note that this transition function can be split in

two ways : $\left[\frac{1}{1-\frac{(x+iy)}{s(z-r)}}\right]\left[(x-iy)\zeta+(z-r)\right]^{-1} := G_- \hat{G}_-^{-1}$

and $\left[\frac{1}{(x-iy)+\frac{(z+r)}{\zeta}}\right]\left[\zeta-\frac{(x+iy)}{z+r}\right] := G_+ \hat{G}_+^{-1}$

The first splitting is valid for $z-r \neq 0$, i.e. away from the positive z-axis, and the second away from the negative z-axis. Now, evaluate the potentials in the usual manner, namely using $G_-^{-1}\,\pi^{A'}\nabla_{A'B'}G_- = \gamma^-_{A'B'}(x)\,\pi^{A'}$ & $G_+^{-1}\,\pi^{A'}\nabla_{A'B'}G_+ = \gamma^+_{A'B'}(x)\,\pi^{A'}$.

We obtain

$$\gamma^+_{A'B'} = \sqrt{2}\begin{pmatrix} 0 & 0 \\ \frac{1}{2r} & \frac{x+iy}{2r(z+r)}\end{pmatrix} \qquad \gamma^-_{A'B'} = \sqrt{2}\begin{pmatrix} 0 & 0 \\ \frac{1}{2r} & \frac{x+iy}{2r(z-r)}\end{pmatrix}$$

Now, $\gamma^{+\,A'B'} = \gamma^{+\,(A'B')} + \gamma^{+\,[A'B']} := \frac{1}{2}\left(A^{+\,A'B'} + \frac{1}{2}\underline{\Phi}^+\varepsilon^{A'B'}\right)$ and similarly for $\gamma^{-\,A'B'}$. Thus,

$$A^{+\,A'B'} = \sqrt{2}\begin{pmatrix} \frac{x+iy}{r(z+r)} & -\frac{1}{2r} \\ -\frac{1}{2r} & 0\end{pmatrix} \quad \& \quad A^{-\,A'B'} = \sqrt{2}\begin{pmatrix} \frac{x+iy}{r(z-r)} & -\frac{1}{2r} \\ -\frac{1}{2r} & 0\end{pmatrix}$$

giving the three-vectors

$$\underline{A}^+ = \left(-\frac{(x+iy)}{r(z+r)}, \frac{i(x-y}{r(z+r)}, -\frac{1}{r}\right) \qquad \underline{A}^- = \left(-\frac{(x+iy)}{r(z-r)}, \frac{i(x-y}{r(z-r)}, -\frac{1}{r}\right)$$

The scalar potentials are $\Phi^+ = \Phi^- = \frac{1}{r}$.

Note that $\underline{A}^+ - \underline{A}^- = \frac{1}{(x-iy)}\nabla(x-iy)$, so that $\{\underline{A}^+, \underline{A}^-\}$ define a connection on a non-trivial line bundle over $\mathbb{R}^3-\{0\}$, with curvature $\underline{B} = \operatorname{curl}\underline{A} = i\frac{\underline{r}}{r^3}$. Thus $\{A^+, A^-\}$ describes a unit magnetic charge at the origin. The potentials satisfy

$$\operatorname{curl}\underline{A}^+ = \operatorname{curl}\underline{A}^- = -i\nabla\Phi^+ = -i\nabla\Phi^- \quad \text{i.e.} \quad E = -iB.$$

These are just the Bogomolny equations for abelian electromagnetic gauge theory. If the electric and magnetic charge were n, the system would correspond to the minitwistor bundle $\mathbb{C}^n$.

References :

[1]. Jones. "Minitwistors" TN 14.

[2]. Sparling & Penrose. "The A.S.D. Coulomb fields' non-Hausdorff twistor space." TN 9.

Phil Jones

A New Approach to Quantum Gravity

by L.P. Hughston

In what follows I wish to outline a new approach to quantum gravity. The approach is based on ideas which arose in connection with my relativistic oscillator model for hadrons (Hughston 1982), which had in turn arisen as a by-product of a twistorial investigation of massive fields (Hughston & Hurd 1981).

