

~~Book~~

Twistor Newsletter (no 16 : 5, August, 1983)

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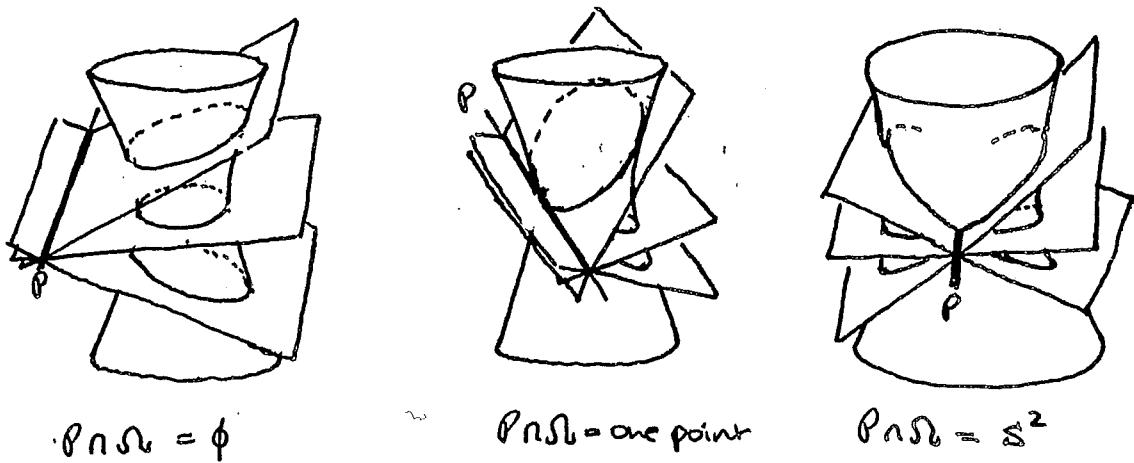
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2.

Cosmological Models in P^5

In [1] and [2], R.P. and W.R. describe a remarkably simple picture of the general Friedmann-Robertson-Walker cosmological models. The point was that being conformally flat these models can be represented as conformal subsets of the Einstein cylinder, or isometrically as sections through a 5-dimensional cone in R^6 , or in terms of a quadratic hypersurface Ω in P^5 .

In the P^5 picture, the general geometry is characterized by the choice of a 3-plane P and the associated pencil of 4-planes which contain P . This pencil foliates the space-time by a family of space-like hypersurfaces, the surfaces of constant 'cosmological time'. The three distinct ways the pencil can slice up the space-time



$$P \cap \Omega = \phi$$

$$P \cap \Omega = \text{one point}$$

$$P \cap \Omega = S^2$$

correspond to the $k=+1, 0, -1$ models respectively. The metric of a general FRW model is now characterized by calculating the conformal factor $\Omega(x)$ (constant on each slice) which relates it to the flat metric. This gives a homogeneity +1 section $\tilde{I}(x) = \Omega(x) I \propto x^\beta$ on P^5 which can be thought of as giving the space-time as the section of the 5-cone in R^6 with $\tilde{I}(x) = 2$.

I just want to point out how, with a slight change of point of view, this fits into the general P^5 framework developed by LPH and me [3]. As usual, take

homogeneous coordinates x^i , $i=0,1,\dots,5$ on P^5 and define the quadric hypersurface Ω_b by the equation $\Omega_{ij}x^i x^j = 0$ where Ω_{ij} has signature $++----$. The conformal structure of flat space is given by the null separation condition $x^i y^j \Omega_{ij} := X \cdot Y = 0$ for points $X, Y \in \Omega_b$. Ω_{ij} acts as a conformal metric 'tensor': to obtain the general conformally flat metric tensor on Ω_b we adjust the conformal weight of Ω_{ij} to -2 by introducing a homogeneity ± 1 section $J(X)$. That is, $g_{ij} = J^{-2} \Omega_{ij}$, or in line element form $ds^2 = J^{-2} \Omega_{ij} dx^i dx^j$.

The curvature of such a metric on Ω_b can be calculated:

$$R_{ij} = -2J^{-1}T_{ij} + \{3J^{-2}J_k J^k - J^{-1}J_{ke}\Omega^{ke}\} \Omega_{ij}$$

$$R = 12J_k J^k - 6J J_{ke}\Omega^{ke} = 6J^3 \nabla \cdot \nabla (J^{-1}) \quad (*)$$

and $G_{ij} = -2J^{-1}T_{ij} + \{-3J^{-2}J_k J^k + 2J^{-1}J_{ke}\Omega^{ke}\} \Omega_{ij}$

where $J_{ke} := \nabla_k \nabla_e J$ and $J_k := \nabla_k J$. The metric of flat space for example derives from $J(X) = 2I_i X^i$ where $I_i = I_{i\bar{\alpha}}$ is the 'infinity twistor'. More generally, the zeroes of J define \mathcal{S} while the singularities of J are curvature singularities.

For a FRW space-time, we choose a pencil \mathcal{P} of 4-planes $\alpha A_i + \beta B_i$, $\alpha, \beta \in \mathbb{C}$ which is real in the sense that $\bar{A}, \bar{B} \in \mathcal{P}$. The intersection of each real 4-plane C_i of this pencil with Ω_b is a 3-dimensional hypersurface which is space-like, null, or time-like if $C \cdot C \geq 0, = 0, < 0$ respectively. Of the possible real pencils, only the 3 cases illustrated above provide space-like 3-surfaces. When $k=\pm 1$, we can without loss of generality choose A_i and B_i to be tangent to Ω_b , i.e. $A^2 = B^2 = 0$. Then the reality and normalization conditions $A = \bar{B}; A \cdot B = +\frac{1}{2}$ (for $k=+1$) $A = \bar{B}, B = \bar{B}; A \cdot B = -\frac{1}{2}$ (for $k=-1$)

4.

can be imposed. (The pencil for a $K=0$ model is spanned by real planes C and D , with $C \cdot C = C \cdot D = 0$ and $D \cdot D = +1$. This case must be dealt with separately).

The assumption of spatial isotropy restricts the general form of $J(x)$ for a $K=\pm 1$ FRW model:

$$J(x) = \sqrt{ab} S(b/a)$$

where $a := 2A \cdot x$, $b = 2B \cdot x$ and S is a free function of one variable. For a J of this form the combinations appearing in (*) can be worked out

$$\begin{aligned} J_K J^K &= S^2 \left[-1 + 4 \left(\frac{b}{a} \right)^2 \left(S'/S \right)^2 \right] \\ J^{-1} J_{ij} &= 4 \left[\left(\frac{b}{a} \right)^2 \left(S''/S \right) + \left(\frac{b}{a} \right) \left(S'/S \right) - \frac{1}{4} \right] \left(\frac{B_i}{b} - \frac{A_i}{a} \right) \left(\frac{B_j}{b} - \frac{A_j}{a} \right) \\ &= J(J_{K\ell} S^{\ell K}) V_i V_j \end{aligned}$$

where V_i is the unit time-like vector normal to the constant time surfaces, and

$$J(J_{K\ell} S^{\ell K}) = 4S^2 \left[\left(\frac{b}{a} \right)^2 \frac{S''}{S} + \left(\frac{b}{a} \right) \frac{S'}{S} - \frac{1}{4} \right].$$

The stress tensor T_{ij} for a FRW model has the general form $T_{ij} = -p(x) g_{ij} + (\mu + p) V_i V_j$ where the scalar fields p and μ are known as the pressure and energy density. Comparison with G_{ij} via the Einstein equation then leads to the following equations:

$$p(x) = 3J_K J^K - 2J J_{K\ell} S^{\ell K} = S^2 \left[-1 + 12 \left(\frac{b}{a} \right)^2 \left(S'/S \right)^2 - 8 \left(\frac{b}{a} \right)^2 S''/S - 8 \frac{b}{a} S'/S \right]$$

$$\mu(x) = -3J_K J^K = S^2 \left[3 - 12 \left(\frac{b}{a} \right)^2 \left(S'/S \right)^2 \right]$$

Particular models are derived from an equation of state such as $p=0$ (dust models), $p=-\mu/3$ (radiation models) etc., which gives a differential equation which can be solved for S and hence leads to $J(x)$.

With less work, one can find $J(x)$ for one's favourite model by choosing coordinates in the following way. For $K=-1$ models, write

$$x^i(\eta, x, \theta, \phi) = e^\eta A^i + e^{-\eta} B^i + Q^i(x, \theta, \phi)$$

where $Q^i = (0, 0; dx, \sin x \cos \theta, \sin x \sin \theta \cos \phi, \sin x \sin \theta \sin \phi)$

satisfies $Q \cdot A = Q \cdot B = 0$. Now $a = 2A \cdot x = -e^\eta$

and $b = 2B \cdot x = -e^{-\eta}$. If we put $\Omega_{ij} = \text{diag}(1, -1; 1, -1, -1, -1)$

then $X \cdot X = -1 + Q^i Q_i = 0$ and the points x^i with $\eta = \text{const.}$ form space-like 3-surfaces. Now

$$dx^i = e^\eta A^i d\eta - e^{-\eta} B^i d\eta + dQ^i$$

$$g_{ij} dQ^i dQ^j = -dx^2 - \sin^2 x [d\theta^2 + \sin^2 \theta d\phi^2] := d\Sigma^2$$

and $J(X) = S(e^{2\eta})$. Therefore

$$ds^2 = J^{-2} \Omega_{ij} dx^i dx^j$$

$$= [S(e^{2\eta})]^2 \{ dx^2 - d\Sigma^2 \}$$

Having put the metric in this standard form, one can read off S and hence $J(X)$ for any cosmological model one wants.

For one example, consider the open Friedmann dust model.

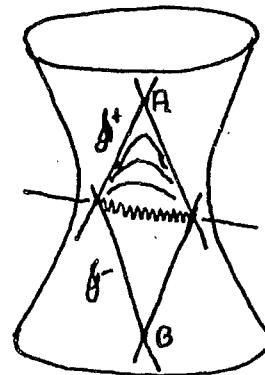
This is the $k=-1$ solution with equation of state $p=0$ and has $S(x) = (\sqrt{x} + \sqrt{x-2})^{-1}$; $J(x) = 2A \cdot x B \cdot x (\sqrt{A \cdot x} - \sqrt{B \cdot x})^2$. The model has

$$f^+ = \text{null cone } A \cdot x = 0$$

$$f^- = \text{null cone } B \cdot x = 0$$

'big bang' = hypersurface

$$(A-B) \cdot x = 0$$



One can now use the general P^5 framework for tensor, spinor and twistor analysis on FRW models, perhaps even for real live cosmology calculations. For example, one could study anisotropic perturbations $g_{ij} = J^{-2} \Omega_{ij} + \epsilon N_{ij}$ in the early universe (such a metric has a first order Weyl tensor $C_{ijkl} = J^{-2} \nabla_i \nabla_k J^2 N_{jl}$). Another direction where this framework might be useful is in the study of more general asymptotic behaviour than asymptotic flatness.

- Refs: [1] R. Penrose 1981 TN12; [2] R. Penrose & W. Rindler S&STS vol. 2
[3] L.P. Hughston & TRH to appear in Phys. Rep.

6.

A note on Sparling's 3-Form

This is just to draw attention to the remarkable fact, noticed by G.A.J.S. a while ago, that the 3-form

$$\Gamma = i d\pi_{A'}^{} d\bar{\pi}_A^{} \wedge dx^{AA'},$$

defined on the spin-vector bundle $(x; \pi_{A'}, \bar{\pi}_A)$ of a curved space-time M is closed iff M is an Einstein vacuum, in which case it is the exterior derivative of the 2-form

$$\Omega = i \pi_{A'}^{} d\bar{\pi}_A^{} \wedge dx^{AA'}$$

(or of the Witten-Nester 2-form $\frac{1}{2}(\Omega + \bar{\Omega})$). [G.A.J.S. would write θ^a for my dx^a and $\theta_{A'}$ for my $d\pi_{A'}$, etc. Here $d\pi_{A'}$ is indeed the "exterior derivative" of $\pi_{A'}$, though $d^2 \neq 0$ on $\pi_{A'}$, since $\pi_{A'}$ has an abstract index. There is no such thing as " x^a " with an abstract index, but $dx^a = \sum_a dx^a$. We have $d^2 x^a = 0$ if the torsion vanishes; otherwise $d^2 x^a = -\frac{1}{2} T^a_{bc} dx^b dx^c$.]

