

# Twistor Newsletter (no 17: 13, January, 1984)

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## Formal Thickenings of Ambitwistors for Curved Space-time

The aim of this note is to investigate a gravitational analogue of the ambitwistor description (due to Green, Isenberg, Yasskin, Witten, Marin, Henkin, Pool, Buchdahl,...) of Yang-Mills fields.

Yang-Mills case: Let  $\mathcal{A}$  denote the space of null rays in  $\mathbb{C}\mathbb{M}$ . Then

$$\mathcal{A} = F_{1,3}(\mathbb{H}) = \{Z \cdot W = 0\} \subset \mathbb{P} \times \mathbb{P}^*$$

and a point  $x \in \mathbb{C}\mathbb{M}$  corresponds to a quadric  $Q_x = L_x \times L_x^\perp$  in  $\mathcal{A}$ . The Ward correspondence generalizes immediately to this case: if  $U$  is a nice (e.g. convex in a standard coordinate patch) open subset of  $\mathbb{C}\mathbb{M}$  and  $V$  is the region in  $\mathcal{A}$  swept out by  $Q_x$  for  $x \in U$ , then there is a 1-1 correspondence:

$$\left\{ (F, \nabla) : F \text{ is a holomorphic vector bundle on } U \text{ and } \nabla \text{ a holomorphic connection for } F. \right\} \leftrightarrow \left\{ E : E \text{ is a holomorphic vector bundle on } V \text{ trivial on all } Q_x. \right\}$$

One can ask: what is the condition, in terms of  $E$ , for  $\nabla$  to satisfy the Yang-Mills equations  $\nabla^* \nabla^2 = 0$ ? The answer is in terms of trying to extend  $E$  to formal neighbourhoods of  $\mathcal{A}$  in  $\mathbb{P} \times \mathbb{P}^*$ . The cohomological machinery for tackling this extension problem is described in TN12 (M.G.E.: non-Abelian sheaf cohomology): if  $E$  has been extended as far as  $V_{(n-1)}$  then the obstruction to extension to  $V_{(n)}$  lies in  $H^2(V, \Omega^*(End E)(-n, -n))$  and  $H^1(V, \Omega^*(End E)(-n, -n))$  acts freely on the choice of extension. This cohomology can be interpreted by the Penrose transform in terms of fields on  $U$ :

Extension to  $V_{(1)}$ :  $H^2(V, \Omega^*(End E)(-1, -1)) = 0$  so  $E$  extends.  $H^1(V, \Omega^*(End E)(-1, -1)) = T(U, \Omega^*(End F)[-1][-1]')$  so if  $E_{(1)}$  is any choice of extension then  $T(U, \Omega^*(End F)[-1][-1]')$  acts on this to produce all possible extensions.

Extension to  $V_{(2)}$ :  $H^2(V, \Omega^*(End E)(-2, -2)) = T(U, \Omega^*(End F)[-1][-1]')$  and claim (rather tricky to prove) that as  $T(U, \Omega^*(End F)[-1][-1]')$  acts on  $E_{(1)}$  the corresponding obstruction in  $T(U, \Omega^*(End F)[-1][-1]')$  just moves under the additive action. In particular, there is exactly one choice of  $E_{(1)}$  which admits an extension  $E_{(2)}$  to  $V_{(2)}$ . Since  $H^1(V, \Omega^*(End E)(-2, -2)) = 0$ , this is unique.

Extension to  $V_{(3)}$ :  $H^1(V, \Omega(\text{End } E)(-3, -3)) = 0$  so if an extension exists it is unique. The obstruction to extension to third order lies in  $H^2(V, \Omega(\text{End } E)(-3, -3)) = \ker \nabla: T(U, \Omega^3(\text{End } F)) \rightarrow T(V, \Omega^4(\text{End } F))$  and turns out to be the current  $\nabla(*\nabla^2)$ . Thus, Yang-Mills fields on  $U$  are in 1-1 correspondence with vector bundles on  $V_{(3)}$  which are trivial on all quadrics  $Q_x$  in  $V$ . All this is well-known.

Gravitational case: Based on the analogy between the twistor description of self-dual Yang-Mills versus self-dual Einstein and on other evidence too, one might expect a gravitational analogue of the above construction. More precisely, suppose  $M$  is a holomorphic manifold with conformal structure (and that  $M$  is geodesically convex for some metric in the conformal class (i.e. shrink  $M$  if necessary)) and let  $A$  be the space of null geodesics of  $M$ . Then  $A$  is a 5-dimensional complex manifold and  $M$  can be reconstructed from it as explained by C.R. LeB. in TIN 9+10, D.Phil thesis (1980), and Trans. A.M.S. 278 (1983), 209-231. It is now natural to ask about formal thickenings of  $A$ , hoping to encode further information on  $M$ .

In TIN 13 and also Letts. Math. Phys. 6 (1982), 345-354 C.R. LeB. proposed an elegant and natural definition of  $A_{(1)}$ , a first order formal thickening of  $A$ . Roughly speaking, the first order information of an embedding  $A \hookrightarrow B$  is the exact sequence on  $A$

$$0 \longrightarrow \mathbb{H} \longrightarrow \hat{\mathbb{H}} \longrightarrow N \longrightarrow 0$$

where  $\mathbb{H}$  is the tangent bundle of  $A$ ,  $\hat{\mathbb{H}}$  is the tangent bundle of  $B$ , and  $N$  the normal bundle of  $A$  in  $B$ . There is, essentially by definition, a natural interpretation of a tangent vector on  $A$  as a Jacobi field on  $M$ . C.R. LeB. observes that a slight weakening of the Jacobi equations allows a 6-dimensional vector space of solutions rather than just 5 for each null geodesic. This is his definition of  $\hat{\mathbb{H}}$  in the curved case and he presents plenty of reasons as to why it is the "correct" definition.

Since the construction of the extended tangent bundle  $\hat{\mathbb{H}}$  is conformally invariant, i.e. depends only on the conformal structure of  $M$ , there should be a method of construction intrinsic to  $A$ . I propose the following as such a method (it feels right and works in the flat case). In his original investigations of  $A$  C.R. LeB. noted that  $A$  carries a natural "universal" line bundle  $L$  generalizing  $\mathcal{O}(-1, -1)$  for  $A$ . It is conformally invariant and has many nice properties such as  $L^3 = \Omega^5$  (the canonical bundle). For C.R. LeB.'s definition of  $\hat{\mathbb{H}}$  via Jacobi fields,  $N = L^*$ . Thus,

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$\hat{\mathbb{H}}$  is a distinguished element of  $\text{Ext}(A, L^*, \mathbb{H}) = H^1(A, \mathbb{H} \otimes L)$ . Unfortunately, the Penrose transform of this group is rather a mess and it is difficult to find this distinguished element directly. Fortunately, the Penrose transform also gives

$$H^0(A, \Omega^2 \mathbb{H} \otimes L) = \ker d: T(M, \mathcal{O}) \rightarrow T(M, \Omega^2) = \mathbb{C}$$

and so  $1 \in \mathbb{C}$  gives a distinguished  $w \in H^0(A, \Omega^2 \mathbb{H} \otimes L)$ . Now use  $w$  as a homomorphism  $\Omega^1 \otimes L^* \rightarrow \mathbb{H}$  to push out the jet exact sequence for  $L^*$ , i.e.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega^1 \otimes L^* & \longrightarrow & J^1(L^*) & \longrightarrow & L^* \longrightarrow 0 \\ & & \downarrow w & & \downarrow & & \parallel \\ 0 & \longrightarrow & \mathbb{H} & \longrightarrow & \hat{\mathbb{H}} & \longrightarrow & L^* \longrightarrow 0 \end{array} \quad \text{where } \hat{\mathbb{H}} = \frac{\mathbb{H} \oplus J^1(L^*)}{\Omega^1 \otimes L^*}$$

It is possible to be more systematic as to the problem of constructing formal thickenings. C.R. LeB has developed an alternative method (private communications) which gives similar results. The machinery described here is inspired by the corresponding supersymmetric problem (see M.G.E. in TIN 15). If  $A$  is embedded as a closed submanifold of  $B$  with ideal sheaf  $\mathcal{I}$  then the  $n^{\text{th}}$  order formal neighbourhood of  $A$  in  $B$  is the ringed space  $(A, \mathcal{O}_{(n)})$  where  $\mathcal{O}_{(n)} = \mathcal{O}_B/\mathcal{I}^{n+1}$ . If  $A$  has codimension 1 in  $B$  then locally  $\mathcal{O}_{(n)} \cong \mathcal{O}[t]/t^{n+1}$ . Thus it is natural to make the following:

Definition. An  $n^{\text{th}}$  order formal thickening of  $A$  is a ringed space  $(A, \mathcal{O}_{(n)})$  with an augmentation  $\mathcal{O}_{(n)} \rightarrow \mathcal{O}$  (a homomorphism of  $\mathbb{C}$ -algebras) such that locally  $\mathcal{O}_{(n)} \cong \mathcal{O}[t]/t^{n+1}$  with the obvious augmentation (setting  $t=0$ ).