I assume the notation and substance of Hughston (1982). According to this scheme the wave function $\psi(x,y)$ of a pair of scalar particles interacting through a relativistic potential $V(q)$ satisfies the equations of motion

$$[X^2 + m_1^2 + V]\psi = 0 \qquad (1)$$

$$[Y^2 + m_2^2 + V]\psi = 0 \quad , \qquad (2)$$

together with the subsidiary equations $\mathcal{M}^2\psi = M^2\psi$ and $\mathcal{S}^2\psi = s(s+1)\psi$. If V is of the form $\frac{\kappa}{q}$ where κ is a parameter, the equations of motion can be solved exactly; a quantization condition emerges, and this is:

$$M^2\left[1 - \frac{(m_1+m_2)^2}{M^2}\right]\left[1 - \frac{(m_1-m_2)^2}{M^2}\right] = -\frac{\kappa^2}{n^2} \qquad (3)$$

where $n > 0$ and $0 \leq s < n$ (n, s integral). Note that κ must have dimensions of mass. For a scalar Coulomb-type interaction it turns out that $\kappa = e\rho$ where e is the electric charge and $\rho = m_1 m_2 / M$ is the relativistic reduced mass. For a scalar gravitational-type interaction one must replace e by Gm_1m_2 ($G = u^{-2}$ where u is the Planck mass). It is instructive to look at the case of equal constituent masses $m_1 = m_2 = m$ (say). Then (3) gives:

$$M^2 = (2m)^2\left[\frac{1}{2} \pm \frac{1}{2}\sqrt{1 - \left(\frac{Gm^2}{n}\right)^2}\,\right] \qquad (4)$$

In order for a state to form it is a condition that $n > Gm^2$. Since $n \sim s$ we should normally therefore expect that $s > Gm^2$ for such a gravitating system.

How might we treat a proper non-linear Einsteinian gravitational interaction within this framework? In (1) and (2) the potential $V(q)$ is a pseudo-differential operator. We shall treat the Schwarzschild solution in a similar manner. Formally, we have:

$$ds^2 = (1-\tfrac{2\gamma}{q})\,d\tau^2 - (1-\tfrac{2\gamma}{q})^{-1}\,dq^2 - q^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{5}$$

with $q > 2\gamma$. To obtain a workable expression, let us write $\tau_a = M^{-1}P_a$, where P_a is the total momentum operator. Then, more precisely, we have:

$$g_{ab}(q) = \eta_{ab} - \frac{2\gamma}{q}\left[\tau_a\tau_b + (1-\tfrac{2\gamma}{q})^{-1}\frac{q_a q_b}{q^2}\right], \tag{6}$$

where $q_a = \xi_a - \tau_a(\xi\cdot\tau)$, $\xi_a = X_a - Y_a$, and $q = \sqrt{-q^2 q_a}$. The parameter γ remains to be determined. By comparison with the linear approximation it emerges, as will be explained below, that a plausible choice for γ is $2G\rho^2/M$.

Let us now associate a connection operator with $g_{ab}(q)$. For any vector $U_a(x,y)$ we define

$$\overset{x}{\nabla}_a U_b = X_a U_b - \overset{x}{\Gamma}{}^c_{ab}\, U_c \tag{7}$$

where

$$\overset{x}{\Gamma}{}^c_{ab} = \tfrac{1}{2}g^{cd}[X_b g_{da} + X_a g_{db} - X_d g_{ab}]. \tag{8}$$

Clearly $\overset{x}{\nabla}_a g_{bc} = 0$. We define $\overset{y}{\nabla}_a$ and $\overset{y}{\Gamma}{}^c_{ab}$ similarly. Since g_{ab} is a function of q alone, it follows that $\overset{x}{\Gamma}{}^c_{ab} = -\overset{y}{\Gamma}{}^c_{ab}$.

Now the idea is to eliminate the potential in (1) and (2), and to replace X^2 and Y^2 with appropriate covariant operators based on the Schwarzschild metric and connection.

For a stationary gravitating state consisting of two particles of masses m_1 and m_2 we therefore take the equations of motion to be:

$$\left.\begin{aligned} [g^{ab}\overset{x}{\nabla}_a\overset{x}{\nabla}_b + m_1^2]\psi &= 0 \\ [g^{ab}\overset{y}{\nabla}_a\overset{y}{\nabla}_b + m_2^2]\psi &= 0 \end{aligned}\right\}, \qquad (9)$$

valid for $q > 2\gamma$, with $g_{ab}(q)$ as in (6). Equations (9) are to be supplemented by the conditions $\mathcal{M}^2\psi = M^2\psi$ and $\mathcal{S}^2\psi = s(s+1)\psi$, where M and s are the total mass and total spin of the state, respectively.