Note that this seems to fit in with a certain pattern of significance in twistor theory:

$$\Sigma = Z^\alpha \bar{Z}_\alpha$$

$$d\Sigma = i \bar{\Phi} - i \Phi$$

where

$$\bar{\Phi} = i Z^\alpha d\bar{Z}_\alpha$$

$$= p_a dx^a = \pi_{A'} \bar{\pi}_A dx^{AA'} \quad \text{on } \Sigma = 0$$

$$d\bar{\Phi} = \bar{\Omega} = i dZ^\alpha \wedge d\bar{Z}_\alpha = dp_a dx^a = d(\pi_A \bar{\pi}_A)_a dx^a$$

$$= i \bar{\Omega}_{\bar{A}} - i \Omega_A$$

$$d\bar{\Omega}_{\bar{A}} = \Gamma \quad \text{in vacuum}$$

Roger Penrose

This d is

$$d\pi_{A'} = V_{B'A'} d\pi^{B'A'}$$

Remarks on Curved-space Twistor Theory and Googlies.

Let M be a complex manifold (though much of what follows works in the real case also). Let $T^*(M)$ be its cotangent bundle, with naturally defined 1-form

$$\underline{\Phi} = p_a dx^a$$

(cf. "A note on Sparling's 3-form" with regard to notation) and a symplectic 2-form

$$\underline{\Theta} = d\underline{\Phi} = dp_a \wedge dx^a.$$

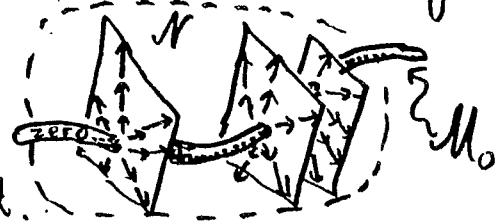
Let N be the manifold $T^*(M)$ but with the zero-section $M_0 (\cong M)$ removed, the structure of N being given by $\underline{\Phi}$. The Poisson structure (dual to $\underline{\Theta}$)

$$\{ , \} = \frac{\partial}{\partial p_a} \wedge \nabla_a,$$

together with $\underline{\Phi}$ serves to define the vector field

$$\underline{V} = p_a \frac{\partial}{\partial p_a}.$$

If M_0 is re-adjoined to N , we can use the integral curves of \underline{V} , "tied together" at the points of M_0 , to reconstruct the fibres of $T^*(M)$, and hence to find the bundle structure of N over M . But without M_0 , the local structure of N is insufficient.



A striking example occurs with projective flat twistor space $M = P\mathbb{T}$. The cotangent bundle $T^*(P\mathbb{T})$ can be described as the space of (Z^α, W_α) under the equivalence

$$(Z^\alpha, \lambda W_\alpha) \sim (\lambda Z^\alpha, W_\alpha), \quad \text{where } Z^\alpha W_\alpha = 0, Z^\alpha \neq 0,$$

(so N is also a 1-bundle over an arbitrary twistor space) and we

8. have

$$\underline{\Phi} = W_\alpha dZ^\alpha$$

(This needs to be invariant under $Z^\alpha \mapsto \lambda(Z^\beta) Z^\alpha$, whence the conditions above.) Note that when we remove M_0 , so $W_\alpha \neq 0$ in addition to $Z^\alpha \neq 0$, then we get a space N which is completely symmetrical between $\mathbb{P}\mathbb{T}$ and $\mathbb{P}\mathbb{T}^*$. Were we to adjoin $\mathbb{P}\mathbb{T}^*$, as M_0 , rather than $\mathbb{P}\mathbb{T}$, then we should obtain the cotangent bundle of $\mathbb{P}\mathbb{T}^*$ rather than of $\mathbb{P}\mathbb{T}$. (Note that

$$\underline{\Phi} = W_\alpha dZ^\alpha = -Z^\alpha dW_\alpha \quad \text{and} \quad \underline{T} = Z^\alpha \frac{\partial}{\partial Z^\alpha} = W_\alpha \frac{\partial}{\partial W_\alpha}$$

This is despite the fact that the adjoined zero-section fits in exactly the same "hole" in N in each case!
(An essentially non-Hausdorff space would arise if both zero-sections were adjoined at once.)

Similar situations arise with any (complex) space-time 4-manifold \mathcal{L} , where now N is to be the space of its null geodesics, with scalings. The "scalings" here mean that a (null) covector p_α is to be associated with the null geodesic, where p_α points along the geodesic and is parallelly propagated along it (a conformally invariant property). The operator \underline{T} generates changes in this scaling:

$$\underline{T} = p_\alpha \frac{\partial}{\partial p_\alpha}$$

— and we have $\underline{\Phi} = p_\alpha dx^\alpha$. Working locally (in the sense of near one of the integral curves of \underline{T}) we find that many varieties of different zero-section can be adjoined to N . Most notably, if we select a hypersurface S in \mathcal{L} , then N may be

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regarded as effectively the cotangent bundle of \mathcal{I} (restrict p_a to tangent directions in \mathcal{I}), or of the projective hypersurface twistor space $P\mathcal{T}(\mathcal{I})$, or of the projective hypersurface dual twistor space $P\mathcal{T}^*(\mathcal{I})$, the only distinction, in each case, being the different choices of adjoined zero-section. We can move \mathcal{I} to $\mathbb{C}\mathbb{P}^+$ and regard N as the cotangent bundle of $\overset{\mathbb{C}\mathbb{P}^+ \text{ or } \mathcal{I}}{\text{projective asymptotic twistor space}} P\mathcal{T}$ or projective asymptotic dual twistor space $P\mathcal{T}^*$. All the structural information seems to lie in the way that the zero-section sits in the "hole" in N .

This symmetry between $P\mathcal{T}$ and $P\mathcal{T}^*$ suggests that the above viewpoint may be helpful for the googly problem. Indeed, the following procedure seems quite hopeful. Consider, first the case of \mathbb{PT} . (Ignoring a sign ambiguity) we can imbed \mathbb{PT}^I (that is, \mathbb{PT} but with the region $\pi_{A'} = 0$ removed) in $T^*(P\mathbb{PT})$ by

$$\underline{z} \mapsto (\underline{z}, \underline{z}\gamma).$$

Note that the scale of \underline{z} (up to a sign) is encoded in the scaling of the covector $\psi = \underline{z}\gamma$ (with $(\lambda\underline{z})\psi = (\lambda\underline{z}, \lambda^2\psi)$). Effectively equivalent is the assignment

$$\underline{z} \mapsto \underline{z}\underline{z}\gamma.$$

In place of I , we have a blown up projective space (with I replaced by a quadric in $P\mathbb{PT}$), since if

$$10. \quad z = \alpha^{-1} \underline{z} + \alpha \underline{\delta} \quad (\underline{z}, \underline{\delta} \text{ fixed})$$

then

$$\underline{z} \underline{z} = \underline{\underline{z}} + \alpha^2 \underline{\delta} \underline{\delta} \rightarrow \underline{\underline{z}}$$

as $\alpha \rightarrow 0$, so both a point on I (\underline{z}) and a plane through it ($\underline{\delta}$) are retained in the limit. Because of singular behaviour of googly maps at I , in the curved case, we seem to require a blow-up. The idea, here, is to adjoin a 3-manifold \mathbb{II} to \mathbb{II}' , which consists just of these above limits.

Circumstantial evidence that finding deformations of the way that \mathbb{II} is adjoined to \mathbb{II}' may be the right thing to do is provided by the following facts: Whereas the natural 4-form

$$d^4z = \frac{1}{24} dz_1 dz_2 dz_3 dz_4 \quad \text{and 3-form } \mathcal{D}z = \frac{1}{6} \underline{\underline{z}} dz_1 dz_2 dz_3$$

have singular limiting values at \mathbb{II} , the forms

$$\underline{z} \underline{z} d^4z \quad \text{and} \quad \underline{z} \underline{z} \mathcal{D}z$$

are well-defined (and non-zero) there. (To see this it seems simplest to choose local coordinates at \mathbb{II} , e.g. $X = \omega^0 \pi_0$, $Y = \omega^0 \pi_1$, $Z = \omega^1 \pi_0$, $W = \pi_0, \pi_0'$.) Note that

$$d\{f(z) \underline{z} \underline{z} \mathcal{D}z\} = ((r+6)f) \underline{z} \underline{z} d^4z$$

so that if these forms are to be deformed (infinitesimally?) at \mathbb{II} by multiplication by some twistor function, it would seem that those of homogeneity -6 have a special role to play. (Compare the "googly photon" construction of $TN(3,5)$ and A.T.T. pp 152-167.)

In the curved case, each " $\underline{z} \underline{z}$ " corresponds to a scaled null geodesic on \mathbb{CP}^+ (lying in a β -plane on \mathbb{CP}^+). The points of \mathbb{II} correspond to scaled generators of \mathbb{CP}^+ .

It would seem that the role of the "zeros" in relation to \mathbb{II} must be important. Work is in progress.

Roger Penrose

The basis of Penrose's 2-surface twistor construction is the generalisation to curved space-time of the right hand side of the identity

$$L[K] \equiv \int_{\Sigma} T^{ab} K^b d\Sigma^a = \frac{1}{4\pi G} \int_{\partial\Sigma} R^{abcd} \Omega^{cd} dS^{ab} \quad (1)$$

where $\Omega^{ab} = \epsilon^{AB} \epsilon^{A'B'}$, $\nabla^{BB'} \omega^{CD} = -i \epsilon^{B(C} K^{D)B'}$, K^a is a killing vector of M , and T^{ab} and R^{abcd} are the energy-momentum and curvature tensors of a weak gravitational field on M . Here Σ is a spacelike hypersurface with boundary $\partial\Sigma$.

It is not known in general, in full general relativity, how the r.h.s. of (1) is related to local expressions involving the energy-momentum tensor. When Σ admits a four dimensional family of 3-surface twistors, one recovers a form of (1). (see Tod⁽²⁾, where this is done explicitly for hypersurfaces in conformally flat space-times.) However, for 3-surface twistors, one requires $B_{ab} = 0 = \bar{B}_{ab}$, where B_{ab} is the magnetic part of the Weyl curvature w.r.t. Σ and \bar{B}_{ab} its normal derivative. Is there a construction which generalises the whole of the identity (1) to curved space-time?

One answer appears to arise from simply doing "the best one can" with the twistors. Let t^α be the unit normal to Σ , ($t^\alpha t_\alpha = 1$), and let $h_{ab} = g_{ab} - t_a t_b$ be the induced 3-metric, $D_a = h_a^b \nabla_b$ be the projection of the full connection onto Σ . Indices may be freely converted from primed to unprimed type using $t^{AA'}$. Thus, if $V_a \in T^*(\Sigma)$

$$V_{AB} = \sqrt{2} t_{(B}^{A'} V_{A)}{}_{A'}, \quad V_{AA'} = -\sqrt{2} t_A^{B'} V_{AB}$$

Let $\Delta = D^\alpha D_\alpha$, G_{ab} be the Einstein tensor, and K_{ab} the extrinsic curvature of Σ . Adjoints are defined by $\alpha^{+A} = \sqrt{2} t^{AA'} \alpha_{A'}$, so that

$$\alpha_A \alpha^{+A} = \sqrt{2} t^{AA'} \alpha_A \bar{\alpha}_{A'} \geq 0$$

The basic identity required is that for all spinors ω^A, α_A

$$-D^{AA'} \left[-3t^{cc'} \alpha_c t_{A'}^B D_{(AB} \omega_{C)} \right] = \frac{3}{2} \left[D^{(AB} \bar{\alpha}^{C)} \right] D_{(AB} \omega_{C)} - \sqrt{2} t^{AA'} \alpha_A \left\{ \begin{aligned} & \Delta \omega^c + \frac{1}{2} t_c^{D'E} t^{A'E} G_{A'D'E} \omega^c \\ & + \sqrt{2} K_{AB} \bar{\alpha}_B D^{AA'} \omega^D \end{aligned} \right.$$

"Doing the best one can" means considering the following variational problem :- Let $S(\partial\Sigma)$ be the set of all spinor fields ω^A with ω^A specified to have some form on $\partial\Sigma$. Minimise, over $\omega^A \in S(\partial\Sigma)$

$$\mathcal{L}[\omega] = \int_{\Sigma} [D^{(AB}\omega^C)]^+ D_{(AB}\omega_C) \sqrt{g} d^3x \quad (3)$$

Using (2), the Euler-Lagrange equations for ω^A are

$$\Delta\omega^c + \sqrt{2} K_{AA'DC} D^{AA'}\omega^D = 4\pi G t_c^{D'} T_{D'BEF} t^{EE'} \omega^B, \quad (4)$$

where $G_{ab} = -\frac{1}{2}\pi G T_{ab}$. With ω^A specified on $\partial\Sigma$, since (3) is finite (if $\partial\Sigma$ is finite) and positive definite, one expects (4) to have solutions in Σ . Attention will be restricted to 2-surface twistor boundary conditions on $\partial\Sigma$. Then the solutions to (4) are Maximal twistor fields on Σ , and by design, reduce to ordinary twistor fields wherever possible.