Hence, a formal thickening is a consistent way of patching  $\mathcal{O}[t]/t^{n+1}$  or, in other words, an element of the cohomology set  $H^1(A, \mathcal{A}_n)$  where  $\mathcal{A}_n = \text{Aut}(\mathcal{O}[t]/t^{n+1})$  the (non-Abelian) group of automorphisms of  $\mathcal{O}[t]/t^{n+1}$  as an augmented  $\mathbb{C}$ -algebra over  $\mathcal{O}$  (i.e. the sheaf thereof). To understand formal thickenings one must therefore understand  $\mathcal{A}_n$ :

Lemma. As an  $\mathcal{O}$ -module  $\mathcal{O}[t]/t^2 \ni a + bt \xrightarrow{\sim} \begin{pmatrix} a \\ b \end{pmatrix} \in \begin{pmatrix} \mathcal{O} \\ \mathcal{O} \end{pmatrix}$ .

Then elements of  $\mathcal{A}_n$  can be represented as matrices  $\begin{bmatrix} 1 & 0 \\ X & g \end{bmatrix}$  where  $g \in \mathcal{O}^*$

$X \in \mathbb{H}$ .

Proof. Since elements of  $\mathcal{A}_n$  are required to preserve the augmentation the first row must be  $(1, 0)$ . If  $a, c \in \mathcal{O}[t]/t^2$  then, since elements of  $\mathcal{A}_n$  are required to preserve multiplication, must have  $ac + X(ac)t = (a + X(a)t)(c + X(c)t)$  i.e.  $X(ac) = aX(c) + cX(a)$  i.e. a derivation i.e.  $X \in \mathbb{H}$ . Similar reasoning for  $bt, dt \in \mathcal{O}[t]/t^2$  shows that the remaining matrix entry must be multiplication by  $g$ .

Finally,  $g$  must be nowhere vanishing so that provides an inverse.  $\square$

$$\begin{bmatrix} 1 & 0 \\ -g^{-1}X & g^{-1} \end{bmatrix}$$

It follows from this lemma that

①  $\mathbb{H}$  is a normal subgroup of  $\mathcal{A}_1$ , and there is an exact sequence  $0 \rightarrow \mathbb{H} \rightarrow \mathcal{A}_1 \rightarrow \mathcal{O}^* \rightarrow 0$ . \*

② The action of  $\mathcal{O}^*$  on  $\mathbb{H}$  by conjugation is  $g \cdot X = gX$ .

③ The sequence \* admits a right splitting  $\begin{bmatrix} 1 & 0 \\ 0 & g \end{bmatrix} \hookrightarrow g$ .

In other words,  $\mathcal{A}_1$  is a semi-direct product  $\mathbb{H} \times \mathcal{O}^*$ .

The machinery of TN12 is designed to deal with ① & ②. By combining this with ③, one can conclude that:

Proposition.  $H^1(A, \mathcal{A}_1) \rightarrow H^1(A, \mathcal{O}^*)$  is surjective and, for each  $L \in H^1(A, \mathcal{O}^*)$ , those formal thickenings which map to  $L$  can be identified with  $H^1(A, \mathbb{H} \otimes L)$ .  $\square$

This agrees with the earlier notion of formal thickenings in terms of the extended tangent bundle  $\hat{\mathbb{H}}$ . As a line bundle  $L$  is the conormal bundle of  $A$  in  $A_{(1)}$ . To create  $\hat{\mathbb{H}}$  directly from  $F \in H^1(A, \mathcal{A}_1)$  let  $\mathcal{A}_1$  act on the sheaf  $\mathbb{H} \otimes \mathcal{O}$  by

$$\begin{bmatrix} 1 & 0 \\ X & g \end{bmatrix} \cdot (P, k) = (P, k) \begin{bmatrix} 1 & 0 \\ X & g \end{bmatrix}^{-1} = (P, k) \begin{bmatrix} 1 & 0 \\ -g^{-1}X & g^{-1} \end{bmatrix}.$$

Then  $F \in H^1(A, \mathcal{A}_1)$  creates a new locally free sheaf  $F(\mathbb{H} \otimes \mathcal{O})$  (as explained in TN12) and this is easily seen to be  $\hat{\mathbb{H}}$ .

This approach easily extends to  $n > 1$ :

Lemma. ① For  $n > 1$  there is a surjective homomorphism  $\mathcal{A}_n \rightarrow \mathcal{A}_{n-1}$  with Abelian kernel isomorphic to  $\mathbb{H} \otimes \mathcal{O}$ , i.e. an exact sequence

$$0 \rightarrow \mathbb{H} \otimes \mathcal{O} \rightarrow \mathcal{A}_n \rightarrow \mathcal{A}_{n-1} \rightarrow 0. \quad \ast\ast$$

② The conjugate action of  $\mathcal{A}_{n-1}$  on  $\mathbb{H} \otimes \mathcal{O}$  descends to an action of  $\mathcal{A}_1$ , namely  $\begin{bmatrix} 1 & 0 \\ X & g \end{bmatrix} \cdot (P, k) = g^n (P, k) \begin{bmatrix} 1 & 0 \\ X & g \end{bmatrix}^{-1}$ .

Proof. It suffices to identify  $\mathcal{A}_n$  as matrices as in the previous lemma. Think of  $\mathcal{O}[t]/t^{n+1}$  as column vectors.

$$\underline{n=2} \quad X \in \mathcal{A}_2 \Leftrightarrow X = \begin{bmatrix} 1 & 0 & 0 \\ X & g & 0 \\ \frac{X^2}{2} + Y & gX + h & g^2 \end{bmatrix} \quad \text{where } g \in \mathcal{O}^* \quad X, Y \in \mathbb{H} \quad h \in \mathcal{O}$$

$$\text{and, moreover, } X \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ P & k & 1 \end{bmatrix} X^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ g^2 P - gkX & gk & 1 \end{bmatrix}$$

which verifies the lemma for  $n=2$ .

$$\underline{n=3} \quad X \in \mathcal{A}_3 \Leftrightarrow X = \begin{bmatrix} 1 & 0 & 0 & 0 \\ X & g & 0 & 0 \\ \frac{X^2}{2} + Y & gX + h & g^2 & 0 \\ \frac{X^3}{6} + XY + Z & g\frac{X^2}{2} + gY + hX + j & g^2X + 2gh & g^3 \end{bmatrix} \quad \text{etc.}$$

$X, Y, Z \in \mathbb{H}$   
 $h, j \in \mathcal{O}$

Unlike  $*$ , the exact sequence  $**$  does not have a right splitting. From TIN12, this lemma, and the manufacture of  $L$  and  $\hat{\Theta}$  from  $H^1(A, \mathcal{A}_1)$  as above, it follows that:

Theorem. For  $F \in H^1(A, \mathcal{A}_1)$  and  $n > 1$  suppose  $F$  has been thickened as far as  $F_{(n-1)} \in H^1(A, \mathcal{A}_{n-1})$ . Let  $L$  and  $\hat{\Theta}$  denote the conformal bundle and thickened tangent bundle on  $A$  constructed from  $F$ . Then the obstruction to extending  $F_{(n-1)}$  to  $H^1(A, \mathcal{A}_n)$  lies in  $H^2(A, \hat{\Theta} \otimes L^n)$  and if such an extension exists then  $H^1(A, \hat{\Theta} \otimes L^n)$  acts freely on the possible choices.  $\square$