To obtain γ we examine the linear approximation. We keep terms only to order one in γ and put

$$g_{ab} = \eta_{ab} + h_{ab}, \quad g^{ab} = \eta^{ab} - h^{ab} \qquad (10)$$

with

$$h_{ab} = -\frac{2\gamma}{q}\left[\tau_a\tau_b + \frac{q_a q_b}{q^2}\right]. \qquad (11)$$

The linearized connection is $\overset{x}{\Gamma}{}^c_{ab} = \overset{x}{\nabla}_{(a} h^c_{b)} - \frac{1}{2}\overset{x}{\nabla}{}^c h_{ab}$. Insertion of these expressions into (9) leads to a system of equations that can be reduced to a single 'radial' equation. If we identify the coefficient of the $\frac{1}{q}$ term in this equation with the corresponding expression arising from (1) and (2), then we obtain the relation $\gamma = 2Gp^2/M$.

Acknowledgements. A number of the ideas discussed here arose, in part, in discussions with A.D. Helfer, T.R. Hurd, and M.C. Sheppard.

References

L.P.H. & T.R.H. Proc. Roy. Soc. A 378 141 (1981)

L.P.H. Proc. Roy. Soc. A 382 459 (1982)

Style: Articles should be written in black ink, pencil, or typewriter, on A4 paper and with margins all around.

Twistor Theory and Harmonic Maps from Riemann Surfaces

Suppose M and N are oriented Riemannian manifolds with M made of rubber and N of stone. If M is constrained to lie on N by means of a smooth mapping $\phi: M \to N$ and then released it will attempt to attain an equilibrium configuration. This may be impossible i.e. M snaps. In many cases equilibrium is always possible. If ϕ is an equilibrium it is called a harmonic map. Geodesics and parametrized minimal surfaces are examples. See Eells & Lemaire, A report on harmonic maps, Bull. L.M.S. 10 (1978), 1-68 for a comprehensive review article. To be more precise, define the energy $E(\phi)$ of ϕ by

$$E(\phi) = \int_M \frac{1}{2} |d\phi|^2 \, dvol$$

where $|d\phi|$ is the Hilbert-Schmidt norm of $d\phi: T_x M \to T_{\phi(x)} N$, i.e. in local coordinates x^i on M, y^α on N, $|d\phi|^2 = \partial y^\alpha/\partial x^i \, \partial y^\beta/\partial x^j \, g^{ij} h_{\alpha\beta}$ where g is the metric on M and h is the metric on N. Then ϕ is said to be harmonic iff it is a critical point of E (if M is not compact then E must be computed on compact subdomains of M with only compactly supported variations allowed). In other words, ϕ is harmonic iff it satisfies the corresponding Euler-Lagrange equations

$$\text{trace} \, \nabla(d\phi) = 0$$

where ∇ is the connection on $\Omega^1(\phi^* TN)$ induced from the Levi-Civita connections on M and N. Bearing in mind the elastic nature of M, $\text{trace} \, \nabla(d\phi)$ is called the tension of ϕ. In local coordinates it becomes

$$g^{ij} \left\{ \frac{\partial^2 y^\alpha}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial y^\alpha}{\partial x^k} + \Omega^\alpha_{\beta\gamma} \frac{\partial y^\beta}{\partial x^i} \frac{\partial y^\gamma}{\partial x^j} \right\}$$

where Γ^k_{ij} and $\Omega^\alpha_{\beta\gamma}$ are Christoffel symbols on M and N respectively. Without the trace, $\nabla(d\phi)$ is the second fundamental form of the mapping ϕ and behaves well with respect to composition. The tension, however, does not in general compose for a composition of mappings.

Perhaps a neater way to write the energy is as

$$E(\phi) = \int_M \frac{1}{2} \, \text{trace} \, d\phi \wedge * d\phi$$

where $*: \Omega^1(\phi^*TN) \to \Omega^{m-1}(\phi^*TN)$ is the Hodge $*$-operator on M and the trace is with respect to h. The Euler-Lagrange equations are then

$$\nabla(*d\phi) = 0$$

where $\nabla: \Omega^{m-1}(\phi^*TN) \to \Omega^m(\phi^*TN)$ is the pull-back of the Levi-Civita connection on N. $\nabla(*d\phi)$ is the Hodge dual of the more usual tension. In local coordinates (or abstract index notation) on N, maintaining a more abstract notation on M

$$\nabla(*d\phi) = d*d\phi^\alpha + \Omega^\alpha_{\beta\gamma}\, d\phi^\beta \wedge *d\phi^\gamma.$$

At this point an obvious analogy with the Yang-Mills action and equations springs to mind (an analogy familiar to many mathematicians and physicists (the latter being interested in harmonic maps under the name of "$\mathbb{C}P_n$ model", "σ-model", or "current algebra")). For harmonic maps $d\phi$ is the analogue of the gauge field F and $\nabla(*d\phi) = 0$ replaces $\nabla(*F) = 0$. Recall that in gauge theories there is the Bianchi identity $\nabla F = 0$. The corresponding identity holds in the harmonic case because $\nabla(d\phi) = \nabla(\phi^*\delta) = \phi^*\nabla\delta$ where $\delta \in \Omega^1(TN)$ on N is the tautological section (Kronecker delta) and $\nabla\delta$ is exactly the torsion of ∇. Now recall that the Yang-Mills equations are special in dimension 4 (being conformally invariant) and that twistor theory gives an illuminating way of looking at them. It is natural to ask whether there is an analogy of this for harmonic maps.