The Π_A^A part of the MTF is defined by

$$\Pi_A^A = \frac{2i}{3} D_{AA'}\omega^A \quad (5)$$

Then a short calculation using (4) shows that

$$D_{AA'}\Pi^A = -i4\pi G t_A^{A'} T_{A'BCC'} t^{CC'} \omega^B \quad (6)$$

Note that in vacuum Π_A^A satisfies Witten's equation $D_{AA'}\Pi^A = 0$ (7). By introducing a potential

$$V(\omega) = -\frac{1}{\sqrt{2}} G_{AA'BB'} t^{AA'} \omega^B \bar{\omega}^{B'} \quad (8)$$

into \mathcal{L} one can arrange that (7) is always satisfied. The coupled

system (5) and (7) is of some interest in its own right and has also been considered by G.T. Horowitz. Here attention will be restricted to the system (4), (5), (6).

Now, if $\omega^{AB} = \omega^A \tilde{\omega}^B$, the vector determined by ω^{AB} is

$$K^{AA'} = \tilde{\omega}^A \pi^{A'} + \omega^A \tilde{\pi}^{A'} \quad (9)$$

The charge associated with K^a is

$$Q[K] = \int_{\Sigma} T_{ab} K^b d\Sigma^a \quad (10)$$

Using (6) and (9) this is just

$$-\frac{i}{2\pi q} \int_{\partial\Sigma} ds^{AA'} t_A^{B'} \tilde{\pi}_{A'} \tilde{\pi}_{B'} \quad (11)$$

The system (4), (5), (9), (10) & (11) could be said to constitute an alternative curved space-time generalisation of (1). (11) is strikingly similar to Tod's expression for the 2-surface kinematic twistor but now $\tilde{\pi}_A$ is not defined by the 2-surface twistor equation, but by the 3-surface contraction (5). The two types of $\tilde{\pi}_A$ agree iff

$$n^{AB} D_{(AB} \omega_{C)} = 0 \quad \text{on } \partial\Sigma \quad (12)$$

where n^a is normal to both t^a and $\partial\Sigma$. Putting $\omega_c = \tilde{\omega}_c$ in (2) shows that if ω^a satisfies (4), if (12) holds then

$$D_{(AB} \omega_{C)} \equiv 0 \quad \text{on } \Sigma \quad (13)$$

Thus (11) defines an expression which one knows will give good answers in several of the examples computed by Tod, but which may be useful in more general situations. Much work is needed to see whether the construction described here is really any use.

Many thanks to R.P., K.P.T. and G.T.H.

Ref. ① R. Penrose T.N. 13 and Proc. Roy. Soc A381 p53 (19)

② K.P. Tod. Prog. Roy. Soc. (to appear)

An Alternative Interpretation of Some Non Linear Gravitons.

In the paper of Tod and Ward [1] a procedure is described for generating "non linear graviton" twistor spaces \mathcal{Y} which admit a global quadratic polynomial q . q gives rise to the vector field

$$\frac{\partial q}{\partial \omega^0} \frac{\partial}{\partial \omega^0}$$

on \mathcal{Y} and this gives a killing vector on the corresponding space-time M whose derivative is anti self-dual. Tod and Ward distinguished two families of twistor spaces:

Type A $q = \omega^0 \omega^1 = \hat{\omega}^0 \hat{\omega}^1$

transition functions $\begin{cases} \hat{\omega}^0 = e^g \omega^0 \\ \hat{\omega}^1 = e^g \omega^1 \end{cases} \quad g = g(q, \pi)$

vector field $\omega^1 \frac{\partial}{\partial \omega^1} - \omega^0 \frac{\partial}{\partial \omega^0}$

Type B $q = P_A^{AB} \omega^A \Pi_B{}^1 \quad P_A^{AB} \text{ non null}$

transition functions $\hat{\omega}^A = \omega^A - (\rho^{AB} \Pi_B{}^1) g$

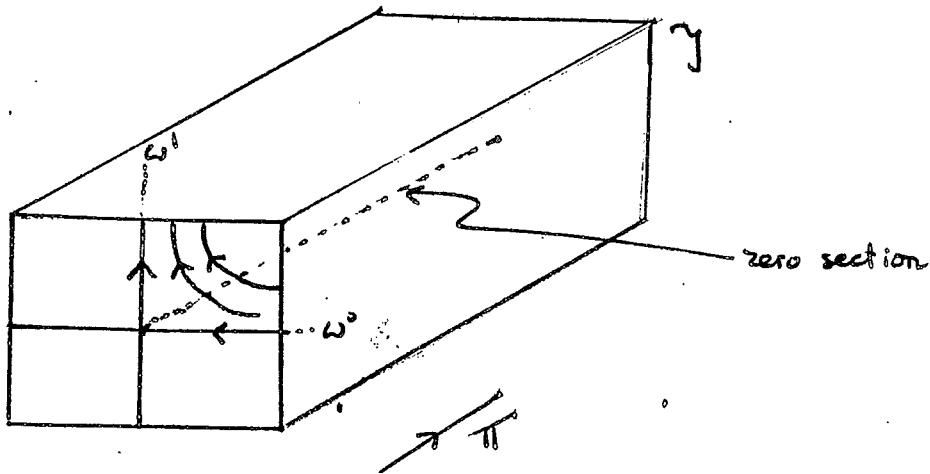
vector field $\rho^{AB} \Pi_B{}^1 \frac{\partial}{\partial \omega^A}$.

Now the idea here is to factor out by the vector field and use q as a coordinate on the factor space (a "minitwistor space"), T . Then, \mathcal{Y} will be described as some kind of bundle over T .

Type A Minitwistor coordinates $(\frac{\Pi_B{}^1}{\Pi_0{}^1}, \frac{\omega^0 \omega^1}{\Pi_0{}^1}) ; (\frac{\Pi_0{}^1}{\Pi_B{}^1}, \frac{\omega^0 \omega^1}{\Pi_B{}^1})$

The integral curves of the vector field on \mathcal{Y} are $\omega^0 \omega^1 = \text{const}$ for fixed $\Pi_B{}^1$. They are well behaved except for $\omega^1 = 0, \Pi = \text{const}$ and $\omega^0 = 0, \Pi = \text{const}$, which meet at the zero section $\omega^1 = \omega^0 = 0$, of \mathcal{Y} . Removing this zero section, the factor space T of \mathcal{Y} by the vector field is a non Hausdorff space - it is the total space of $O(2) \rightarrow \mathbb{R}$ with its zero section doubled up.

T is obtained by identifying two copies T_0 and T_1 of $O(2)$ everywhere except along their zero section. T_0 comes from factoring out $\mathcal{Y} - \{\omega^0 = 0\}$ by the vector field, and T_1 from $\mathcal{Y} - \{\omega^1 = 0\}$.



Over the two patches $\{\pi_0, \tau_0\}$ and $\{\pi_1, \tau_1\}$ of the minitwistor space T_0 , we can use $\frac{\omega}{\pi_0}, \frac{\omega}{\pi_1}$ as coordinates along the fibres of $\mathcal{Y} \rightarrow T_0$. The transition function is $\frac{\pi_1}{\pi_0} e^{-\beta}$. The fibres are copies of \mathbb{C}^* .

As explained in [2], T_0 corresponds to a copy of \mathbb{C}^3 , and a \mathbb{C}^* bundle over T_0 encodes a solution of the Bogomolny equations on it. To evaluate this field, perform the splitting

$$g(q(x,\pi), \pi) = h(x, \pi) - \hat{h}(x, \pi) \quad x \in \mathbb{C}^3$$

Then, the potential $A_{A'B'}(x)$ is given by

$$\Pi^{A^1} \nabla_{A^1 B^1} h = A_{A^1 B^1} \Pi^{B^1}$$

and its symmetric and skew parts give rise to vector and scalar potentials ω and V on \mathbb{C}^3 . In [1] the metric is shown to be given by

$$ds^2 = V^{-1} (dt - \omega \cdot d\bar{x})^2 + V d\bar{x} \cdot d\bar{x}$$

Type B Minkowski coordinates $(\frac{\pi_1}{\pi_0}, \frac{P_0 \omega^0 \pi_0}{\pi_0^2})$; $(\frac{\pi_0}{\pi_1}, \frac{P_0 \omega^0 \pi_0}{\pi_1^2})$

As fibre coordinates along the integral curves of the vector field, we use $\xi := \frac{P_0'' G^A}{\Pi_0}$; $\tilde{\xi} = \frac{P_0'' \tilde{G}^A}{\tilde{\Pi}_0}$.

Using $Z = \frac{P_{\alpha}^{(n)} \omega^{(n)}}{\Pi_{\alpha}^{(n)}} = \frac{P_{\alpha}^{(n)} \omega^{(n)}}{\Pi_{\alpha}^{(n)}}$, which holds since the deformation preserves q , we obtain

$$\begin{pmatrix} - & \text{sign} \\ - & \end{pmatrix} = \begin{pmatrix} - & \text{sign} - \frac{\pi i}{T_0} Z \\ 0 & - \end{pmatrix} / \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

with trivial translation bundle

Thus \mathcal{Y} has been exhibited as an affine line bundle over the factor space T . Such bundles provide an alternative

way of encoding "Bogomolny fields" on \mathbb{C}^3 . To extract the field we split g (the term $\frac{T_{\mu\nu}}{2}$ does not contribute since it is a coboundary) and obtain the same Maxwell field as before. Tod and Ward show that the metric is again given by

$$ds^2 = V^{-1} (dt - \omega_{\mu} dx^{\mu})^2 + V d\bar{z} dz.$$

To summarise, the Tod and Ward "non linear gravitons" of types A and B are respectively the principal \mathbb{C}^* and affine bundles on minitwistor space which encode the solution of the Bogomolny equations appearing in the expression for the metric.

The Eguchi Hanson twistor space can be described in these terms as a \mathbb{C}^* bundle over a non Hausdorff minitwistor space. The E.H. solution is described in terms of charges placed at a finite number of points of \mathbb{R}^3 . The \mathbb{C}^* bundle corresponds to the Coulomb field of these charges.

Thanks to M.G.E.

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- Phil Jones

Three-Surface Twistors and Conformal Embedding

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Abstract

The three-surface twistor equation is defined for an arbitrary three-surface Σ in an arbitrary curved space M .

It is proved that three-surface twistors exist on Σ if and only if Σ can be embedded in a conformally flat space-time with the same first and second fundamental forms.

(to appear in G.R.G.)

On the Density of Elementary States

Introduction: Recall that an elementary state (or, more precisely, a first order elementary state) of homogeneity -2 is a cohomology class represented as a Čech cocycle by the twistor function $1/(A.Z)(B.Z)$ for fixed dual twistors A and B . As a twistor diagram this is written as $\overline{A} \vee \overline{B}$. The corresponding massless field is defined away from the null cone of the point \overline{AB} and has the simplest possible singularity on this cone. A singularity of the next order up (a second order elementary state) corresponds to a twistor function of the form $(C.Z)/(A.Z)^2(B.Z)$ i.e. a twistor diagram $\overline{A} \vee \overline{B} \wedge \overline{C}$. The general pattern (for other homogeneities too) is explained in [1] where it is also pointed out that any cohomology class on $\mathbb{P} - \overline{AB}$ can be expanded as a series $\sum e_n$ where e_n is an n^{th} order elementary state based on the line \overline{AB} . The sense in which this expansion is valid is left rather vague in [1]. We will show how to make this precise but, more importantly, we will show that a similar expansion is possible for any class in $H^1(\mathbb{P}^+)$ provided that the line \overline{AB} is contained in \mathbb{P}^+ . This justifies the folklore that to prove things about cohomology classes in general it suffices to consider only elementary states (which can be manipulated using twistor diagrams). Note that there is also the possibility of considering just the first order elementary states but allowing the line \overline{AB} to vary. The methods to be explained shortly also apply to this possibility showing that they too are dense (but an expansion is not possible).

Motivation: It is possible to give [2], in terms of twistor theory, a rather natural definition of the scalar product $\langle | \rangle$ of massless fields. In particular, this definition is manifestly $SU(2,2)$ -invariant. Unfortunately, it is not at all clear why $\langle | \rangle$ is positive definite. Even on Minkowski space this is mysterious for integral helicity (e.g. Maxwell) and the usual argument is to expand using plane waves. Once there is a precise meaning to the density of elementary states it is possible to prove directly that the scalar product is positive. An advantage of this approach is that it is not limited to $SU(2,2)$. An analogous construction [3] provides unitary representations of $SU(p,q)$ for all p and q .