As for the Yang-Mills case one should interpret these cohomology groups on  $M$  by the Penrose transform:

n=2  $H^2(A, \hat{\Theta} \otimes L^2) = 0$  so  $A_{(1)}$  always admits thickenings to second order. To identify  $H^1(A, \hat{\Theta} \otimes L^2)$  think of the conformal metric on  $M$  as a metric with values in a line bundle  $g_{ab} : (\mathcal{O}^{(ab)}) \rightarrow \mathcal{F}$  (in the flat case  $\mathcal{F} = \mathcal{O}[1][1]'$ ). Then  $H^1(A, \hat{\Theta} \otimes L^2) = T(M, 2\mathcal{F}^*)$ , roughly speaking two conformal factors since if  $s \in T(M, \mathcal{F}^*)$  then  $sg_{ab}$  is a genuine (where  $s \neq 0$ ) metric in the conformal class of  $g_{ab}$ .

n=3  $H^1(A, \hat{\Theta} \otimes L^2) = 0$  so if thickenings exist they are unique (this happens for higher order too) but there is the possibility of genuine obstructions: there are exact sequences

$$\begin{aligned} 0 \rightarrow K \rightarrow H^2(A, \hat{\Theta} \otimes L^3) \rightarrow T(M, 2\mathcal{F}^*) \rightarrow 0 \\ 0 \rightarrow T(M, 2\Omega^4) \rightarrow K \rightarrow T(M, \Omega^2) \rightarrow 0 \end{aligned}$$

so that the obstruction comprises primary, secondary and tertiary pieces as follows: First there is a primary obstruction in  $T(M, 2\mathcal{F}^*)$ . By analogy with the Yang-Mills case I would anticipate that the freedom in choosing a second thickening shows up precisely in this primary obstruction. In other words:

Conjecture.  $A_{(1)}$  admits a canonical thickening  $A_{(2)}$  characterized by the vanishing of the primary obstruction to third order thickening. Furthermore, vector bundles  $E$  on  $A$ , trivial on all quadrics  $Q_x$ , admit unique extensions  $E_{(2)}$  to  $A_{(2)}$ .  $\square$

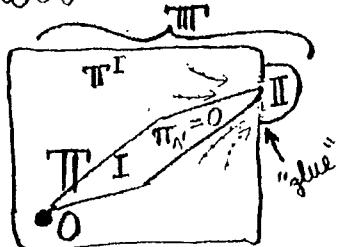
Things now get a little more speculative. Having eliminated the primary obstruction, there is a secondary obstruction, namely a two-form on  $M$ . Surely (B.P.J. & K.P.T.) this must vanish (being conformally invariant). The final tertiary obstruction is two four-forms. A reasonable guess is  $\text{tr}(C^+ \wedge C^+)$  and  $\text{tr}(C^- \wedge C^-)$  where  $C^+$  (resp.  $C^-$ ) is the self-dual (resp. anti-self-dual) Weyl curvature. So maybe:

Conjecture.  $A_{(2)}$  admits a thickening to  $A_{(3)}$  iff  $C^+ \wedge C^-$  have equianharmonic principal null directions i.e. at the vertices of an infinite regular hyperbolic tetrahedron.  $\square$

The primary obstruction to fourth order thickening is two closed 3-forms  $? ? ? ? ? ?$

## Relative Cohomology, Googlies and Deformations of $\mathbb{I}$

In TN 16, pp 9, 10, I suggested that a possible home for the gravitational "googly" information might lie in the differing (allowable) ways that  $\mathbb{I}$  (= the non-projective blown-up line  $\mathbf{I}$ ) can be attached ("glued") to  $\mathbb{T}$  (or to  $\mathbb{T}^* = \mathbb{T} - \mathbf{I}$  — it makes little difference). (Note:  $\mathbf{I} = \{(\omega^A, \pi_{A'}) \mid \pi_{A'} = 0\}$ .) I assume that the twistor space  $\mathbb{T}$  is flat, i.e. that the associated space-time is self-dual, with the usual handedness conventions ( $\Psi_{ABCD} = 0$ ,  $\tilde{\Psi}_{ABC'D'} \neq 0$ ).



Recall that  $\mathbb{I}$  is given by the finite limits of

$$(\omega^A \pi_{B'}, \pi_A, \pi_{B'})$$

as  $\pi_{A'} \rightarrow 0$ ,  $\omega^A \rightarrow \infty$ . We expect that the allowable deformations of the gluing

of  $\mathbb{I}$  to  $\mathbb{T}$  will be alterations in the holomorphic structure of  $\mathbb{III} = \mathbb{T} \cup \mathbb{I}$  (not affecting which points of  $\mathbb{I}$  are topological limits of <sup>which</sup> point sequences in  $\mathbb{T}$ ). An infinitesimal allowable deformation of this gluing would be expected to show up as a relative cohomology group (relative to  $\mathbb{I}$ ) — in fact an  $H_{\mathbb{I}}^2$  of holomorphic vector fields in  $\mathbb{III}$  which are tangent to  $\mathbb{I}$  at  $\mathbb{I}$ . I shall address the problem of how such an  $H_{\mathbb{I}}^2$  might be obtainable from an  $H^1(\dots, \mathcal{O}(-6))$  — which represents a linearized self-dual gravitational field in the now standard way.

Consider the exact sequence

$$0 \rightarrow \widehat{d\Omega^2} \xrightarrow{i} \widehat{\Omega^3} \xrightarrow{d} \Omega^4 \rightarrow 0 \quad (1)$$

where  $\widehat{\Omega^3}$  denotes the sheaf of holomorphic 3-forms on  $\mathbb{III}$  which, on  $\mathbb{T}$ , are proportional to  $DZ = \frac{1}{6} \sum dz_i d\bar{z}_j dz_k$ , those which are closed being denoted  $\widehat{d\Omega^2}$ ; and where  $\Omega^4$  denotes holomorphic 4-forms on  $\mathbb{III}$ . Recall that  $\pi_A, \pi_B, DZ$  is finite on  $\mathbb{III}$  and

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$$d(\pi_A, \pi_B, f \circ \varphi) = \pi_A, \pi_B, d^4Z (T+6)f \quad (2)$$

( $T$  being the Euler operator and  $d^4Z = \frac{1}{24} d\zeta_1 d\zeta_2 d\zeta_3 d\zeta_4$ ). From (1),

$$\rightarrow H^1(2, \widehat{\Omega^3}) \rightarrow H^1(2, \Omega^4) \xrightarrow{\iota} H^2(2, d\widehat{\Omega^2}) \rightarrow \dots \quad (3)$$

is exact,  $2$  being some suitable neighbourhood of  $\mathbb{I}$  (or perhaps part of  $\mathbb{I}$ ) in  $\mathbb{T}$ . Note that the map  $\iota$  does indeed involve an integration of quantities of the form  $g d^4Z$  (think of what the connecting homomorphism actually means) — and for regularity at  $\mathbb{I}$  we must have two  $\pi_i$ 's in  $g$ ; so what is involved is the inverse operation in (2), i.e. " $(T+6)^{-1}$ ", on quantities of that form for which this operation does not globally exist; i.e. degree -6 functions. For the other homogeneity degrees in  $H^1(2, \Omega^4)$ , these parts come from the left in (3), whereas the -6's go to the right. (Note:  $(T+6)^{-1}$  acting on a -6 function is that function  $\times$  a logarithm. The logarithm would need to extend to  $\mathbb{I}$  which is why the  $H^2$  comes in instead here.)

Now from the relative cohomology sequence we have

$$\rightarrow H^1(2-\mathbb{I}, d\widehat{\Omega^2}) \rightarrow H_{\mathbb{I}}^2(2, d\widehat{\Omega^2}) \rightarrow H^2(2, d\widehat{\Omega^2}) \rightarrow \dots$$

Again it is -6 functions that are relevant here  $\uparrow$ , since these provide  $d\widehat{\Omega^2}$  finite on  $\mathbb{I}$ , with zero on the right in (2) for closedness. The idea is that the appropriate part of  $H_{\mathbb{I}}^2(2, d\widehat{\Omega^2})$  should be  $H^2(2, d\widehat{\Omega^2})$  which by (3) is provided by -6 homogeneity twistor functions ( $H^1$ 's). (It thus appears that the required infinitesimal deformations can be provided by -6 functions  $\times$  logarithms.) Note that the 3-forms in  $\widehat{\Omega^3}$  (and in particular  $d\widehat{\Omega^2}$ ) "point" in the direction of  $T$ , which is tangent to  $\mathbb{I}$  at  $\mathbb{I}$ , as required. (We can divide a 3-form by a 4-form — canonically  $d^4Z$ , in  $\mathbb{T}$  — to get a vector field!)