The special dimension for harmonic maps is $\dim M = 2$ for then the $*$-operator is conformally invariant when acting on 1-forms. Thus, the harmonic field equations $\nabla(*d\phi) = 0$ are manifestly conformally invariant. For Euclidean Yang-Mills in dimension 4 there are special solutions namely $\pm$ self-dual, $*F = \pm F$, which satisfy the equations as a consequence of the Bianchi identity. In dimension 2, however, $*^2 = -1$ so the analogue of $\pm$self-dual can only be $*d\phi = \pm i d\phi$. For this to make sense TN must be a complex bundle so N should be an almost complex manifold and more specially it is easiest to take N complex. From now on N is made from complex stone (more rigid than the strongest granite). But now, even if N has a nicely compatible metric, i.e. an Hermitian one, ∇ is not necessarily complex linear. Indeed ∇ being complex linear is

equivalent to N being Kähler so ... Kähler stone from now on. Now, if $d\phi$ is $\pm$ self-dual,

$$\nabla(*d\phi) = \nabla(\pm i d\phi) = \pm i \nabla(d\phi) = 0$$

as required. By the Korn-Lichtenstein theorem the $*$-operator on M is equivalent to a complex structure where $*$ becomes multiplication by i. Thus, $*d\phi = id\phi$ is precisely the condition that $d\phi$ be complex linear, i.e. $d\phi$ is self-dual means ϕ is holomorphic (and anti-self-dual means antiholomorphic).

Now to the question: what has this to do with twistor theory? Twistor theory can be viewed as a way of doing conformal geometry in dimension 4 (for a conformally right-flat space). This can be clearly seen after <u>complexification</u>. More precisely, if the original 4-fold is $\mathbb{R}$-analytic then the conformal structure takes on a geometric significance, namely through α-planes, after "thickening" into the complex. The Ward correspondence, for example, becomes a simple geometric construction in the complexification. The same procedure works equally well in the 2-dimensional $\mathbb{R}$-analytic case (where there is no integrability condition analoguous to "right-flat"). In particular, the existence of the complex structure induced by a conformal structure in the $\mathbb{R}$-analytic category (due to Gauss), becomes a rather simple geometric observation. The Cauchy-Riemann equations then take on a geometric interpretation which leads to holomorphicity in exactly the same way that Maxwell's equations lead to the twisted photon – Maxwell's equations should be regarded as a 4-dimensional version of the Cauchy-Riemann equations. Without going into details it is clear that one should hope that complexification will throw geometric light on harmonic mappings from Riemann surfaces. The purpose of this article is to explain further upon this hope.

A harmonic mapping between $\mathbb{R}$-analytic Riemannian manifolds is automatically $\mathbb{R}$-analytic (by ellipticity). Thus, in our case of M a Riemann surface and N Kähler, one can complexify to obtain $\phi : \mathbb{C}M \to \mathbb{C}N$ on some neighbourhood of M. Then ϕ will satisfy a complexified version of the harmonic field equations. To identify these equations it is best to rewrite

$\nabla(*d\phi) = 0$ as follows. The derivative $d\phi$ may be split into its self-dual and anti-self-dual parts $d\phi = \partial\phi + \bar\partial\phi$ usually called the (1,0) and (0,1) parts. Bearing in mind that $\nabla(d\phi) = 0$ it follows that ϕ is harmonic $\Leftrightarrow$

$$\nabla(\partial\phi) = 0 \quad \text{or equivalently} \quad \nabla(\bar\partial\phi) = 0.$$