Analogy: As an example of the general case of $SU(p,q)$ and also as a guiding analogy for $SU(2,2)$ it is a good idea to keep in mind the case of $SU(1,1) = SL(2, \mathbb{R})$ acting on homogeneous functions on the northern hemisphere of the Riemann sphere or, equivalently, the disc $\{|z| \leq 1\}$. This displays all features of the twistor case. There is a scalar product constructed from the spinor transform (given by an integral formula as for the twistor transform (but rigorously in this case as we are dealing with functions rather than cohomology classes)) which gives rise to a series of unitary representations known as the discrete series. For homogeneous functions of degree zero (to be compared with anti-self-dual Maxwell fields) the spinor transform coincides with exterior derivative and the formula for the inner product becomes

$$\langle f | g \rangle = \frac{i}{2\pi} \oint_{|z|=1} f \bar{d}g$$

which is Hermitian by integration by parts. Actually, there is a feature displayed by this example which is absent in the usual twistor case (but see [4]). This feature is that the spinor transform is not an isomorphism since it has the constant functions as kernel. Consequently the constant functions are orthogonal to all other functions with respect to the inner product above. However, if we factor out the space of all holomorphic functions on the disc by the constant functions (or, in other words consider $\langle \cdot | \cdot \rangle$ as defined on holomorphic 1-forms in which case the functions should be thought of as "potentials" for the 1-forms as "fields" (which makes a lot of sense because the Penrose transform is degenerate in this case)) then it turns out that the scalar product is positive definite. To see that this is the case expand f as a Taylor series in z^n for $n \geq 1$. It is easy to verify that the scalar product is positive on each z^n :

$$\begin{aligned} \langle z^n | z^n \rangle &= \frac{i}{2\pi} \oint_{|z|=1} z^n n \bar{z}^{n-1} d\bar{z} \\ &= \frac{1}{2\pi i} \oint_{|z|=1} n \frac{dz}{z} = n , \end{aligned}$$

and that they are mutually orthogonal. Note that it is important that f is "positive frequency" (i.e. defined on the northern hemisphere rather than the southern). For $f = 1/z$, for example, $\langle f | f \rangle = -1$. For odd homogeneities (= "Fermions") this is irrelevant just as in the twistor case. The idea is to imitate this proof of positivity in the twistor case by using elementary states instead of z^n . A final remark for $SL(2, \mathbb{R})$ is that the discrete series is usually constructed on L^2 holomorphic functions (w.r.t. the hyperbolic measure and with suitable weightings and transformation rules for these "functions" (they are really sections of a power of the Hopf bundle)). An analogous possibility, i.e. using L^2 cohomology, exists in the twistor case but the construction given in [2] seems a lot simpler.

Topologies on cohomology and massless fields: Although, from the point of view of the scalar product, it is the cohomology $H^1(\overline{\mathbb{P}^+})$ i.e. massless fields on $\overline{\mathbb{M}^+}$ (i.e. holomorphic in a neighbourhood of \mathbb{M}^+ i.e. positive frequency and real analytic) that is relevant, it is easier first to discuss the topologies on $H^1(\mathbb{P}^+)$ and on massless fields on \mathbb{M}^+ . In general, for any complex manifold X and any holomorphic vector bundle V over X , there are various natural topologies on $H^p(X, \mathcal{O}(V))$. Perhaps the easiest to describe is to realize $H^p(X, \mathcal{O}(V))$ by means of the Dolbeault resolution of $\mathcal{O}(V)$ and to endow (for details see [5]) each space of forms $T(X, \mathcal{E}^{0,2}(V))$ with the topology of uniform convergence with all derivatives (with respect to some (irrelevant) choice of local trivialization) on compact sets. The $\bar{\partial}$ -operator is then a continuous linear operator between Fréchet spaces. Its kernel is therefore closed and thus a Fréchet space. Its image is not necessarily closed (there is an example in [5]). Hence, if we endow $H^p(X, \mathcal{O}(V))$ with the quotient topology, then it is not necessarily Hausdorff but $H^p(X, \mathcal{O}(V))/\{0\}$ is Fréchet. It is also possible to endow $H^p(X, \mathcal{O}(V))$ with a topology by using the Čech definition with respect to a Leray cover and giving the cocycles a Fréchet topology derived from the topology of uniform

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on compact sets in the intersections (see [6] for details). It is shown in [6] that these two definitions give the same answer on cohomology (and hence that the choice of cover is irrelevant). There is another possibility, the corresponding weak topology, but this is unnecessary here. There is always a pairing

$$H^p(X, \mathcal{O}(V)) \otimes_{\mathbb{C}} H_*^{n-p}(X, \Omega^n(V^*)) \rightarrow \mathbb{C}$$

where n is the dimension of X and H_* denotes cohomology with compact supports. It may be given, for example, by cup product of Dolbeault representatives followed by integration over X . Serre [5] shows that if $H^{p+1}(X, \mathcal{O}(V))$ is Hausdorff then this pairing identifies $H_*^{n-p}(X, \Omega^n(V^*))$ as the topological dual of $H^p(X, \mathcal{O}(V))$. Laufer [6] shows (more than) the converse, i.e. if the pairing identifies $H_*^{n-p}(X, \Omega^n(V^*))$ as the topological dual of $H^p(X, \mathcal{O}(V))$ then $H^{p+1}(X, \mathcal{O}(V))$ is Hausdorff i.e. is Fréchet. Armed with this information together with the vanishing of second cohomology of \mathbb{P}^{\pm} and $\overline{\mathbb{P}}^{\pm}$ (which can be deduced from results of Andreotti and Grauert [7]) it is straightforward to show that

$H^1(\mathbb{P}^+, \mathcal{O}(k))$ is a Fréchet space with dual

$$H_*^2(\mathbb{P}^+, \Omega^3(-k)) = H^1(\overline{\mathbb{P}}^-, \Omega^3(-k))$$

For example, $H^1(\mathbb{P}^+, \mathcal{O}(k))$ is Fréchet because $H^0(\mathbb{P}^+, \mathcal{O}(k)) = H^0(\mathbb{P}, \mathcal{O}(k))$ has $H^3(\mathbb{P}, \Omega^3(-k)) = H_*^3(\mathbb{P}^+, \Omega^3(-k))$ as dual. A choice of twistor ϵ corresponds to a choice $\Omega^3 \simeq \mathcal{O}(-4)$ and so it follows that $H^1(\mathbb{P}^+, \mathcal{O}(-n-2))$ is a Fréchet space with topological dual $H^1(\overline{\mathbb{P}}^-, \mathcal{O}(n-2))$, the pairing being the dot product [2]. For a fixed helicity it is clear that the holomorphic massless fields form a Fréchet space under the topology of uniform convergence on compact sets. It is also clear that the Penrose transform is continuous because to control a massless field on a compact set $K \subset \mathbb{M}^+$ it suffices to control the cohomology class on the corresponding compact set $K' \subset \mathbb{P}^+$. Hence, by the open mapping theorem (which is valid for Fréchet spaces) the Penrose transform is a topological isomorphism also.

Thus, the twistor transform is a topological isomorphism (there are other proofs of this). A similar discussion can be given for $H^1(\overline{\mathbb{P}^+}, \mathcal{O}(-n-2))$ with similar conclusions. The only difference is that $H^1(\overline{\mathbb{P}^+}, \mathcal{O}(-n-2))$ has the topology of the strong dual of the Fréchet space rather than a Fréchet space. The isomorphism $H^1(\overline{\mathbb{P}^+}) = H_*^2(\mathbb{P}^-)$ is an isomorphism of Hausdorff topological vector spaces (see [6] for more details of the topologies) and

$$H^1(\overline{\mathbb{P}^+}, \mathcal{O}(-n-2)) \otimes_{\mathbb{C}} H^1(\overline{\mathbb{P}^-}, \mathcal{O}(n-2)) \rightarrow \mathbb{C},$$

given by the dot product, is a perfect pairing (identifies each with the topological dual of the other (giving the strong topology to the dual)). Since both the twistor transform and the dot product are continuous (use the Čech definitions of topology and dot product) it follows that the scalar product $\langle \phi | \psi \rangle = \phi \cdot \bar{\psi}$ is a continuous function on $H^1(\overline{\mathbb{P}^+}) \times H^1(\overline{\mathbb{P}^-})$.

Proposition The elementary states based on a line $L \subset \mathbb{P}^-$ are dense in $H^1(\overline{\mathbb{P}^+})$ and $H^1(\overline{\mathbb{P}^-})$.

Proof. Restrict to homogeneity -2. Other homogeneities are similar. Denote by x the point in \mathbb{M}^- corresponding to the line L . Since the Hahn-Banach theorem is valid for Fréchet spaces the elementary states will be dense in $H^1(\overline{\mathbb{P}^+}, \mathcal{O}(-2))$ if every continuous linear functional which vanishes on all the elementary states is identically zero. Hence we would like to show that:

$F \cdot e = 0$ for all elementary states e based on L ,

where $F \in H^1(\overline{\mathbb{P}^-}, \mathcal{O}(-2))$, implies that $F = 0$. If e_1 is the first order elementary state (suitably normalized) then $F \cdot e_1 = \phi(x)$ where ϕ is the solution of the wave equation corresponding to F . Similarly, second order elementary states yield the derivatives of ϕ . More precisely, choosing (non-standard) coordinates so that x is the origin, the second order elementary states are $T\mathcal{A}' \frac{\partial}{\partial w_A} e_1$.

Integrating by parts (this makes good cohomological sense),

$$F \cdot \pi_{A'} \frac{\partial}{\partial w^A} e_1 = - \pi_{A'} \frac{\partial}{\partial w^A} F \cdot e_1 = (\nabla_{A'A} F)(x).$$

Using higher order elementary states implies all derivatives vanish at ∞ . Thus, by analyticity, ϕ , and hence F , is zero. The proof for $H^1(\bar{P}^+, \mathcal{O}(-2))$ is similar. \square

Application: If $\frac{A}{A+B} \subset P^-$ then $A, B \in P^{*+}$ and so

$$\langle \overset{A}{\circ} \overset{B}{\circ} | \overset{A}{\circ} \overset{B}{\circ} \rangle = 1 / \frac{A}{A+B} > 0.$$

Similarly, it is straightforward to verify that $\langle \cdot | \cdot \rangle$ is positive on all elementary states. By the previous proposition the elementary states are dense in $H^1(\bar{P}^+)$ and earlier it was observed that $\langle \cdot | \cdot \rangle$ is continuous. It follows that $\langle \cdot | \cdot \rangle$ is non-negative definite. To show that $\langle \cdot | \cdot \rangle$ is actually positive definite one must check moreover that the orthogonal complement of the elementary states is just $\{0\}$. This follows by arguing as in the proof of the proposition.

Remark: The Hilbert space completion of $H^1(\bar{P}^+)$ w.r.t. $\langle \cdot | \cdot \rangle$ is called the space of normalizable states and lies between $H^1(\bar{P}^+)$ and $H^1(P^+)$ in much the same way that L^2 lies between smooth functions and distributions or, more accurately, between real analytic functions and hyperfunctions.

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The Norm on Superficial Twistor Space

B P Jeffries

The quasi-local angular-momentum twistor introduced by R.P. in T.N 13 still has many problems associated with it, aside from the generally computationally difficult procedure for evaluating it. In particular no satisfactory norm or infinity twistor have yet appeared, aside from in special cases, leaving as yet no reason for the 10 complex components of $A_{\alpha\beta}$ to reduce to 10 real numbers. This paper is devoted to a description of the norms that appear to be correct in limited cases, and a suggestion of a generic norm

Although integral expressions for a norm have been suggested - see Penrose(1982) - these seem unsatisfactory as they provide no link with the norm of 'ordinary' twistor space, namely

$$\sum = \omega^A \bar{\Pi}_A + \bar{\omega}^{A'} \bar{\Pi}_{A'} \quad (1)$$

an expression which is conformally invariant. Any discussion on the constancy of a norm is handicapped by our lack of knowledge of all the derivatives of $\bar{\Pi}_A$. To recap, using the notation of GHP(1972), we write

$$\omega_A = \omega^0 \sigma_A + \omega^1 \tau_A \quad \bar{\Pi}_A = \bar{\Pi}_1 \bar{\sigma}_A - \bar{\Pi}_0 \bar{\tau}_A \quad (2)$$

$$\partial' \omega^0 = \sigma' \omega^1 \quad \partial' \omega^1 = \sigma \omega^0 \quad (3)$$

$$-i \bar{\Pi}_0 = \partial' \omega^1 - \rho \omega^0 \quad -i \bar{\Pi}_1 = \partial \omega^0 - \rho' \omega^1 \quad (4)$$

by commuting $[\partial, \partial']$ on ω^0 and ω^1 we have

$$-i(\partial \bar{\Pi}_0 + \rho \bar{\Pi}_1) = X_2 \omega^1 + X_1 \omega^0 \quad (5)$$

$$-i(\partial' \bar{\Pi}_1 + \rho' \bar{\Pi}_0) = X_3 \omega^1 + X_2 \omega^0 \quad (6)$$

where

$$X_1 = \psi_1 - \phi_{01}$$

$$X_2 = \psi_2 - \phi_{11} - \lambda$$

$$X_3 = \psi_3 - \phi_{21}$$

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We may wonder whether in general an expression of the form

$$\delta' \Pi_{01} = A \Pi_{01} + B \Pi_{11} + C \omega^0 + D \omega^1 \quad (6a)$$

can be found. The following argument that one can be found is due to Paul Todd.