All this clearly needs much sharpening up. Work is in progress. The required deformed gluings of  $\mathbb{I}$  to  $\mathbb{T}$  are supposed to determine which are the correct googly maps by a regularity condition at  $\mathbb{I}$ . A possible line of approach for obtaining insights is to examine how twistor functions (perhaps of the dual Hefer variety) can be used to yield test fields on the googly space-time via contour integrals involving these googly maps. There appears to be some interesting topology. (More work in progress.)

Thanks to T. N. B. especially. — Roger Penrose

A Note On The Spurting 3-form or The Hamiltonian at G.R.

The purpose of this note is to show

how the Spurting 3-form,  $\bar{\Gamma} = i \partial \bar{\Pi}_{A^1 A^2} \partial \bar{\Pi}_{A^3 A^4} dX^{AA'}$

can be interpreted as the hamiltonian density of  
G.R..

In the canonical formalism of G.R. we take the space of the det. 3-metrics  $g_{ij}$  on a hypersurface  $\Sigma$  to be the configuration space and the corresponding canonically conjugate momenta are given by symmetric tensor densities  $\Pi_{ij} = \sqrt{g} (g_{ij} \text{tr} K - K_{ij})$  where  $K_{ij}$  is the extrinsic curvature. ( $i, j = 1, 2, 3$ ).

The hamiltonian density is usually given as:

$$H_{\text{total}} = H_N + H_i N^i$$

Where  $N$  is the lapse function,  $N^i$  the shift vector field,  $H$  is the superhamiltonian,  $H_i$  is the supermomentum.  $\partial L = G_{\perp\perp} \delta g$  and  $H_i = G_{\perp i} \delta g$

where  $G_{\mu\nu}$  is the Einstein tensor and  $\perp$  denotes the direction orthogonal to the given hypersurface. (1)

If  $\Sigma$  is asymptotically flat we cannot just use  $\int_{\Sigma} H_{\text{total}} d^3x$  as the hamiltonian. The variation

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of  $H$  must not contain surface integrals. So we must add a surface integral to  $\int g H_{\text{total}} d^3x$ :

When  $N$  and  $N_i$  are asymptotically constant the surface integral yields  $P_0 N + P_i N_i$ , where  $(P_0, P_i)$  is the A.O.M momentum (2).

If we take null lapse & shift, (ie  $N^2 = N^i N_i$ ) then we can write  $(N, N_i) = \psi^A \bar{\psi}^{A'}$  for some spinor field  $\psi^A$ , and the surface integral can be written (see (3)):

$$\oint_{\Sigma} i \psi_A \nabla_{\mu} \bar{\psi}_A dx^{\mu} dx^{A'}$$

So our expression for the hamiltonian  $H$  can be written:  $H = \int_{\Sigma} \bar{\psi}^A \psi^{A'} G_{AA'B} d\Sigma^B + \oint_{\Sigma} i \psi_A \nabla_{\mu} \bar{\psi}_A dx^{\mu} dx^{A'}$

However in (4) we are given the equation:

$$\Gamma = i D\bar{\psi}_A \wedge D\bar{\psi}_A dx^{A'} = \pi^{A'} \bar{\pi}^A G_{AA'B} d\Sigma^B + d(i\bar{\pi}_A D\bar{\psi}_A dx^{A'})$$

$$\text{where } d\Sigma^B = \epsilon^{\mu\nu\sigma} dx^{\mu} dx^{\nu} dx^{\sigma}$$

So we see that if we integrate  $\Gamma$  over the  
given by  
section of the spin bundle  $\lambda \psi_A$  over  $\Sigma$ , we  
will obtain the correct hamiltonian with the surface  
integral automatically incorporated.

Remarks: (a)  $H$  is still regarded as a functional  
of  $g_{ij}, \pi_{ij}, N_i$ , and  $N$ . However when dealing with

Spinors it is convenient to extend the phase space by using the set of orthonormal tetrads on  $\Sigma$  as configuration space; in this formalism we must use 6 extra constraints and add an extra term to the hamiltonian (which is constrained to vanish).

The interesting question is whether we can introduce twistorial coordinates on the gravitational phase space using the fermion metric, and treat hamilton's equations for G.R. twistorially.

(b) Related descriptions of the gravitational hamiltonian have been found by A.Ashikhin (G.R.G 10 seminar < 5>) and Nestor (6). The idea is implicit in (7).

Lionel Mason.

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### TWISTORS AND CAUSAL RELATIONS IN MINKOWSKI SPACE

Recall that in  $M$  (which here means non-compactified Minkowski space) causal relations are particularly easy to understand. As a matter of convenience, we make the canonical identification of  $M$  with  $T_0 M$ , the tangent space to  $M$  at the origin, a point  $x \in M$  will be regarded as indistinguishable from its position vector, and we will write

$$x = x^a = x^{AA'}$$

when  $x$  is to the future of the origin - written  $x \in I^+(0)$  (see ref. 1 for notation) iff  $x^a$  is future-pointing and timelike. In order to make this more amenable to a twistor-theoretic approach, we observe that this is equivalent to saying

$$x^a p_a > 0$$

for any future pointing null vector  $p^a$ , or, in spinor notation,

$$x^{AA'} \pi_{A'} \bar{\pi}_A > 0$$

for any  $\pi_{A'} \in \mathbb{C}^2 \setminus \{(0,0)\}$ .

More generally, we can see that if  $x^a$  and  $y^a$  are two points in  $M$ , then

$$(x^a - y^a) p_a > 0 \quad \forall \text{ future pointing null } p^a$$

iff  $x \in I^+(y)$ , with strict inequality if  $x \in I^+(y)$ .

Furthermore,  $(x^a - y^a) p_a$  takes on positive, zero and negative values as  $p_a$  varies over the set of future-

pointing null vectors iff the interval between  $x$  and  $y$  is spacelike, and finally we get  $(x^a - y^a) p_a \leq 0$  for all future pointing null  $p^a$  iff  $x \in J^-(y)$ , etc.

These relations can be interpreted in twistor space in the following two ways (which will only be sketched here).

### 1) Algebraic Version

Consider  $\mathbb{P}N \subset \mathbb{P}\mathbb{T}^*$  as the space of null geodesics in  $M^\#$  (compactified Minkowski space), and let  $\mathbb{P}N^I$  be  $\mathbb{P}N \setminus I$ , where  $I$  represents the light cone at infinity, so  $I = \{(\omega^A, 0) : \omega^A \in \mathbb{C}^2 \setminus \{0, 0\}\}$  in homogeneous coordinates.

Now pick  $x \in M$ , and let  $X^{\alpha\beta}$  be the line in  $\mathbb{P}N^I$  corresponding to  $x$ , and define

$$\Xi^\alpha{}_\beta(x) := -i X^\alpha I_{\beta\beta}$$

Then if  $Z^\alpha = (iy^{AA'}\pi_A, \pi_B)$  for some  $y \in M$   $\Rightarrow Z^\alpha \in \mathbb{P}N$ . It is easy to show that

$$\bar{Z}_\alpha Z^\beta \Xi^\alpha{}_\beta(x) = (x^{AA'} - y^{AA'}) \pi_A \bar{\pi}_B = (x^a - y^a) p_a$$

We thus have a division of  $\mathbb{P}N$  into

$$\mathbb{P}N_{<+} := \{Z^\alpha \in \mathbb{P}N : \bar{Z}_\alpha Z^\beta \Xi^\alpha{}_\beta(x) > 0\}$$

$$\mathbb{P}N_{=0} := \{Z^\alpha \in \mathbb{P}N : \bar{Z}_\alpha Z^\beta \Xi^\alpha{}_\beta(x) = 0\}$$

$$\mathbb{P}N_{>} := \{Z^\alpha \in \mathbb{P}N : \bar{Z}_\alpha Z^\beta \Xi^\alpha{}_\beta(x) < 0\}$$

and  $\sim$  if  $L_y$  is a twistor line in  $\mathbb{P}N^I$  corresponding to  $y \in M$ , we see that

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$$L_y \subset PN_{x^c}^{\ell} \Leftrightarrow y \in I^+(x)$$

$$L_y \subset PN_{x^c}^{\ell} \cup PN_{x^c}^o \Leftrightarrow y \in J^+(x)$$

$L_y$  intersects both  $PN_{x^c}^{\ell}$  and  $PN_{x^c}^o \Leftrightarrow$  the interval between  $x^c$  and  $y$  is spacelike.