In this form it is straightforward to complexify. Since M and N are already complex, their complexifications are easy to identify: $\mathbb{C}M = M \times \bar M$, for example, where $\bar M$ denotes the smooth manifold M but with the conjugate holomorphic structure and $M \hookrightarrow \mathbb{C}M$ is the diagonal $(m \mapsto (m, \bar m))$. If $\phi : M \to N$ is holomorphic then its complexification $\mathbb{C}\phi : \mathbb{C}M \to \mathbb{C}N$ is given by $\mathbb{C}\phi(p, \bar q) = (\phi(p), \bar\phi(q))$. In general a $\mathbb{R}$-analytic $\phi : M \to N$ will complexify to only a small neighbourhood of M in $\mathbb{C}M$ so it is best to maintain $\mathbb{C}M$ as a general notation for any neighbourhood of M in $M \times \bar M$. A holomorphic $\psi : \mathbb{C}M \to \mathbb{C}N$ is the complexification of $\phi : M \to N$ iff ψ satisfies the "reality" condition $\psi(p, \bar q) = \psi(q, \bar p)$. The splitting $d\phi = \partial\phi + \bar\partial\phi$ is now manifested geometrically: after complexification, ∂ refers to differentiation along M whilst holding the $\bar M$ variable fixed and $\bar\partial$ refers to differentiation along $\bar M$ with M occurring parametrically. In other words (1,0)-forms complexify to the cotangent bundle to M and (0,1)-forms complexify to the cotangent bundle to $\bar M$. The connection ∇ on N complexifies (assuming it is $\mathbb{R}$-analytic) to a holomorphic torsion free connection on $\mathbb{C}N$. More generally, if L is any holomorphic manifold with holomorphic torsion free connection ∇ then we define $\phi : \mathbb{C}M \to L$ (holomorphic) to be <u>harmonic</u> iff $\nabla(\partial\phi) = 0$. To interpret this geometrically let $\mu : \mathbb{C}M \to M$ be projection onto the first factor. Then $\nabla : \Omega^{1,0}(\phi^*TL) \to \Omega^2(\phi^*TL)$ can be interpreted as a relative connection ∇_μ on $\mu^*\Omega^1(\phi^*TL)$ i.e. $\nabla_\mu : \mu^*\Omega^1(\phi^*TL) \to \Omega^1_\mu(\mu^*\Omega^1(\phi^*TL))$ satisfying $\nabla_\mu(fs) = f\nabla_\mu s + \bar\partial f \otimes s$. Thus $\mu^*\Omega^1(\phi^*TL)$ and hence ϕ^*TL maybe canonically regarded as the pull-back of a bundle on M (since, by dimension reasons (plus some simple topological restrictions on the fibres of μ) ∇_μ is relatively flat). Therefore ϕ is harmonic iff $\partial\phi \in \mu^*\Omega^1(\phi^*TL)$ is covariant constant on the fibres of μ or, in other words, pushes down to a section of the bundle

on M. In particular, this proves

Proposition: The zero set of $\partial\phi$ consists of the fibres of $\mu : \mathbb{C}M \to M$ over a discrete set of points and, in particular, by rescaling, $\partial\phi$ defines a complex direction at $\phi(x)$ even if $\partial\phi(x) = 0$ (unless $\partial\phi \equiv 0$). □

This proposition is one of the key steps in the recent classification of harmonic isotropic maps from Riemann surfaces into complex projective spaces (Din & Zakrzewski, General classical solutions in the $\mathbb{C}P^{n-1}$ model, Nucl. Phys. B174 (1980) 397-406. Din & Zakrzewski, Properties of the general classical $\mathbb{C}P^{n-1}$ model, Phys. Lett. 95B (1980), 419-422. D. Burns, unpublished. Eells & Wood, Harmonic maps from surfaces to complex projective spaces, Univ. of Warwick preprint 1981). Actually, the whole classification theorem complexifies rather well. Because it would make this article rather too lengthy to go into detail I will just describe some of the key points. Firstly the complexification of complex projective space. An Hermitian form on V gives rise to the Fubini-Study metric on $\mathbb{P}(V)$. The Hermitian form on V is the same as an isomorphism $\bar{V} \cong V^*$ and so $\mathbb{P}(V)$ can be complexified to $\mathbb{P}(V) \times \mathbb{P}(V^*)$. The natural pairing $\langle , \rangle : V \otimes V^* \to \mathbb{C}$ gives rise to a $\mathbb{C}^2$-valued metric on $\mathbb{P}(V) \times \mathbb{P}(V^*)$ induced by $\langle (a,b), (c,d) \rangle = (\langle a,c \rangle, \langle b,d \rangle)$ on $V \otimes V^*$. This is the complexification of the Fubini-Study metric and there is a corresponding connection. In general, a mapping $\phi : \mathbb{C}M \to L$ is said to be isotropic iff $\partial\phi$ and higher derivatives along M are orthogonal to $\bar{\partial}\phi$ and higher derivatives along $\bar{M}$. This is the case for example if $\bar{\partial}\phi = 0$. In the case of $L = \mathbb{P}(V) \times \mathbb{P}(V^*)$ it is essentially just algebra (albeit cunning algebra) to show that if $\phi : \mathbb{C}M \to \mathbb{P}(V) \times \mathbb{P}(V^*)$ is harmonic and isotropic then so is $D\phi$ defined by $D(R,S) = (\partial R - \langle \partial R, S \rangle R/\langle R,S \rangle , \bar{\partial} S - \langle R, \bar{\partial} S \rangle / \langle R,S \rangle)$ where $(R,S) : \mathbb{C}M \to V \times V^*$ is a lift of ϕ (make local choices and patch). The proposition above is used to show that D is well-defined provided ϕ avoids the quadric $\{(R,S) \in \mathbb{P}(V) \times \mathbb{P}(V^*)$ s.t. $\langle R,S \rangle = 0\}$ (D is essentially ∂). Hence the classification.