Consider the determinant

$$D = \begin{vmatrix} \omega^0 & \omega^0 & \omega^0 & \omega^0 \\ \omega^1 & \dots & \dots & \dots \\ \Pi_{01} & & & \\ \Pi_{11} & & & \end{vmatrix} \quad (7)$$

where $\omega^0, \omega^1, \dots$ etc are the four independent solutions.

Let $K = \frac{\delta' D}{D}$. Since derivatives of $\omega^0, \omega^1, \Pi_{01}, \Pi_{11}$, contain no $\omega^0, \omega^1, \Pi_{01}, \Pi_{11}$, terms respectively we may write

$$\begin{vmatrix} \omega^0 & \dots & & \\ \omega^1 & \dots & & \\ \omega^0 & & & \\ \Pi_{01} & & & \\ (\delta' - K) \Pi_{01} & (\delta' - K) \Pi_{11} & \dots & \end{vmatrix} = 0 \quad (8)$$

implying (6a).

If our two-surface (including normals) is embedable in conformally flat real Minkowski space we might expect that Π_{01} obeys

$$\tilde{\nabla}_{AB} \tilde{\Pi}_{01} = 0 \quad \text{for some suitably conformally transformed } \tilde{\Pi}_{01} \text{ & } \tilde{\Pi}_{11}.$$

Alternatively

$$-i \nabla_{AA'} \Pi_{01} = P_{AA'BB'} \omega^B \quad P_{AA'BB'} = \overline{P_{A'B'A'B}}$$

implying that the other two Π derivatives are of the form

$$-i (\delta' \Pi_{01} + \bar{\sigma}' \Pi_{11}) = C \omega^0 + D \omega^1 \quad (10)$$

$$-i (\delta' \Pi_{11} + \bar{\sigma}' \Pi_{01}) = C' \omega^1 + D' \omega^0 \quad (11)$$

Applying $[\delta, \bar{\sigma}]$ to Π_{01} and Π_{11} leads to the relations

$$X_2 = \bar{X}_2 \Rightarrow \bar{\psi}_2 = \bar{\psi}_2 \quad (12)$$

$$C = \bar{X}_1 \quad (13)$$

$$C' = \bar{X}_3 \quad (14)$$

$$D\sigma - \bar{\sigma}D' = \delta' X_1 - \delta' \bar{X}_1 \quad (15)$$

$$D'\sigma' - \bar{\sigma}D = \delta' X_3 - \delta' \bar{X}_3 \quad (16)$$

$$\delta' D = \delta' X_2 + \sigma' X_1 + \bar{\sigma} \bar{X}_3 - p' \bar{X}_1 - p \bar{X}_3 \quad (17)$$

$$\delta' D' = \delta' X_2 + \sigma' X_3 + \bar{\sigma}' \bar{X}_1 - p' \bar{X}_3 - p \bar{X}_1 \quad (18)$$

However even should we have a two-surface on which $\mathcal{V} = \bar{\mathcal{V}}$ then we have no guarantee that D and D' as given by (15,16) will satisfy (17,18).

In fact since X_1 and X_3 are given by $X_1 = \sigma - \bar{\sigma}\rho$, $X_3 = \sigma' - \bar{\sigma}'\rho'$ and we are free to choose $\sigma, \rho, \sigma', \rho'$, in general (17,18) will not be satisfied, hence no expression like (9) can be found. It can be shown also that if the norm (1) is constant the (12-18) are satisfied and vice-versa, since (12-18) are just the conditions that X_1, X_2, X_3 may be removed by a real conformal transformation and we see that the following 3 statements are equivalent

1. (10,11) are true for some C, C', D, D'
2. the norm (1) is constant
3. the two surface is related by a real conformal factor to a surface in real Minkowski space.

We may ask how terms such as the imaginary part of \mathcal{V} may be mimicked by some form of conformal transformation on real Minkowski space. As suggested by R.P. this may be achieved by allowing \mathcal{E}_{AB} and $\mathcal{E}_{AB'}$ to scale differently under a conformal transformation, but keeping the standard form of conformal metric, to wit $\mathcal{E}_{AB} = \sqrt{-g} \mathcal{E}_{AB}$ $\mathcal{E}_{AB'} \rightarrow \sqrt{-g} \mathcal{E}_{AB'}$,

$$\hat{\nabla}_A \xi_B = \nabla_{AA'} \xi_{B'} + \gamma_{BA'} \xi_A \quad (19)$$

$$\hat{\nabla}_{A'} \xi_{B'} = \nabla_{AA'} \xi_{B'} + \bar{\gamma}_{B'A'} \xi_A$$

In this case we have

$$-i \nabla_{AA'} \Pi_{B'} = i(\gamma_{AB'} - \bar{\gamma}_{B'A'}) \Pi_{A'} + P_{AA'B'B'} w^B \quad (20)$$

$$\sum = w^A \bar{\Pi}_A + \bar{w}^{A'} \Pi_{A'} + i(\gamma_{AA'} - \bar{\gamma}_{AA'}) w^A \bar{w}^{A'} \quad (21)$$

and $P_{AA'B'B'}$ is no longer Hermitian.

$$\text{Putting } (\gamma_{AA'} - \bar{\gamma}_{B'A'}) = (0, \bar{\gamma}_B, i \mathcal{F}(2)) + (i \bar{\gamma}_B, i \mathcal{F}(2)) - (0, \bar{\gamma}_B, i \mathcal{F}(2)) - (i \bar{\gamma}_B, i \mathcal{F}(2)) \quad (22)$$

and comparing coefficients with (5) and (6) we see that $\mathcal{F} = \bar{\mathcal{F}} = 0$.

In addition the two 'unknown' derivatives are of the form

$$-i(\gamma' \Pi_{B'} + \bar{\gamma}' \bar{\Pi}_{B'}) = B \Pi_{B'} + C w^0 + D w^1 \quad (23)$$

$$-i(\gamma \Pi_{B'} + \bar{\gamma} \bar{\Pi}_{B'}) = B' \Pi_{B'} + C' w^1 + D' w^0 \quad (24)$$

where for the moment we do not assume $B = 2i\gamma\theta, B' = 2i\bar{\gamma}\theta$.

Note that the coefficient of $\Pi_{B'}$ in (23) is constrained to be $\bar{\gamma}$ and

that of $\bar{\Pi}_0$ in (24) is $\bar{\sigma}'$. Applying commutators to $\bar{\Pi}_0$ and $\bar{\Pi}_1$ leads to

$$\gamma \theta = \gamma' \theta' = \sqrt{2} - \bar{\sqrt{2}} \quad (26)$$

thus implying θ and θ' are of the form (25) and $\gamma \theta' = \text{Im} \sqrt{2}$. (27)

This equation has only constants as kernel, which will not affect the curvature. We are now lead to

$$C = \bar{X}_1 + 2ip\gamma'\theta' + 2i\bar{\sigma}\theta \quad (28)$$

$$C' = \bar{X}_3 + 2ip'\gamma\theta + 2i\bar{\sigma}'\theta' \quad (29)$$

plus similar equations to (15,16,17,18). In fact we are also lead to much the same conclusions. The following statements are equivalent

1. A norm of the form (21) is constant
2. $\bar{\Pi}$ derivatives are of the form (23,24)
3. The two-surface is real conformally imbedable in real Minkowski space with generated by

The fact that we do not have a general norm despite introducing torsion to imitate $\text{Im} \sqrt{2}$ can be interpreted as due to inadequate freedom in choice of conformal factor. In $X_1, X_3, X_2 + \bar{X}_2$ we have 5 real functions. To mimic these terms we have 3 real functions $\varrho, \bar{\varrho}, \varrho'$. Should we be allowed complex conformal factors then we naively have enough freedom, ie 6 real functions to use, however there are difficulties. We cannot arrange for both ϱ and ϱ' to be real simultaneously, though by fixing a complex boost we may have one real. The answer appears to be to reintroduce 'phase' torsion of the form (19) and we may arrange for the conformally rescaled ϱ & ϱ' to be real, since $\varrho \rightarrow \varrho - i\bar{\varrho}'$, $\varrho' \rightarrow \varrho' - i\bar{\varrho}$. plus scaling.

Although we may arrange $\varrho = \bar{\varrho}$, $\varrho' = \bar{\varrho}'$ we cannot arrange that the rescaled σ & $\bar{\sigma}$ will be complex conjugates. Thus what should appear as the coefficient of $\bar{\Pi}_1$ in (23) will not be $\bar{\sigma}'$ but what would be $\bar{\sigma}'$, were we in a complex conformally rescaled real Minkowski space. The picture is still unclear but it appears that an expression similar to (21) will be constant, but with $\tilde{\omega}^A$ replacing $\bar{\omega}^A$ and $\tilde{\Pi}_A$ replacing $\bar{\Pi}_A$. $\tilde{\omega}^A \neq \bar{\omega}^A$ but

rather satisfies equations such as

$$\delta' \tilde{\omega}^0 = \delta \tilde{\omega}^0 \quad \delta \tilde{\omega}' = \delta \tilde{\omega}' \quad (30)$$

$$-i\tilde{\pi}_{11} = \delta' \tilde{\omega}^0 - p' \tilde{\omega}' \quad -i\tilde{\pi}_0 = \delta \tilde{\omega}' - p \tilde{\omega}^0 \quad (31)$$

The basis for $\tilde{\omega}^A$ is chosen so as to agree with that of $\tilde{\omega}^A$ after suitable complex rescaling. If this cloudy picture emerges as correct it provides a correspondence between the $A_{\alpha\beta}$ of the curved space and $A_{\alpha\beta}$ of a transformed real Minkowski space, and should thus show that A has only 10 real components.

Thanks are due as ever to Paul Tod and Roger Penrose for help and inspiration.

*Ben Jeffys
21/7/83*

Refs.

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Penrose TN 13

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Tod Proc Roy Soc A, to appear

Quasi-local charges in Yang-Mills theory

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(Communicated by R. Penrose, F.R.S. — Received 6 April 1983)

A definition of quasi-local charges in Yang-Mills theory, in the spirit of Penrose's quasi-local momentum in general relativity, is proposed.

The Generalized Penrose-Ward Transform

Michael Eastwood

Abstract

The Penrose transform and Ward correspondence are closely related and both generalize considerably. This article explains carefully the relationship between these two transforms and also how to construct them for more general homogeneous spaces. Many examples (all the usual and more exotic) are discussed.

Conformal Killing Vectors and Reduced Twistor Spaces.

If one takes the time translation killing vector $\frac{\partial}{\partial t}$ on affine Minkowski space then the space of integral curves is naturally a complex affine three space, \mathbb{A} , with Euclidean metric. $\frac{\partial}{\partial t}$ gives rise to a vector field on \mathbb{PT}^1 , namely $\pi_0 \frac{\partial}{\partial w} + \pi_1 \frac{\partial}{\partial \bar{w}}$, and factoring out by this gives a two dimensional complex manifold, "minitwistor space" denoted MT [1]. In fact the correspondence between MT and \mathbb{A} can be set up directly i.e without going via space time and twistor space. This is achieved by considering \mathbb{A} as the parameter space of rational holomorphic curves of normal bundle $\mathcal{O}(2)$ in MT . This correspondence was generalised by Hitchin [2] as follows:

Let T be a two dimensional complex manifold containing a rational curve C with normal bundle $N \cong \mathcal{O}(2)$. Then $H^1(C, N) = 0$ so by Kodaira's theorem C belongs to a locally complete family of such curves, parametrised by some complex manifold H . Moreover tangent vectors to H at $x \in H$ are represented by global sections of the normal bundle of the corresponding curve C_x . Clearly $\dim H^0(C_x, N) = 3$, hence H is three dimensional. Give H a conformal structure by calling a tangent vector at x null if and only if the corresponding section of the normal bundle of C_x vanishes to second order at one point.