In slightly more generality, if  $x \in \text{CM}$ , we can form  $\Psi^{\alpha\beta}(x)$  as before, but now if  $Z^a = (iy^{nn'}, \pi_{n'})$  for some  $y \in \text{CM}$ , we get

$$\bar{Z}_a Z^b \Psi^{\alpha\beta}(x) = (\text{Re}(x^{nn'}) - \text{Re}(y^{nn'})) \pi_{n'} \bar{\pi}_{n'}$$

and so, in an obvious notation, we get  $P\mathbb{T}_{x^c}^{\ell}$ ,  $P\mathbb{T}_{x^c}^o$  and  $P\mathbb{T}_{x^c}^P$ , with  $PN_{x^c}^{\ell, o, P} = PN \cap P\mathbb{T}_{x^c}^{\ell, o, P}$ ,

Note that we only get information about the separation of the real parts of two points of CM.

## 2) Geometric Version

In this section, for simplicity we take  $x^c$  to be the origin, 0,  $\sim x = L_0 = \{(0, \pi_n) : \pi_n \in \mathbb{C}^2 \setminus 0\}$ . (If necessary, we perform a Poincaré transformation taking  $x^c$  to 0.)  $I$  is still  $\{(\omega^a, 0) : \omega^a \in \mathbb{C}^2 \setminus 0\}$ .

Now, consider the transformation induced on  $P\mathbb{T}^I$  by the linear transformation  $T$  acting on  $\mathbb{T}^I$  (ie the homogeneous coordinates) which has the matrix

$$\begin{bmatrix} i & & 0 \\ & i & \\ 0 & & 1 \end{bmatrix}$$

This transformation is of order 4 - i.e  $T^4 = \mathbb{1}$ , and it leaves  $L_0$  and  $I$  pointwise fixed, and so can be regarded as a rotation by  $\pi/2$  about the axes  $L_0$  and  $I$ .

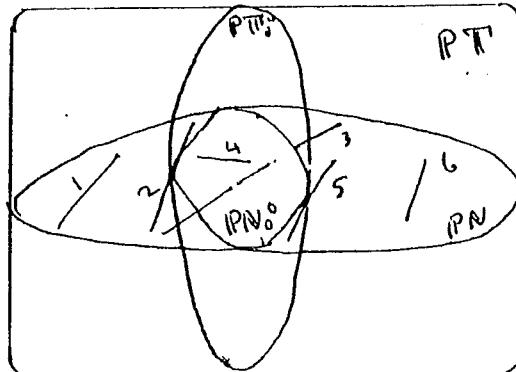
The point of this is that under  $T$ , the future tube,  $\mathbb{P}\mathbb{T}^+$ , maps to  $\mathbb{P}\mathbb{T}_0^+$ ,  $\mathbb{P}N$  maps to  $\mathbb{P}\mathbb{T}_0^0$  and the past tube,  $\mathbb{P}\mathbb{T}^-$ , maps to  $\mathbb{P}\mathbb{T}_0^-$ , thus giving us the geometry of the regions  $\mathbb{P}\mathbb{T}_0^+$ ,  $\mathbb{P}\mathbb{T}_0^0$  and  $\mathbb{P}\mathbb{T}_0^-$ . Using this, we can prove the following

Proposition. Let  $L_z$  be a twistor line in  $\mathbb{P}\mathbb{T}$ , corresponding to a point  $z \in \mathbb{CM}$ . Then we have the following possibilities:

- 1)  $L_z \subset \mathbb{P}\mathbb{T}_0^+$
- 2)  $L_z \subset \mathbb{P}\mathbb{T}_0^+ \cup \mathbb{P}\mathbb{T}_0^0$  and  $L_z \cap \mathbb{P}\mathbb{T}_0^0$  is a singleton
- 3)  $L_z$  intersects each of  $\mathbb{P}\mathbb{T}_0^+$ ,  $\mathbb{P}\mathbb{T}_0^0$ ,  $\mathbb{P}\mathbb{T}_0^-$ , and  $L_z \cap \mathbb{P}N_0$  is a 1-real-parameter family
- 4)  $L_z \subset \mathbb{P}N_0$
- 5)  $L_z \subset \mathbb{P}\mathbb{T}_0^-$
- 6)  $L_z \subset \mathbb{P}\mathbb{T}_0^- \cup \mathbb{P}\mathbb{T}_0^0$  and  $L_z \cap \mathbb{P}\mathbb{T}_0^0$  is a singleton.

Proof: ref 2.

Pictorially, we have, restricting to  $z \in M$



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and the cases correspond, in turn, to 1)  $z \in I^+(0)$ , 2)  $z$  is null separated from and to the future of 0, 3)  $z$  is spacelike separated from 0, 4)  $z$  is the origin (or  $I$ ), as these are the only two twistor lines lying in  $\text{PN}_0^\circ$ , 5)  $z$  is in  $I^-(0)$ , and 6)  $z$  is null separated from and to the past of 0.

The next step to show that.

Proposition

$$\text{PN}_0^\circ = \{ \text{null geodesics in } M \text{ which do not enter} \\ I^+(0) \cup I^-(0) \}$$

Proof : ref 2.

This gives a nice picture of the way that  $\text{PN}_0^\circ$  splits off the future of 0 from its past, for the set of points lying on these geodesics consist of all the points in  $M$  which are either spacelike or null separated from the origin, and the complement of that set is the two connected components  $I^+(0)$  and  $I^-(0)$ . In fact the family of null rays gives a sheeting of  $M \setminus (I^+(0) \cup I^-(0))$  by hyperboloids ruled by the null rays, degenerating to the light cone at  $L_0$  in  $\text{PN}_0^\circ$ .

Now, pushing on to the main result of this note, we observe that the set of points on the light cone of the origin is represented in  $\text{PN}$  by a 3 real parameter family of twistor lines, with a 1 real parameter family of them through each point of  $L_0$ . Each line in the family

intersects  $PN_0^\circ$  in precisely one point - ie the point at which it intersects  $L_0$  - and, but for that point, lies entirely in  $PN_0^f$  or  $PN_0^b$ , according as the line represents a point to the future or past of  $0$ . Furthermore, these lines fill out  $PN_0^f$  and  $PN_0^b$ , and so we come to the following result.

#### FACT

A point  $y \in I^+(0)$  is represented by a twistor line  $Ly \subset PN_0^f$ , lying in such a way that precisely one of the above-mentioned family of lines through each point on  $L_0$  intersects it, and this line intersects it in a singleton, and furthermore every point of  $Ly$  lies on precisely one such line.

Note: for people who feel compelled to do things with bundles, I suppose you could say that the 1-parameter family of lines through each point of  $L_0$  (ie those sticking into  $PTI_0^f$ ) form the fibres of a bundle over  $L_0$ , of which the fibres give a foliation of  $PN_0^f$ , and points in  $I^+(0)$  are given by holomorphic sections over  $L_0$ , but personally, I feel that that's going a bit far.

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Robert Low

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#### Acknowledgement.

Talks with R. Penrose and H.I. Ford helped me considerably when I was working on this.

Applications of the Geometry of  $SO(8)$  Spinors  
To Laplace's Equation in Six Dimensions

by L.P. Hughston

In TN 14 (p.46) it was suggested that in even dimensions the conformally invariant Laplace's equation

$$\left(\frac{\partial^2}{\partial x^1 \partial x^1} + \frac{\partial^2}{\partial x^2 \partial x^2} + \dots\right)\phi(x^a) = 0 \quad a = 1 \dots 2n$$

could be solved by consideration of the appropriate 'pure spin-space' for the group  $SO(2n+2)$ . Here the case of six dimensions will be discussed in more detail. I hope to treat the matter elsewhere at greater length [6].