In the best of worlds there would be a harmonic analogue of the Isenberg-Yasskin-Green & Witten ambitwistor description of Yang-Mills. $\mathbb{C}M$ is the ambitwistor space for M.

Michael Eastwood.

MORE TWISTOR FUNCTIONS FOR SOURCED FIELDS

In which a conjecture from the articles of R.P and T.N.B in TIN 14 p.p. 9-23 becomes a Theorem, and we see why relative cohomology describes sourced fields.

The results and notation of TIN 14 p.p. 9-23 are assumed throughout.

The Left-handed Monopole

We recall from TIN14 that although a wide variety of fields with source on the analytic world-line $y^a(s)$ are elements of $H^1_R(U, \mathcal{O}(-4-2))$, the left-handed Maxwell monopole seems not to be. Instead, the following conjecture was made:

Theorem

There is an element of $H^1_R(U, \mathcal{O}^*)$ which corresponds naturally to the left-handed Maxwell monopole. To evaluate the field one "takes logs" and uses the usual contour integral. This can only be done locally, the global obstruction $(\in H^2_R(U, \mathbb{Z}))$ being the charge.

Proof

Let s_α be such that $\alpha_A \xi^A(s_\alpha) = 0$. We will write ξ^A_α for $\xi^A(s_\alpha)$ and similarly for $y^{AA'}_\alpha$ etc.

$$\text{Let} \quad f(Z) = \frac{\beta_A \xi^A_\alpha}{g(\pi)}$$

where $g(\pi)$ is some homogeneity $+1$ function of $\pi_{A'}$.

Then $\log f(\dot{z})$ is a twistor function for the left-handed monopole, as can be seen by substituting in the contour integral formula and using $\partial s_\alpha / \partial w_A = \left(i y_\alpha^{BB'} \alpha_B \pi_{B'} \right)^{-1} \alpha_A$ to reduce it to a form recognisable from TTN 14.

Furthermore it is clear by analogy with the "justification" of the conjecture in TTN 14 pp 20-21 that these expressions (with different $g(\pi)$) give a cocycle in $H^1_{\mathcal{L}}(U, \mathcal{O}^*)$.

It may be of interest to note that $\beta_A \xi^A_\alpha$ can be generated by an integral similar to those in TTN 14: up to constants,

$$(\beta_A \xi^A_\alpha)^{-1} = \oint_\alpha \frac{\alpha \cdot \dot{\xi} \, ds}{\alpha \cdot \xi \;\; \beta \cdot \xi}$$

Why Relative Cohomology describes sources

Let U and U' be two copies of a neighbourhood of a piece of ruled surface $\mathcal{L}$. Let X be the space obtained by identifying U and U' except on $\mathcal{L}$. Suppressing the sheaf of your choice, the Mayer-Vietoris sequence for U, U' looks like:

$$\to H^p(X) \longrightarrow H^p(U) \oplus H^p(U') \longrightarrow H^p(U \setminus \mathcal{L}) \longrightarrow H^{p+1}(X) \to$$

which we can write as the direct sum of the relative cohomology sequence

$$\to H^p_{\mathcal{L}}(U) \longrightarrow H^p(U) \longrightarrow H^p(U \setminus \mathcal{L}) \longrightarrow H^{p+1}_{\mathcal{L}}(U) \to$$

with the nearly trivial sequence

$$\to H^p(U') \longrightarrow H^p(U') \longrightarrow 0 \longrightarrow H^{p+1}(U') \to .$$

Thus we see (by the Five lemma if you must) that

$$H^p(X) = H^p_L(U) \oplus H^p(U').$$

The relevance of all this is that fields with sources on $y^a(s)$ have branch singularities at places where the advanced and retarded points on the world-line coincide. They are thus fields on a <u>double covering</u> of a neighbourhood of the world-line with these points deleted, being (say), the advanced field on one sheet and minus the retarded on the other.