We now define a projective connection on H . Pick a non null vector in $T_x H$. This defines a section of the normal bundle of C_x which vanishes at two points. There is a one parameter family of $\mathcal{O}(2)$ curves in H which pass through these points, and this defines a curve in the parameter space H which starts off in the direction given by the chosen vector. If this vector were chosen to be null, the section of the normal bundle of C_x would vanish at one point and the relevant curve in H would correspond to the one parameter family of curves tangent to C_x at the point. We thus have a projective connection on H for which the curves defined above are geodesics. This connection is clearly compatible with the conformal structure, since a geodesic which starts out in a

null direction remains null. This is sufficient [2] to determine an affine connection on H , and a manifold with these structures is called a Weyl space.

There is another way to obtain Weyl spaces. Let (M, g) be a four dimensional space time with conformal killing vector V . Then, the space H of trajectories of V can be given the structure of a Weyl space.

A point $x \in H$ corresponds to a trajectory V_x of V in M . A tangent vector X at x corresponds to a normal Jacobi field J_X along V_x . Define a conformal metric on H by requiring X to be null if and only if $g(J_X, J_X) = 0$. ($\mathcal{I}_V g \propto g$ so this makes sense).

For X, Y tangent vector fields on H , as a candidate for the covariant derivative of Y in the direction X , one might try $\nabla_{J_X} J_Y$ (where ∇ is the metric connection on M). However, $\nabla_{J_X} J_Y$ is not lie propagated along V . One can correct for this by using

$$\nabla_{J_X} J_Y^b - \mathcal{D}_{ac}^b J_X^a J_Y^c$$

where $\mathcal{D}_{ac}^b = \frac{1}{2} \{ \delta_a^b \omega_c + \delta_c^b \omega_a - g_{ac} \omega^b \}$ and ω^b is chosen s.t. $\mathcal{I}_V \omega_b = \nabla_b V$. This is now lie propagated but is not orthogonal to V , therefore one must apply the projection operator $h^b_c = (\delta^b_c - V_c V^b / V^2)$. Finally, we adopt

$$(D_X Y)^b = (\nabla_{J_X} J_Y^c - \mathcal{D}_{ad}^c J_X^a J_Y^d) h^d_c$$

as the connection on H . A simple calculation shows it to be compatible with the conformal structure. There is some freedom in the choice of ω_b , the only requirement being $\mathcal{I}_V \omega_b = \nabla_b (\frac{1}{2} \nabla \cdot V)$. A fairly natural choice is $\omega_b = \nabla_b (\ln V^2)$, and this gives a connection on H that is invariant under conformal rescaling of (M, g) .

Suppose now we start off with a right flat space time (M, g) with conformal killing vector V , where M is sufficiently restricted so that V does not vanish on M . We can then factor out by V and obtain a Weyl space as above. Alternatively, we can construct the twistor space P_M for M , on which V induces a vector field \tilde{V} . Let T denote the factor space of P_M by \tilde{V} . Then we have:

Proposition

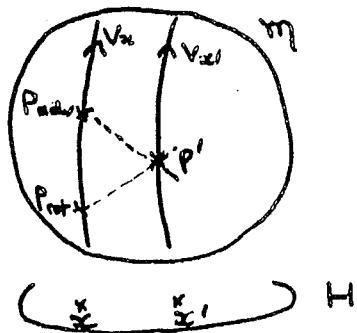
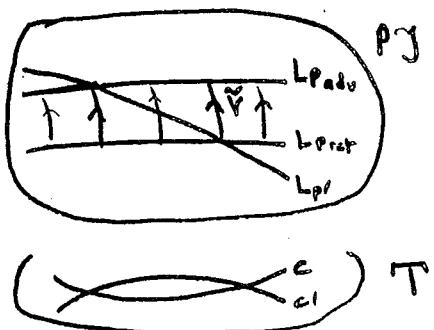
T is a manifold of the type used in the Hitchin construction and H is the parameter space of curves with normal bundle $D(2)$.

Proof

A point x in H lifts to a trajectory of V in M . Via the twistor

30.

Correspondence this defines a ruled surface $R \subset \mathbb{P}^1$, ruled by lines in one direction and integral curves of \tilde{V} in another. R therefore projects to a rational curve C in T . To determine the normal bundle of C , let x' be infinitesimally separated from x in a non null direction. Then each point p' on the trajectory over x' is null separated from two points on the trajectory over x , called Paud & Prat



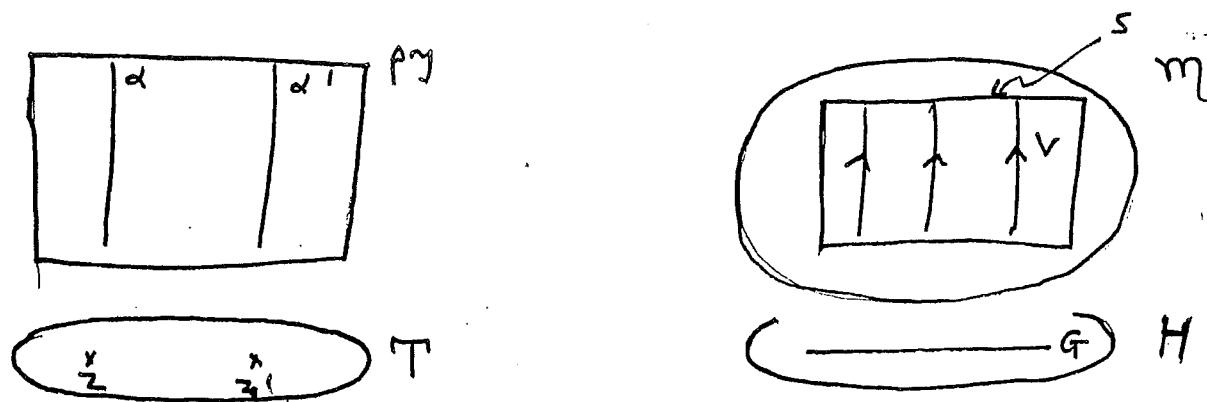
Therefore the line $L_{p'} \subset \mathbb{P}^1$ intersects the ruled surface R (corresponding to the conf. killing vector trajectory V_{x_0}) in two points. In fact the whole ruled surface R' (corresponding to the trajectory V_{x_1}) meets R in two integral curves of \tilde{V} and hence the projected curves c and c' in T intersect in two points. We deduce that the curves in T parametrized by points in H have self-intersection number two, hence their normal bundle is $B(2)$ and the proposition is proved. Applying the Hitchin construction to T gives H a Weyl structure. Does this agree with the Weyl structure inherited from M ?

Conformal Metric.

Suppose x and x' are infinitesimally null separated with respect to the conformal structure inherited from M . Then a point on the trajectory $V_{x'}$ is now separated from only one point on the trajectory V_x . The ruled surfaces in \mathbb{P}^1 meet in one integral curve of \tilde{V} and the projected curves in T are tangent. Thus x and x' are null separated with respect to the Hitchin conformal structure. The converse is also clear.

Projective Connection.

Pick a pair of minitwistors $z, z' \in T$. These lift to a pair d, d' of integral curves of \tilde{V} in T . These in turn correspond to a pair of one-parameter families of α -planes in M . The intersection points of this collection of α -planes form a 2-surface in M , ruled by the integral curves of V . This projects along V to a curve G in H , which is geodesic with respect to the Hitchin connection.



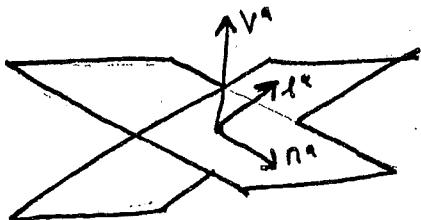
We now address the question of whether G is also geodesic with respect to the connection induced from M :

Fixing an α -plane in one family, its intersections with the other family give a null curve in M with tangent vector

$$\ell^a = (V^2)^{\frac{1}{2}} V^{AB'} \rho_{B'} \pi^A$$

Doing the same for an α -plane in the other family gives a null curve with tangent vector

$$n^a = (V^2)^{\frac{1}{2}} V^{AB'} \pi_B \rho_A$$



$$\text{Then } \ell^a - n^a = (V^2)^{\frac{1}{2}} V^a (\rho \cdot \pi)$$

Set $J^a = \ell^a + n^a$. J^a is orthogonal to V^a , tangent to S and lie propagated and hence corresponds to the tangent vector to the curve G . To show that G is geodesic with respect to the connection inherited from M , we need to show

$$(\nabla_J J^c - \gamma_{ad}^c J^a J^d) h_c^b \propto J^b$$

i.e. $\nabla_J J^c - \gamma_{ad}^c J^a J^d$ is a lin. comb. of $J^c \circ V^c$

This reduces to showing that

$$\ell^a \nabla_a \ell^b + n^a \nabla_a n^b = \alpha \ell^b + \beta n^b \text{ some } \alpha, \beta$$

To do this, introduce the vectors

$$m^a = (V^2)^{\frac{1}{2}} V^{AB'} \pi_B \rho_A ; \tilde{m}^a = (V^2)^{\frac{1}{2}} V^{AB'} \rho_B \pi_A$$

so that ℓ, m, \tilde{m}, n make a null tetrads. Then we have to show that

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$$m_b l^a \nabla_a l^b + m_b n^a \nabla_a n^b = 0; \quad \tilde{m}_b l^a \nabla_a l^b + \tilde{m}_b n^a \nabla_a n^b = 0.$$

Substituting for l, m, \tilde{m}, n we obtain the condition

$$V^B c' V^A o' \tilde{\Phi}_{AB} + \frac{1}{2} V^2 \tilde{\Phi} c' o' = 0$$

(where $\nabla_a V_b - \nabla_b V_a = \tilde{\Phi}_{AB} \Sigma_A{}^c o^c + \phi_{AB} \Sigma_{AB}$), which is exactly the condition that V is hypersurface-orthogonal. Thus the connection inherited from \mathcal{M} agrees with the Hitchin connection when V is H.S.O. Unfortunately, a simple calculation reveals that a right flat space time with a H.S.O. conformal killing vector must be flat. However, a simple modification of the connection inherited from \mathcal{M} makes it agree with the Hitchin connection in general:

Our connection coefficients γ_{bc}^a contain ω_b , which is chosen so that $\mathcal{L}_V \omega_b = \frac{1}{2} \nabla_b (\nabla \cdot V)$. So far we have used $\omega_b = \nabla_b (\ln V^2)$. But, we can modify ω_b by $\omega_b \rightarrow \omega_b + K_b$ where $\mathcal{L}_V K_b = 0$. Substituting $\nabla_b (\ln V^2) + K_b$ for ω_b in our previous calculation shows that for G to be a Hitchin geodesic, we require

$$K_b = \frac{1}{V^2} (V^A o' \tilde{\Phi}_{AB} - V_B o' \phi_{AB}).$$

$$= \frac{i}{V^2} \epsilon_{ba}^{cd} V^a \nabla_c V_d$$

Therefore, the expression we use for ω_b in $\gamma_{ac}^b + \frac{1}{2} (\delta_a^b \omega_c + \delta_c^b \omega_a - g_{ac} \omega^b)$

$$\text{is } \nabla_b (\ln V^2) + \frac{i}{V^2} \epsilon_{ba}^{cd} V^a \nabla_c V_d.$$

$$= \frac{2V^a}{V^2} (\nabla_b V_a + i (\nabla V)_{ba})$$

$$= \frac{2V^a}{V^2} (\nabla V)_{ba} + \frac{2V_b}{V^2}$$

$(\nabla V)_{ba}^+ : \text{self dual part of } (\nabla_{[b} V_{a]})$

To summarise

If \mathcal{M} is a right flat space time with non null conformal killing vector V^a then the factor space H is a Weyl space corresponding to the minitwistor space T obtained by factoring the twistor space by the appropriate vector field. If we make tangent vectors X on H correspond to lie propagated Jacobi fields J_X^a orthogonal to V^a then the Weyl structure ~~obtained~~ obtained from T by the Hitchin correspondence is given by

Conformal metric: X null $\Leftrightarrow g(J_X, J_X) = 0$ (g = metric on \mathcal{M})
affine connection

$$(D_X Y)^b = (\nabla_{J_X} J_Y^c - \gamma_{ad}^c J_X^d J_Y^a) h_c^b$$

where $\gamma_{ad}^c = \frac{1}{2} \{ \delta_a^c \omega_d + \delta_d^c \omega_a - g_{ad} \omega^c \}$

$$\omega_b = \frac{2V^a}{V^2} (\nabla V)_{ba}^+ + \frac{2V_b}{V^2}$$

Thanks to K.P.T. for lots of useful ideas.

References

- [1] Jones "Minitwistors" TN14
- [2] Hitchin "Complex Manifolds and Einstein's equations" - lectures given at Primorsko 1980 (preprint).

Phil Jones

Some examples of Penrose's quasi-local mass construction

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(Communicated by R. Penrose, F.R.S. - Received 14 March 1983)

Penrose's quasi-local mass (Penrose 1982) is calculated for a variety of two-surfaces in particular space-times. The results are compared with other definitions of mass where these are available.