1. Laplace's Equation in Six Dimensions. A conformally invariant formulation will be used here, analogous to the treatment for four dimensions described in [3,4,5,6]. Let  $X^i$  ( $i=1 \dots 8$ ) be coordinates for  $C^8$  (or  $R^8$ ), and homogeneous coordinates for the associated  $P'$ . Let  $\Omega_{ij} X^i X^j = 0$  be a non-singular quadric in this  $P'$ . Then, if  $\phi(X^i)$  is homogeneous of degree -2 in  $X^i$ , the condition

$$P_\Omega \nabla_i \nabla^i \phi = 0, \quad (1)$$

where  $P_\Omega$  denotes restriction to  $\Omega$ , depends only upon the value of  $\phi$  on  $\Omega$ . This is the global conformally invariant form of Laplace's equation in 6 dimensions; the homogeneity -2 is the 'conformal weight'.

2.  $SO(8)$  Spinors. Our approach is essentially that of Cartan [7]. The 'primed' and 'unprimed' spin-spaces for  $SO(8)$  are each 8-dimensional. Let  $W^\alpha$  and  $Z^\alpha$  be coordinates for these spaces ( $\alpha'=1 \dots 8$ ,  $\alpha=1 \dots 8$ ). The spin spaces come naturally endowed with a pair of non-singular quadratic forms  $\Omega_{\alpha\beta}$  and  $\Omega_{\alpha'\beta'}$ . A spinor  $Z^\alpha$  is called 'pure' if it is null, i.e.  $\Omega_{\alpha\beta} Z^\alpha Z^\beta = 0$ . Similarly  $W^\alpha$  is pure if  $W_\alpha W^\alpha = 0$ . The quadratic forms are used to

raise and lower indices. The projective pure primed (resp. unprimed) spinors form a 6-dimensional quadric in the projective primed (resp. unprimed) spin-space.

There is a fundamental trilinear form  $\Gamma_{\alpha\beta\gamma}$  with one index of each type. It satisfies

$$\Gamma_{\alpha\beta\gamma} \Gamma_{\alpha'\beta'}^{\gamma'} = \Omega_{\alpha\beta} \Omega_{\alpha'\beta'}^{\gamma'} \quad (2)$$

$$\Gamma_{\alpha\beta\gamma} \Gamma_{\alpha'\beta'}^{\gamma'} = \Omega_{\alpha\beta} \Omega_{\alpha'\beta'}^{\gamma'} \quad (3)$$

$$\Gamma_{\alpha'\beta\gamma} \Gamma_{\alpha\beta'}^{\gamma'} = \Omega_{\alpha\beta} \Omega_{\alpha'\beta'}^{\gamma'} , \quad (4)$$

(where equality is up to a non-vanishing multiplicative factor). The curved lines under the indices denote symmetrization. Multiplication between tensors of different types can be achieved with  $\Gamma_{\alpha\beta\gamma}$ . Thus  $A^i B^\alpha \Gamma_{\alpha\beta\gamma} = C_\gamma$ , etc.. Note that if either  $A^i$  or  $B^\alpha$  is pure (null) in this expression, then  $C_\gamma$  is necessarily pure. Proof:

$$\begin{aligned} C_\gamma &= \Gamma_{\alpha'\beta\gamma} \Omega^{\alpha'\beta'} \\ &= \Gamma_{\alpha'\beta\gamma} \Gamma_{\alpha'\beta'}^{\alpha''} A^i B^\alpha A^j B^\beta \\ &= \Omega_{\alpha'\beta\gamma} \Omega_{\alpha'\beta'}^{\alpha''} A^i B^\alpha A^j B^\beta \\ &= \Omega_{\alpha'\beta\gamma} \Omega_{\alpha'\beta'}^{\alpha''} A^i B^\alpha A^j B^\beta \quad (\text{by eq. 4}) \\ &= A^2 B^2 \\ &= 0 \quad (\text{since either } A^i \text{ or } B^\alpha \text{ is pure}). \end{aligned}$$

Similarly a vector  $V^i$  is null iff it can be expressed in the form  $V^i = \Gamma_{\alpha\beta\gamma} P^\alpha Q^\beta$  where either  $P^2 = 0$  or  $Q^2 = 0$ . (This generalizes the space-time result that a vector  $v^a$  is null iff it is of the form  $v^a = \delta_{AA'} \gamma^A \gamma^{A'}$ .)

3. Isotropic Planes The quadric  $\Omega_{ij} X^i X^j = 0$  has two systems of isotropic (totally null) 3-planes in it, called  $\alpha$ -planes and  $\beta$ -planes. The  $\alpha$ -planes (resp.  $\beta$ -planes) are in 1-1 correspondence with the projective pure unprimed

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(resp. primed) spinors. The  $\alpha$ -planes and  $\beta$ -planes are self-dual and anti-self-dual, respectively.

From  $\Gamma_{\alpha\beta\gamma}$  we can construct a set of spinor-valued differential forms ('Clifford forms'). These are denoted as follows:

$$\Gamma_{ij}^{\alpha\beta} := \underbrace{\Gamma_i^{\alpha\alpha'}}_i \Gamma_j^{\beta} \quad \Gamma_{ij}^{\alpha'\beta'} := \Gamma_i^{\alpha'\alpha} \underbrace{\Gamma_j^{\beta'}}_j \quad (5)$$

$$\Gamma_{ijk}^{\alpha'\alpha} := \underbrace{\Gamma_i^{\alpha'} \Gamma_j^{\beta}}_i \Gamma_k^{\gamma'\alpha} \quad (\text{Other forms can be constructed by permuting the three types of indices.}) \quad (6)$$

$$\Gamma_{ijkl}^{\alpha\beta} := \underbrace{\Gamma_i^{\alpha} \Gamma_j^{\alpha'}}_i \Gamma_l^{\beta} \Gamma_k^{\beta'} \quad (7)$$

$$\Gamma_{ijk\ell}^{\alpha'\beta'} := \underbrace{\Gamma_i^{\alpha'} \Gamma_j^{\alpha}}_i \Gamma_k^{\beta'} \Gamma_\ell^{\beta} \quad (8)$$

PROPOSITION The skew tensor  $Z_{ijkl} := Z^\alpha Z^\beta \Gamma_{\alpha\beta;ijkl}$  is simple ( $Z_{[ijkl} Z_{m]nop} = 0$ ) and null ( $\Omega^{im} Z_{ijkl} Z_{mnop} = 0$ ) if and only if  $Z^\alpha$  is pure ( $Z^\alpha Z_\alpha = 0$ ).

Thus  $Z_{ijkl}$  represents the  $\alpha$ -plane, in the quadric  $X^i X_i = 0$ , corresponding to the projective spinor  $Z^\alpha$ .  $X^i$  lies on  $Z_{ijkl}$  iff  $Z_{[ijkl} X_{m]} = 0$ . Note that non-projectively the tensor  $Z_{ijkl}$  determines the spinor  $Z^\alpha$  only up to a sign.

PROPOSITION An  $\alpha$ -plane  $A^\alpha$  and a  $\beta$ -plane  $B^\beta$  meet generically in a point in  $\Omega$ , given by  $A^\alpha B^\beta \Gamma_{\alpha\beta;i}$ . If  $A^\alpha B^\beta \Gamma_{\alpha\beta;i} = 0$  then  $A^\alpha$  and  $B^\beta$  meet in a  $P^2$ . Hence  $A^\alpha B^\beta \Gamma_{\alpha\beta;i} = 0$  iff the tensor  $C_{ijk} = \Gamma_{\alpha\beta;ijk} A^\alpha B^\beta$  is simple ( $C_{[ijk} C_{l]mn} = 0$ ) and totally null ( $\Omega^{il} C_{ijk} C_{lmn} = 0$ ), which thus corresponds to the 2-plane of intersection.

Similarly a host of other geometrical relations can be expressed by use of the  $\Gamma$ -forms.