X, as defined above is precisely the "twistor space" corresponding to this double covering (provided one considers only lines which intersect L once in each copy) and so one should expect that fields on the double cover correspond to elements of $H^1(X)$. But as we saw above

$$H^1(X) = H^1_L(U) \oplus H^1(U')$$

which gives a decomposition into sources on the world-line represented by $H^1_L(U)$ and free fields (which dont "need" the double cover) represented by $H^1(U')$.

I think this viewpoint clarifies the rôle of $H^1_L(U)$ in describing sources.

Toby Bailey

Manifestly Conformally Invariant Inverse Twistor Fns.

The $\mathbb{CP}^5$ calculus developed recently by LPH and TRH lends itself naturally to a facile translation between zrm fields on spacetime and twistor fns. In TN 14, I discussed the correspondance twistor fn $\longrightarrow$ spacetime field; here I shall outline the direction spacetime field $\longrightarrow$ twistor fn.

<u>The Scalar Field.</u> Let $\varphi(X^{\alpha\beta})$ be a twist -1 fn on a nbhd of the quadric Ω in $\mathbb{CP}^5$ representing a zrm field:

$$\rho_\Omega \nabla^2 \varphi = 0$$

Pick a point $\mathcal{O}^{\alpha\beta} = P^{[\alpha} Q^{\beta]}$ in the nbhd. Then

$$\omega = dw^\alpha \, \partial/\partial z^\alpha \, \varphi(w^{[\alpha} z^{\beta]})$$

is closed and $f(z^\alpha) = \int_P^Q \omega$ is a twistor fn for φ.

Proof. $d\omega = \partial/\partial w^\beta \, \partial/\partial z^\alpha \, \varphi \, dw^\alpha \wedge dw^\beta$.

$$= w\rho \, z^\sigma \cdot \nabla_{\rho\alpha} \nabla_{\sigma\beta} \varphi \, dw^\alpha \wedge dw^\beta = \nabla^2 \varphi \; \overline{w \, z \, dw \wedge dw} = 0$$

so ω is closed. To show that this produces the correct twistor fn, it suffices to consider an elementary state

$$\varphi = 1/X \cdot A \,, \quad \underset{\shortparallel}{A} = \underset{+\,+}{L\,M}$$

Then

$$\int_P^Q \omega = \int_P^Q \underset{w\,dw}{\underset{+\,+}{L\,M}} \Big/ \left(\underset{w\,z}{\underset{+\,+}{L\,M}}\right)^2$$

$$= \int_P^Q d\left(\begin{matrix} L \\ | \\ w \end{matrix} \Big/ \begin{matrix} L \\ | \\ z \end{matrix} \; \underset{w\,z}{\underset{+\,+}{L\,M}}\right) = \frac{\underset{P\,Q}{\underset{+\,+}{L\,M}}}{\underset{P\,z}{\underset{+\,+}{L\,M}} \quad \underset{Q\,z}{\underset{+\,+}{L\,M}}}$$

which is indeed the correct twistor fn for φ.

<u>Helicity $\pm 1/2$.</u> Let $g^{-1}(z) \leftrightarrow \nu_\alpha^{-1}(X)$, $h^{-3}(z) \leftrightarrow \mu^{\alpha\,-1}(X)$ be twistor fns and their corresponding helicity $\pm 1/2$ fields. For any A^α, B_α, we have

$$\left.\begin{aligned} A \cdot \partial/\partial z \; g &= \int_P^Q dw \cdot \partial/\partial z \; A^\alpha \nu_\alpha \\ B \cdot z \quad h &= \int_P^Q dw \cdot \partial/\partial z \; B_\alpha \mu^\alpha \end{aligned}\right\}$$

whence

$$\left.\begin{aligned} \partial/\partial z^\alpha \; g &= \int_P^Q dw \cdot \partial/\partial z \; \nu_\alpha \\ z^\alpha \quad h &= \int_P^Q dw \cdot \partial/\partial z \; \mu^\alpha \end{aligned}\right\} \quad (1)$$

Now, contracting with z^α and $\partial/\partial z^\alpha$, respectively,

$$\left.\begin{aligned} g &= -\int_P^Q dw \cdot (\partial/\partial z \; \nu_\alpha) z^\alpha \\ h &= \int_P^Q dw \cdot \partial/\partial z \, (\partial/\partial z^\beta) \mu^\beta \end{aligned}\right\}$$