The Generalized Twistor Transform and

Unitary Representations of $SU(p,q)$

Michael Eastwood

Abstract

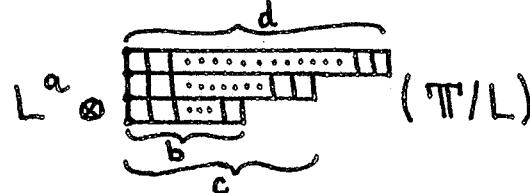
The twistor transform gives rise to the scalar product on massless fields, a natural series of unitary representations of $SU(2,2)$. This construction is generalized so as to apply to $SU(p,q)$ for all p and q . The scalar product is defined on $H^{p-1}(\overline{\mathbb{P}^+}, \mathcal{O}(k))$ for $\mathbb{P}^+ \subset \mathbb{P}_{p+q-1}(\mathbb{C})$. For $p=q=1$ this gives the (mock) discrete series.

The Penrose Transform for Homogeneous Bundles

Complex projective 3-space \mathbb{P} may be regarded as a homogeneous space

$$GL(4, \mathbb{C})/H, \text{ where } H = \left\{ \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix} \in GL(4, \mathbb{C}) \right\}.$$

A finite dimensional representation of H gives rise to a homogeneous vector bundle on \mathbb{P} . Representations of H are determined by restriction to $\mathbb{C}^* \times GL(3, \mathbb{C})$ or further to $U(1) \times U(3)$ (see e.g. [1]) and the irreducible such may be listed by means of the symbols $(a|b, c, d)$ for integers a, b, c, d with $b \leq c \leq d$. For $b \geq 0$ the homogeneous bundle for this symbol has as fibre



over each line $L \in \mathbb{P}$ (for $b < 0$ modify with $(\bigoplus (\mathbb{T}/L))^b$ etc.). The usual Penrose transform interprets $H^1(\mathcal{O}(-n-2))$ as solutions of massless field equations of helicity $n/2$. The line bundle $\mathcal{O}(-n-2)$ is the homogeneous bundle $(n+2|0, 0, 0)$. It is natural to ask for a similar interpretation of $H^1(\mathcal{O}(a|b, c, d))$ in general. Such an interpretation can be given by using the methods described in [2]. For example, $H^1(\mathcal{O}(4|-1, 0, 0))$ is isomorphic to solutions of

$$\nabla^{AA'} \nabla^{BB'} \phi_{A'B'C'} = 0 \quad (\text{physical sig.?})$$

for $\phi_{A'B'C'} \in T(\mathcal{O}_{(A'B'C')})$. Similarly, $H^1(\mathcal{O}(-1|-1, -1, 0))$ gives a potential modulo gauge description (as in [3]) of the "dual" equation (interchange primed and unprimed). Thus, there is a twistor transform and scalar product. In general I conjecture that $\mathbb{D}: \tilde{H}^1(\mathbb{P}^+, \mathcal{O}(E)) \rightarrow \tilde{H}^1(\mathbb{P}^{*-}, \Omega^3(E^*))$ where \tilde{H}^1 denotes "reduced" cohomology (factor out $H^1(\mathbb{P})$ by $H^1(\mathbb{P})$) (cf. [4]).

1. D.E. Littlewood: The theory of group characters and... Clarendon Press 1950
 2. M.G. Eastwood: The generalized Penrose-Ward transform. (abstract, this TN)
 3. N.P. Buchdahl: A generalized deRham sequence. TN 10
 4. M.G. Eastwood: The generalized twistor transform and... (abstract, this TN)
- Many thanks to Nick Buchdahl for useful chat. Michael Eastwood.

Philosophy

A googly structure (photon, graviton) on \mathbb{P} should be equivalent to a leg-break structure on \mathbb{P}^* . I shall start with a leg-break structure on \mathbb{P}^* and attempt to translate it to something on \mathbb{P} .

This approach has the advantage that it is deductive, and is therefore correct (though it may be inconvenient). Its disadvantage is that the natures of the structures that one deduces on \mathbb{P} are not always obvious. However, it is not too much to hope that these will be understood eventually.

Starting points

Throughout this paper the idea will be to describe a leg-break structure on \mathbb{P}^* cohomologically, break it up into smaller cohomological pieces, translate the pieces into cohomological pieces on \mathbb{P} , and then reassemble the translated pieces. I shall consider only the photon explicitly. The constructions given here have analogs (sometimes a bit more mysterious) for the graviton.

The Ward space is a bundle

$$\begin{array}{ccc} \mathcal{Z}^* & & \\ \downarrow & & \\ \mathcal{L}^* & \subset & \mathbb{P}^* \end{array}$$

which is a deformation of a region of \mathbb{T}^* according to

$$\hat{w} = w \exp i e f(w) \quad (*)$$

for two patches U, \hat{U} which cover \mathcal{Z}^* . ($w, \hat{w} \in$ pull-back of U, \hat{U} to \mathbb{T}^* .)

A scalar zrm field of unit charge minimally coupled to the electromagnetic field described by this Ward space is an element of

$$H^1(\mathcal{Z}^*, \mathcal{O}(-1) \otimes \mathcal{I}^*). \quad (**)$$

For short, put $\tilde{\mathcal{O}}(-2) = \mathcal{O}(-1) \otimes \mathcal{I}^*$. This is also meant to emphasize that $\tilde{\mathcal{O}}(-2)$ is to be thought of as a deformation of $\mathcal{O}(-2)$.

Let's analyze the cohomology group $(**)$ with the M-V sequence:

$$\begin{aligned} \tilde{\mathcal{O}}(-2): \quad 0 &\rightarrow H^0(U) \oplus H^0(\hat{U}) \xrightarrow{\psi} H^0(U \cap \hat{U}) \xrightarrow{\delta} H^1(\mathcal{Z}^*) \rightarrow 0 \\ & (\tilde{\alpha}(\tilde{w}), \tilde{\beta}(\tilde{w})) \xrightarrow{k} \tilde{\alpha}(\tilde{w}) - \tilde{\beta}(\tilde{w}) \end{aligned}$$

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I have written \tilde{W} to emphasize that these are abstract W 's living in the patched space appropriate to $\tilde{\mathcal{O}}(-2)$. Really \tilde{W} is the pair (W, \hat{W}) subject to the relation (*) where defined. To make this more explicit, suppose that U, \hat{U} are nice sets so that we can 'trivialize' $\tilde{\mathcal{O}}(-2)$ over them (and hence over their intersection). Then we have

$$H^0(U, \tilde{\mathcal{O}}(-2)) \cong H^0(U, \mathcal{O}(-2)), \quad H^0(\hat{U}, \tilde{\mathcal{O}}(-2)) \cong H^0(\hat{U}, \mathcal{O}(-2))$$

$$\alpha(\tilde{w}) \leftrightarrow \alpha(w) \quad \tilde{\beta}(\tilde{w}) \leftrightarrow \hat{\beta}(\hat{w})$$

$$H^0(U \cap \hat{U}, \tilde{\mathcal{O}}(-2)) \cong H^0(U \cap \hat{U}, \mathcal{O}(-2))$$

$$\tilde{\gamma}(\tilde{w}) \leftrightarrow \gamma(w) = \hat{\gamma}(\hat{w}) \exp ie f(w)$$

which allows us to re-write the M-V sequence as

$$0 \rightarrow H^0(U, \mathcal{O}(-2)) \oplus H^0(\hat{U}, \mathcal{O}(-2)) \rightarrow H^0(U \cap \hat{U}, \mathcal{O}(-2)) \rightarrow \mathbb{Z} \rightarrow 0$$

$$(\alpha(w), \hat{\beta}(w)) \rightsquigarrow \alpha(w) - \hat{\beta}(w) \exp ie f(w)$$

Here \mathbb{Z} stands for the zrm fields and the squiggly arrow is meant to emphasize that the map is not merely the difference after restriction. Of course, this sequence is nothing more or less than writing out the Čech description of the cohomology group representing the zrm fields minimally coupled to the Ward background.

The point of all this is that we can describe the various H^0 's in the above sequence in terms of various cohomological goodies on \mathbb{P} . Consider the case where the zrm field in question is an elementary state. (One has to be a little bit careful about what an elementary state is, in this curved twistor space, in fact, but that is not important just here.) Then we can take $U = \mathbb{P}^* - A$, $\hat{U} = \mathbb{P}^* - B$, where A, B are \mathbb{P}_2 's in \mathbb{P}^* . For our elementary state we take $(A.W B.W)^{-1/2}$. It is important to remember that when we think of the state as an H^1 , we only know where the line AB is, the individual planes A, B being allowed to flap around to express the cohomological freedom; but when we think of the state as an H^0 we know where the particular planes A, B are.

Proposition. $H^0(\mathbb{P}^* - A, \mathcal{O}(-2)) \cong H^2(\mathbb{P} - A, \mathcal{O}(-2)).$

$a \mathbb{P}_2 \text{ in } \mathbb{P}^*$ a point in \mathbb{P}

Sketch of proof. We can get this by doing integrals of the form

$$\begin{aligned} A \xrightarrow{z} L &= z \cdot \frac{\partial}{\partial A} \quad A \xrightarrow{z} M \\ &= z \cdot \frac{\partial}{\partial A} \frac{(ALMN)^2}{A \bar{z} LM \bar{A} \bar{z} MN \bar{A} \bar{z} NL} \end{aligned}$$

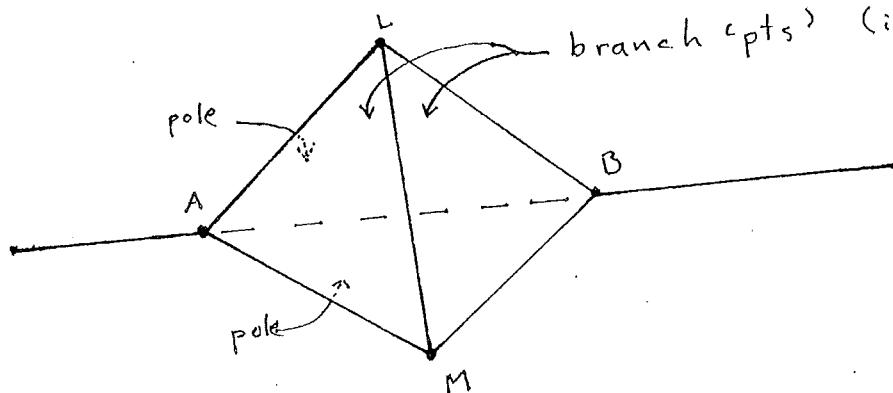
The freedom in choosing L, M, N is the cohomological freedom of the H^2 .

Remark. More generally, there seems to be a correspondence between $H^{n-k-1}(\mathbb{P}^n - \mathbb{P}^k)$ and $H^k(\mathbb{P}^n * -\mathbb{P}^{n-k-1})$.

This proposition enables us to identify the groups we will factor out by. In order to identify the googly version of the elementary state in $H^0(\mathbb{P}^* - A - B)$, we compute

$$\begin{aligned} A \xrightarrow{z} L &= \int_0^\infty dt \quad A + tB \xrightarrow{z} M \\ &= \frac{ABL}{zABL zABM} \log \left(\frac{zALM}{zBLM} \right) + \text{cyc. perm. } \begin{matrix} L \\ N \\ M \end{matrix} \quad (\#*) \end{aligned}$$

$\mathbb{P}:$



Here we are interested in the equivalence class generated by allowing the points L, M, N to move; any fixed set of values for them defines only a particular representative.

Note that this contains the right amount of information: it knows where A, B are, but no more. The cohomology

38.

logical interpretation of this equivalence class is as an element of the dot product

$$H^1(\overline{IP} - \overline{AB}, \mathcal{O}(-2)) \circ H^0(\overline{IP} - \text{cut from } \overline{ALM} \text{ to } \overline{BLM}, \mathcal{O}) \\ \rightarrow H^1(\overline{IP} - \text{cut from } A \text{ to } B \text{ along } \overline{AB}, \mathcal{O}(-2))$$

(thanks to MGE and RP).

Remarks. (1) This cohomological interpretation is a little bit glib since it involves simultaneously taking two equivalence classes: that due to cohomology and that due to the different possible allowed cuts. This needs to be worked out more carefully. (2) In the untwisted case, the map from $H^2(\overline{IP}\text{-cut})$ to the H^1 is a period integral. (3) Suppose that, in addition to the googly field we are constructing, there were a leg-break field present on \overline{IP} . In the above dot product, then, we could replace $H^1(\overline{IP}-\overline{AB})$ by $H^0(\overline{IP}-\overline{ABL}-\overline{ABM})$ modulo $H^0(\overline{IP}-\overline{ABL})$ $H^0(\overline{IP}-\overline{ABM})$ with a leg-break patching relation. Thus expressions of the form (***)^{***}, modulo suitable equivalences, may represent zrm fields coupled to ambidextrous photons.