#### 4. A Contour Integral Formula for Laplace's Equation

Let  $F(Z^\alpha)$  be a holomorphic function, homogeneous of degree  $-4$ , on a region of the quadric  $\Omega_{\alpha\beta} Z^\alpha Z^\beta = 0$ . We can parametrize  $Z^\alpha$  by writing  $Z^\alpha = \Gamma_{i\alpha}^{\alpha'} X^i W^{\alpha'}$ . Now consider the following integral:

$$\phi(X^i) = \oint F(\Gamma_{i\alpha}^{\alpha'}, X^i W^{\alpha'}) X^j X^k \Gamma_{jk\alpha'\beta'\gamma'} W^{\alpha'} dW^{\beta'} dW^{\gamma'} dW^{\delta'} \quad (9)$$

where  $\Gamma_{\alpha'\beta'\gamma'\delta'}^{ijk} = \underbrace{\Gamma_{i\alpha}^j \Gamma_{\alpha\beta}^k \Gamma_{\beta\gamma}^l \Gamma_{\gamma\delta}^m}_{\Gamma_{i\alpha}^j \Gamma_{\alpha\beta}^k \Gamma_{\beta\gamma}^l \Gamma_{\gamma\delta}^m}$ . Since  $F(Z^\alpha)$  is

homogeneous of degree  $-4$  it follows that  $\phi(X^i)$  is homogeneous of degree  $-2$ . The contour is required to lie in the quadric  $\Omega_{\alpha\beta} W^{\alpha'} W^{\beta'} = 0$ .

THEOREM The differential form

$$F(\Gamma_{i\alpha}^{\alpha'}, X^i W^{\alpha'}) X^j X^k \Gamma_{jk\alpha'\beta'\gamma'} W^{\alpha'} dW^{\beta'} dW^{\gamma'} dW^{\delta'} \quad (10)$$

has vanishing exterior derivative, and thus  $\phi(X^i)$  is independent of the choice of contour.

Proof. This follows by a straightforward calculation, with use of the fact that  $F(Z^\alpha)$  is homogeneous of degree  $-4$ , together with the identity:

$$\Gamma_{[\alpha'\beta'\gamma']}^{ijk} \Gamma_{\epsilon'}^{i\alpha} = \Omega^{ij} \Gamma^k_{\alpha'\beta'\gamma'} \epsilon_{\rho'\sigma'\tau'\alpha'\beta'\gamma'\epsilon'}, \quad (11)$$

where  $\epsilon_{...}$  is the totally skew tensor, and

$$\Gamma_{\rho'\sigma'\tau'}^{k\alpha} = \Gamma_{\rho'\tau'}^k \Gamma_{\sigma'}^\alpha \Gamma_{\alpha'}^{i\alpha}. \quad \square \quad (12)$$

THEOREM  $\phi(X^i)$  satisfies  $\rho_\Omega \nabla^i \nabla_i \phi = 0$ .

Proof. This follows with some elementary calculation, with use of the identities:

$$\Omega^{ij} \Gamma_{j\alpha' p' q' s'} = 0 , \quad (13)$$

and

$$\Gamma^k_{j\alpha' p' q' (s'} \Gamma^{j\alpha}_{s' e')} = + \Gamma^{ka}_{[\alpha' p' q'} \Omega_{s']e']} + \Gamma^{ka}_{[\alpha' p' q'} \Omega_{s']s'} \square . \quad (14)$$

Details will be noted in a future publication [6].

Some other relevant references are noted in [8-13].

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### Higher Dimensions

In TN15 I raised the question of whether the twistor treatment of self-dual Yang-Mills generalizes to dimensions  $> 4$ . (Readers will be aware that higher dimensions are very fashionable at present; at the time of writing, eleven is all the rage.) I conjectured that there are no twistor-solvable non-linear hyperbolic/elliptic equations in dim  $> 4$ . (By "hyp/ell" I mean that there are as many equations as unknowns, and that the characteristic surfaces are quadratic cones. The sd YM equations are hyp/ell in this sense, if one takes gauge freedom into account. Also, they imply the Yang-Mills equations.) But there is a lot one can do in higher dimensions, and I want to report on some work in this direction. More details may be found in "Completely Solvable Gauge-Field Equations in Dimension Greater than Four", Durham preprint 1983.

The sd YM equations are integrability conditions for a linear system

$$\pi^{A'} D_{AA}, \Psi = 0, \quad (*)$$

where  $D_a = \partial_a + A_a$  is the gauge-covariant derivative. Thinking of them in this way leads directly to the twistor construction. So one should try to generalize (\*) to higher dim. This can be done in many different ways: I shall now give two examples in dim. eight.

Example A. Let unprimed capital indices be 4-dimensional, while primed indices remain 2-dim. So  $x^a = x^{AA'}$  are now coords on  $\mathbb{C}^8$ . Let us restrict to the Euclidean slice  $\mathbb{R}^8$ . The metric is  $g_{ab} = \epsilon_{AB} \epsilon_{A'B'}$  as usual, and the symmetry group of the setup is  $[\mathrm{Sp}(2) \times \mathrm{Sp}(1)]/\mathbb{Z}_2$ , a subgroup of  $\mathrm{SO}(8)$ . (The Sp's occur because they are what preserve the  $\epsilon$ 's.) The integrability of (\*) then gives us a system of equations on  $A_a$ . These (it turns out) imply the YM equations, but are not hyp/ell, being overdetermined (18 eqns in 7 unknowns).

The picture here is that twistor space is  $\mathbb{CP}^5$ , the lines in which correspond to points in compactified  $\mathbb{C}^8$ . The basic equation is  $\omega^A = i x^{AA'} \pi_A$ , as usual.  $H^1(\mathcal{O}(-2))$  corresponds to solutions of  $\partial_{[A} \partial_{B]} A' \phi = 0$  which implies, but is stronger than,  $\square \phi = 0$ . Vector bundles over the twistor space correspond to solutions of the system of 18 equations mentioned above.

Example B. Both sorts of indices remain 2-dimensional, but now  $x^a = x^{AA'B'C'}$  (totally sym in  $A'B'C'$ ) are coords on  $\mathbb{R}^8$ . The Euclidean line-element is  $dx^{AA'B'C'} dx_{AA'B'C'}$  and the symmetry group is  $\mathrm{SO}(4) \subset \mathrm{SO}(8)$ . This time take as linear system

$$\pi^{A'} \pi^{B'} \pi^{C'} D_{AA'B'C'}, \Psi = 0.$$

The integrability conditions now form a well-determined system (7 equations in 7 unknowns), but the characteristic cone is<sup>†</sup> of degree  $> 2$  (ie. not quadratic). The YM eqns are not automatically implied.

The twistor space is  $\mathbb{CP}^7$ , with basic equation  $\omega^{AA'B'} = i x^{AA'B'C'} \pi_{C'}$ .  $H^1(\mathcal{O}(-2))$  corresponds to solutions of  $\boxtimes \phi = 0$ , where  $\boxtimes$  is<sup>†</sup> a scalar differential operator of order greater than 2. (The two statements marked with † are tentative, as I haven't worked through the algebra.)

Richard Ward.

### Cohomology of the Quadric and Homogeneous Z.R.M. Fields

In [1], a way of developing a "Penrose transform" between cohomology of minitwistor space  $M\mathbb{P}$  (the total space of the bundle  $\Theta(2)$  on  $P_1$ ) and solutions of Z.R.M. equations on  $\mathbb{C}^3$  was described. This note will describe a similar correspondence, between cohomology on another well-known minitwistor space - the quadric - and solutions of ZRM. equations on the complement of a quadric in  $P_3$ .

Denote by  $M^0$  the complement of the null cone of the origin in complex affine Minkowski space.  $M^0$  corresponds to the complement of a pair of skew lines in  $P\mathbb{I}$ . The dilatation vector field on  $M^0$  corresponds to a vector field along the transversals to the pair of lines in  $P\mathbb{I}$  (fig 1), and factoring out by this gives a quadric  $Q$  on one side, and the complement,  $H$ , of a quadric in  $P_3$  on the other.