With the aid of the zrm equation for ν_α (TRH thesis) and a little algebra, one can get the elegant equation:

$$g = \int_P^Q \nu_\alpha \, dw^\alpha$$

Helicity ±1. Let $j(\overset{\circ}{z}) \leftrightarrow \varphi_{\alpha\beta}(\overset{-1}{x})$, $k(\overset{-4}{z}) \leftrightarrow \psi^{\alpha\beta}(\overset{-1}{x})$ be twistor fns and their corresponding helicity ± 1 fields. The analog of (1) is

$$\left.\begin{aligned} \partial/\partial z^\alpha \, j &= \textstyle\int_P^Q dw \cdot (\partial/\partial z \, \varphi_{\alpha\beta}) z^\beta \\ z^\alpha \, k &= \textstyle\int_P^Q dw \cdot \partial/\partial z \; \partial/\partial z^\beta \, \psi^{\alpha\beta} \end{aligned}\right\} \quad (2a,b)$$

but the trick of contracting with z^α, $\partial/\partial z^\alpha$ fails since $z \cdot \partial/\partial z \, j = 0 = \partial/\partial z \cdot (z\,k)$. Instead, we have

$$\left.\begin{aligned} j &= \int^z dz^\alpha \int_P^Q dw \cdot (\partial/\partial z \, \varphi_{\alpha\beta}) z^\beta \\ k &= \tfrac{1}{w \cdot z} \textstyle\int_P^Q dw \cdot \partial/\partial z \; w_\alpha \partial/\partial z^\beta \, \psi^{\alpha\beta} \end{aligned}\right\} \quad (3a,b)$$

|Helicity|>1. The trick works again, and we have

$$\left.\begin{aligned} z^\rho \cdots z^\sigma \textstyle\int^z dz^\beta \int_P^Q z^\alpha \, w \cdot \partial/\partial z \, \varphi_{\alpha\beta\rho\cdots\sigma} \\ \partial/\partial z^\rho \cdots \partial/\partial z^\sigma \; \tfrac{1}{w \cdot z} \textstyle\int_P^Q w \cdot \partial/\partial z \; w_\alpha \partial/\partial z^\beta \, \psi^{\alpha\beta\rho\cdots\sigma} \end{aligned}\right\} \quad (4a,b)$$

Remarks.

1) It is easy to cohomologize this locally.

2) The transition from 2a to 3a in effect introduces the first potential of the asd electromagnetic fields. The twistor fns of the higher helicity fields (4a) are also effectively in terms of first potentials. In the Ward and leg-break graviton constructions, these potentials are important because one starts from

$$\pi^{A'} \nabla_{AA'} \, z^\alpha = 0$$

(covariant derivative)

and discovers a patching relation of the form (infinitesmally) $\hat{z} \sim z + f \sim z + \int$ potential on α-plane

One is tempted to interpret the transition (2a) → (3a) as telling us what structures of spacetime to look at in order to construct a leg-break space. Perhaps there is a hint in the transition (2b) → (3b) of how to construct a googly space.

Thanks to TRH.

- Adam Helfer

References: TRH thesis; LPH & TRH, back issues of GRG.

Integrals for general-relativistic sources: a development from Maxwell's electromagnetic theory.

by

Roger Penrose

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Abstract

Maxwell's pioneering work on electromagnetism produced a paradigm - a field theory of astonishing accuracy and of remarkable mathematical elegance and simplicity. One direct consequence of this theory was that the charge surrounded by a closed 2-surface S can be computed by integrating field components over S. In general relativity, by introducing a twistor concept associated with S (taken to be spacelike topological 2-sphere), the gravitational field near S (spin 2) may be reduced to the Maxwell case (spin 1). The resulting "charge" integrals provide suggestive definitions for the total mass-momentum and angular momentum surrounded by S.

Donaldson's moduli space: a "model" for quantum gravity?

by

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Abstract

An unorthodox viewpoint is presented for a quantum gravity scheme in which linear quantum mechanics and classical general relativity are each supposed to arise as different limiting cases of a single non-linear theory. It is suggested that Donaldson's moduli-space for SU(2)-instantons may provide a tentative analogy for such a scheme.

Spinors and Torsion in General Relativity

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Abstract

Conforaml rescalings of spinors are considered, in which the factor Ω, in $\epsilon_{AB} \mapsto \Omega\epsilon_{AB}$, is allowed to be complex. It is argued that such rescalings naturally lead to the presence of torsion in the space-time derivative ∇_a. It is further shown that, in standard general relativity, a circularly polarized gravitational wave produces a (non-local) rotation effect along rays intersecting it similar to, and apparently consistent with, the local torsion of the Einstein-Cartan-Sciama-Kibble theory. The results of these deliberations are suggestive rather than conclusive.