Space-Time

So far we have seen how to describe background-coupled zrm fields googly. We do not yet know how to evaluate these fields at a given space-time point though. It is important to know how to do this, especially in the case of the graviton, as in that case it is the only way I know of describing the points of the space-time !

A point in space-time describes a linear map from \mathbb{Z} to the complex numbers. This is thus an element of the dual of \mathbb{Z} , which may be analyzed by dualizing individually the spaces in the M-V sequence at the bottom of p.35. At present, this is a bit difficult computationally, but works at least to first order in charge in the electromagnetic case.

— Adam Helfer

Thanks to MGE, RP. Also LPH, TRH for P⁵ which enormously speeds some of the calculations.

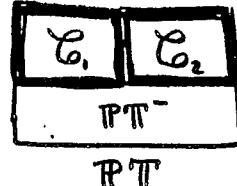
On Closed-Set Coverings for \overline{PT}^+ — A Correction

In a note in TN3, p.-1 (unfortunately also reprinted in Adv. in Twistor Theory, pp. 35, 36) I made the suggestion that the closed-set covering of \overline{PT}^+ given by the 2 closed sets

$$C_1 : |Z^0 + Z^2| \geq |Z^1 + Z^3|, Z^\alpha \bar{Z}_\alpha \geq 0$$

and

$$C_2 : |Z^0 + Z^2| < |Z^1 + Z^3|, Z^\alpha \bar{Z}_\alpha \geq 0$$

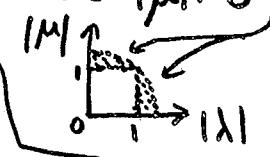


would suffice for all positive-frequency analytic massless wave functions (defined on \overline{CM}^+), claiming that an explicit inverse twistor function construction would show this. However this is actually not so, it being necessary that the field be defined in a somewhat larger region, and the claim, as stated, is false.

Any twistor function defined in a neighbourhood of $C_1 \cap C_2$ would yield a field at any point of CM corresponding to a line in PT which meets $C_1 \cap C_2$ in an S'. It is not hard to see that, for example, the line joining $(1+\lambda, 0, 1-\lambda, 0)$ to $(0, 1+\mu, 0, 1-\mu)$ has this property provided that $|\lambda|^2 + |\mu|^2 \leq 2$, whereas it extends into PT^- whenever $|\lambda| > 1$ or $|\mu| > 0$. So the resulting field extends to a fixed region outside CM^+ .



But it is clear



that not all massless wave functions defined on CM^+ can extend to such a region (e.g. the positive frequency part of a nowhere analytic field on the real compactified space M does not extend outside CM^+ , and by an arbitrarily small conformal transformation, CM^+ can be transformed to CM^+ together with an arbitrarily narrow open "collar" extending into CM^-). No such simple 2-set cover will work generally (and to have a reasonable chance might have to be based on such as the "snail contour" of TN1., p. 22, BDB & RP. — unfortunately not reprinted in A.T.T.). Thanks to Miles Eastwood for his perceptive scepticism and to Alex Pilato for reminding me of this problem. — Roger Penrose

Remarks on Pure Spinors

Let σ be an impure (but reduced) spin space for $SO(2n)$. σ is a complex v.s. of dimension 2^{n-1} . The pure spinors of type σ in $SO(n)$ are a system of null n -planes in the vector rep. These constitute a subvariety π in σ of complex projective dimension $n(n-1)/2$. π is an intersection of quadrics in σ (taken projectively). These quadrics are evidently elements of

$$(\sigma \otimes \sigma)^* = \frac{1}{2} \binom{2^n}{n} \oplus \Lambda^{n-2} \oplus \Lambda^{n-4} \oplus \dots$$

The Λ 's are of course exterior powers of the vector rep; $\frac{1}{2} \binom{2^n}{n}$ is some other rep of that dimension. π is defined by the intersection of the quadrics corresponding to

$$\Lambda^{n-4} \oplus \Lambda^{n-8} \oplus \dots$$

Thus for $n \geq 8$ π is defined by a reducible rep. of $SO(2n)$, and we can also consider the invariantly defined "partial spin spaces" got by looking at the intersections of quadrics corresponding to just some of $\Lambda^{n-4}, \Lambda^{n-8}, \dots$. Thus there is more to the geometry of $SO(2n)$ than pure spinors.

If $SO(2n)$ is a gauge group for a YM theory, then $\frac{1}{2} \binom{2^n}{n}, \Lambda^{n-4}, \Lambda^{n-8}, \dots$ give rise to invariant quadratic couplings. This suggests that perhaps we could use pure spinors to generate mass matrices.

Refs: SBP TN 14, 15, MSc thesis. Thanks to SBP. — Adam Helfer

Twistor Theory and the energy-momentum and angular momentum of the gravitational field at spatial infinity

by

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Penrose's (1982) "Quasi-local mass and angular momentum" is investigated for 2-surfaces near spatial infinity in both linearised theory on Minkowski space and full general relativity. It is shown that for space-times which are radially smooth of order one in the sense of Beig and Schmidt (1982), with asymptotically electric Weyl curvature, there exists a global concept of a twistor space at spatial infinity. Global conservation laws for the energy-momentum and angular momentum are obtained, and the ten conserved quantities are shown to be invariant under asymptotic co-ordinate transformations. The relationship to other definitions is discussed briefly.

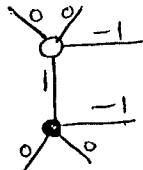
To

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New ideas in twistor diagram theory

A brief resumé of recent developments

Stephen Huggett (D.Phil., 1980) showed the existence of hitherto unknown contours for



These contours have now been identified constructively, and then (as S.A.H. suggested) used to define a previously unrecognised contour for the Möller scattering diagram. This can be evaluated explicitly (for elementary states.) The result shows a dependence upon the twistor functions used, and not only upon the corresponding fields. But this representative-dependence can be considered as residing entirely in the forward direction where the conventional QFT amplitude is infinite. In other directions the result is the "right" answer. So we have a way of making the infinity finite, at the cost of introducing 'states' which contain more information than the space-time fields do.

From the elementary twistor diagram point of view, this new contour is constructed by a process in which only one external line at each vertex is surrounded by an S^1 . The representative-dependence arises from the choice that is made as to which external line, i.e. which singularity region of the twistor function, is so treated. We may ask for a more abstract description of this process, i.e. ask what our amplitude is really a functional of. It is not a functional of the standard H^1 's, but on the other hand it does not contain all the information in the twistor functions. RP suggests that the intermediate object that is really being used can actually be described by relative cohomology.

This is interesting because the same phenomenon appears in the construction of the so-called "hard" contours for the box diagram. It suggests the germ of a theory in which twistor "states" are actually more general than the space-time fields. On the other hand, the position is somewhat confused at present because the discovery of this new contour has also indicated quite another direction for the development of the theory, which suggests a different way of removing the infra-red divergence, and has implications for the treatment of mass.

In this development we make a simple but radical modification to the whole diagram formalism. We change the definition of all the internal lines by an (essentially inhomogeneous) displacement, thus wherever $w.z$ appears, we replace it by

$$t'w.z - k$$

where k is some (pure number) constant. Note that for every 'old' diagram that could be integrated over a (projectively defined) contour, there will exist a (non-projective) contour for the corresponding 'new' diagram, which yields the same result. But the converse is not true. There are strictly more contours in the new, inhomogeneous, spaces. In particular the 'new' Møller scattering diagram has a contour which yields a finite functional of the fields, equal to the conventional QFT amplitude but this time with the infinite amplitude in the forward direction simply subtracted entirely. The vital point here is the inhomogeneity, which is used in an essential way. It means that the twistor contour cannot be regarded as the translation of a space-time integral. Thus we have a different way in which to get something new out of twistor geometry.

If this introduction of inhomogeneity were a trick that applied in this calculation alone, it might be of rather limited interest. However, once we start on this road, further developments rather naturally suggest themselves.

Firstly, we can drop the "boundary" definition of negative-integer diagram lines. We can revert to the logarithmic idea which, historically, came first anyway. Take the example of the spin-1 inner product which in ordinary diagram theory is

$$\oint_{w.z=0} f_{-4}(z^\alpha) g_{-4}(w^\alpha) D^4 w \wedge D^4 z$$

The standard modification, described above, makes this

$$\oint_{t'w.z=k} f_{-4}(z^\alpha) g_{-4}(w^\alpha) D^4 w \wedge D^4 z$$

without affecting the result. But we can replace this, again leaving the result unchanged, by

$$\frac{1}{2\pi i} \oint f_{-k}(z^\alpha) \log(\bar{k}^w z - k) g_{-k}(w_\alpha) D^4 w \wedge D^4 z$$

In this particular case, the boundary and the logarithm are interchangeable. This is because the logarithm is essential to the contour, which means that changing the factor to

$$\log(c(\bar{k}^w z - k)) \text{ where } c \text{ is some constant,}$$

would make no difference. The contour sees only the branch point, i.e. the boundary.

But it is not difficult to write down integrals in which the logarithm is not essential. In this case the logarithm cannot be replaced by a boundary, and the value of the constant c does play a role. Clearly we have a more internally consistent scheme if such integrals are allowed, since then a (-1) -line, like a 0 -line, can be either essential or inessential. If the boundary definition is used, a (-1) -line must always be essential. But how do we fix upon the right constant c ?

There is in fact a natural choice. We start with the definition of the pole factors:

$$w \underline{\wedge^n} z = \frac{1}{2\pi i} \frac{(-1)^n n!}{(\bar{k}^w z - k)^{n+1}}$$

and then extend in the negative direction, preserving the relation

$$\bar{k} \frac{\partial}{\partial w_\alpha} w \underline{\wedge^n} z = z^\alpha w \underline{\wedge^{n+1}} z$$

by

$$w \underline{\wedge^{-n}} z = \frac{1}{2\pi i} \frac{(\bar{k}^w z - k)^{n-1}}{(n-1)!} \left\{ \log(\bar{k}^w z - k) - \gamma \right\} - \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n-1} \right)$$

where γ is Euler's constant, i.e. $\Gamma'(0)$. Clearly it is γ that plays the role of the constant c , the other terms being fixed by the derivative relation. It is determined by the condition

$$w \underline{\wedge^0} z = \sum_{n=-\infty}^{+\infty} w \underline{\wedge^n} z = "0"$$

in the sense of asymptotic series. We can write these factors and relations in many ways, e.g. as

$$w \underline{\wedge^n} z = \int_s \frac{(\bar{k}^w z - k)^{-s-1}}{(-s-1)!} \Big|_{s=n}$$

which then holds for all n , positive and negative.

4.4:

Note that the negative terms sum to the "Euler function":

$$\sum_{w=-\infty}^{-1} w - z = E(k^{-1} w \cdot z - k)$$

where $E(x) \equiv \int_{-\infty}^{\infty} \frac{e^{-w}}{w+x} dw$

and the properties of this function, in connection with the "universal bracket factor", have been noted before. However, the scheme suggested here is essentially different from that proposed by RP (in Advances in Twistor Theory, page 249). The Euler function replaces not the divergent series of poles, but the convergent series of boundary-definitions. The divergence must be handled in some other way.

Now, as suggested on pages 250-1 of A.T.T., we may try to get the Hankel function out of the Euler functions and a mass pole. This indeed constitutes the second natural development of the freedom offered by the new inhomogeneous factors. We find that

$$F(x, z) = \oint f(w_x) g(y_x) \cdot \frac{U(w_x k^{-1} - k) U(y_x k^{-1} - k)}{(x z w_y - m^2/2)}$$

where $U(\)$ represents the "universal" factor, does indeed give us the "right" mass eigenstates out of a compact integral. That is, there is a contour such that the Hankel function arises when f and g are elementary. For this it is necessary that we employ the logarithmic definition of the negative-integer lines and a contour in which those lines are not essential. If we stuck to the old boundary-definitions, then we should still get solutions of the Klein-Gordon equation all right, but not the pure Hankel function solution, i.e. not with the right asymptotic behaviour at infinity. It should also be noted that the contour is such that only a finite number of the "pole" terms in the universal factors can give a non-zero contribution.

We also find that we can replace the mass-shell enforcing denominator by

$$(x z - m_1)(w_y - m_2) \quad (m_1 m_2 = m^2/2)$$

which would seem to give a role to "partial masses" in a new, suggestive way. Another very important point is that now that the mass has been introduced in a geometrical way, it is possible to write down an inner-product formula which seems to work iff the masses coincide. (That is, a blown-up space exists in just this case. See A.T.T. pages 271,272 for an early statement of this general idea.)

Andrew Hodges