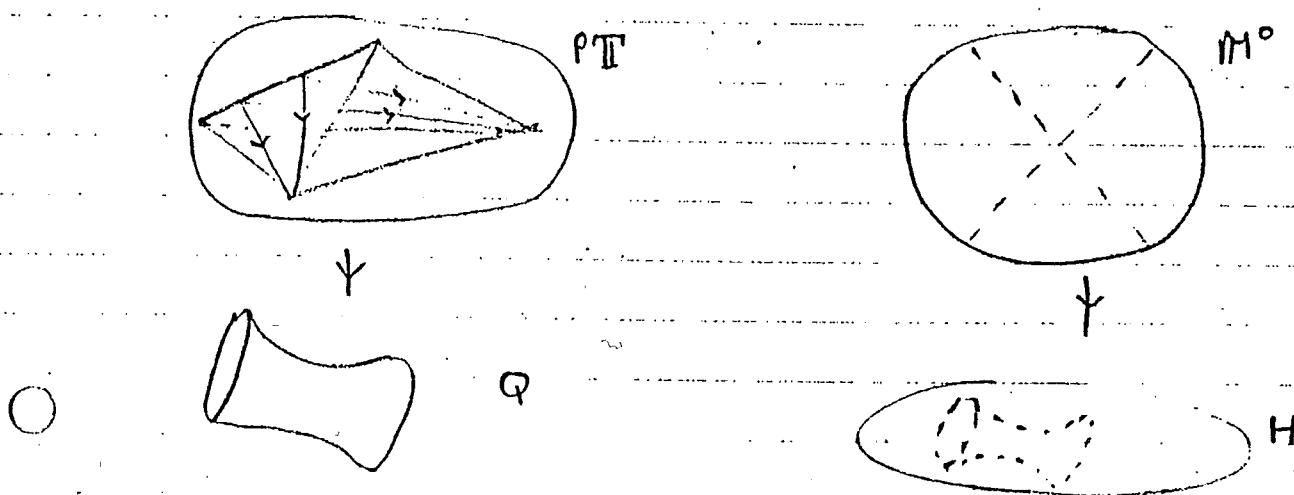


fig 1.

$H$  is the space of (non-tangent) plane sections of  $Q$ . These are rational curves of normal bundle  $\Theta(2)$ , and we thus have a usual minitwistor set up.

If we think of  $Q$  as embedded in a  $P_3$ , then  $H$  is  $P_3^*$  minus the dual quadric  $Q^*$ . As a correspondence space for the E.P.W. machine we use the bundle,  $C$ , of tangent planes to  $Q^*$ . Over a point in  $Q^*$ , the fibre of  $C$  is  $P_2$  minus a line pair. There is a fibration  $\mu: C \rightarrow Q$  given by mapping  $c \in C$  into the point  $q \in Q$  corresponding to the tangent plane (to  $Q^*$ ) in which  $c$  sits. Also, there is obviously a fibration

$\nu: C \rightarrow H$  since  $C$  is a bundle of subspaces of  $H$ .  $\nu$  has fibre  $P_1$ , since there is a  $P_1$ 's worth of tangent planes to  $Q^*$  through a given point  $h \in H$ . We thus have a double fibration

$$Q \xleftarrow{\mu} C \xrightarrow{\nu} H$$

Using homogeneous coordinates  $x_{\alpha\beta}$  on the  $P_1$ 's in which  $Q$  is embedded, we can introduce coordinates  $(w_\alpha, \pi_\alpha)$  on  $Q$  subject to  $(w_\alpha, \pi_\alpha) \sim (\lambda w_\alpha, \mu \pi_\alpha)$ . The embedding is  $(w_\alpha, \pi_\alpha) \mapsto w_\alpha \pi_\alpha$ .

Denote by  $\mathcal{O}(m, n)$  the sheaf of functions on  $Q$  satisfying  $f(\lambda w_\alpha, \mu \pi_\alpha) = \lambda^m \mu^n f(w_\alpha, \pi_\alpha)$ . We can now take an open set (which is regular in the appropriate sense)  $U \subset H$  and the corresponding set  $U'' \subset Q$  and ask for an interpretation of  $H'(U'', \mathcal{O}(m, n))$  in terms of field equations on  $U$ . Applying the E.P.W. machine (see my forthcoming D.Phil. thesis for details) we obtain

$$\begin{aligned} H'(U'', \mathcal{O}(-m, -n)) &\cong \ker \nabla_A^P : \Gamma(U, \mathcal{O}_{(A\dots A)}(-n)[-1]) \rightarrow \Gamma(U, \mathcal{O}_{A(A\dots A)}(-1-n)[-2]) & m+n > 2 \\ H'(U'', \mathcal{O}(-m, -n)) &\cong \ker \square : \Gamma(U, \mathcal{O}(-n)[1]) \rightarrow \Gamma(U, \mathcal{O}(-n-2)[2][-1])' & m+n = 2 \\ H'(U'', \mathcal{O}(-m, -n)) &\cong \ker \nabla^{AA} : \Gamma(U, \mathcal{O}_A(-1-n)[-1]) \rightarrow \Gamma(U, \mathcal{O}^A(-2-n)[-2][-1])' & m+n = 1 \\ H'(U'', \mathcal{O}(m, n)) &\cong \ker \nabla^{AA} : \Gamma(U, \mathcal{O}_A^{(A\dots L^M)}(n)[-1]) \rightarrow \Gamma(U, \mathcal{O}^{(A\dots L^M)}(n-2)[-2][1])' & m+n > 0 \\ \text{Im } \nabla_A^P : \Gamma(U, \mathcal{O}^{(A\dots L)}(n)) &\rightarrow \Gamma(U, \mathcal{O}_A^{(A\dots L^M)}(n)[-1]) \end{aligned}$$

(figures in round brackets are homogeneities in  $\alpha$ ).

For the case  $m=n=0$ , one can then, as in the time-translation case, take the potential modulo gauge and construct from this a solution of the  $C^*$  Bogomolny equations

$$*d\omega = d\Phi$$

on  $U$ , where  $\omega$  and  $\Phi$  are respectively a one-form and a scalar on  $U$ .  $\omega$  is represented by a function  $w_{\alpha\beta}(x)$  which is homogeneous of degree  $-1$  and satisfies  $x^{\alpha\beta} w_{\alpha\beta} = 0$ .  $\Phi$  is given by a function  $\Phi(x)$  homogeneous of degree zero. The Bogomolny equations can be written as

$$2i x_\alpha^\beta \nabla_{(\alpha}^\gamma \omega_{\beta)\gamma} = \nabla_{AB} \Phi.$$

A solution is defined up to gauge transformations of the form  $w_{\alpha\beta} \mapsto w_{\alpha\beta} + \nabla_{\alpha\beta} f$  where  $f$  is homogeneous of degree zero. The solution can be constructed from a potential-modulo-

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gauge  $\Psi_{AA'}(x)$  (homogeneous of degree -1) according to

$$\bar{\Phi} = -i(x \cdot r)$$

$$W_{AA'} = \Psi_{AA'} - \frac{(x \cdot r)}{(x \cdot x)} x_{AA'}$$

The Bogomolny equations then follow from the anti self duality equations on  $\Psi_{AA'}$ .

Thanks to Mike Eastwood.

### Reference

- [1]. Jones "Minitwistors." TN 14.

- Phil Jones

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### Erratum for "An Alternative Interpretation of Some Non-linear Gravitons." (in TN 16)

In my description of the Tod and Ward type B non linear graviton as an affine bundle, I pointed out that in the transition function

$$\begin{pmatrix} 1 & \lg -\frac{T_{\mu\nu}}{T_{11}} z \\ 0 & 1 \end{pmatrix}$$

the term  $\frac{T_{\mu\nu}}{T_{11}} z$  does not contribute. Of course it does contribute since it corresponds to the solution  $W=0, V=1$  of the C\* Bogomolny equations and substituting into  $V^{-1}(dt - W dx)^2 + V dx \cdot dx$  gives the Euclidean metric on  $\mathbb{R}^4$ .

- Phil Jones

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### SOME DUAL RELATIONS IN TWISTOR THEORY \*

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### ABSTRACT

The option of employing twistors or dual twistors in integral representations, etc., is considered. In particular, dual-space analyses are presented which relate to the problem of background electromagnetic fields, and to the inverse transformation.

Extended Regge trajectories\* were introduced in [1] to make use of the extra freedom arising from the fact that the spin  $J$  of a particle is not a good quantum number. Only  $I^2$  can be measured. Thus, with some inspiration from twistor theory, one can define a new quantum number  $j$  by

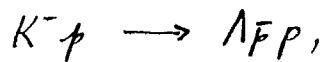
$$j + \frac{1}{2} = \epsilon (J + \frac{1}{2}), \quad \epsilon = \pm 1.$$

Remarkably straight Regge trajectories were obtained by plotting  $j$  versus  $\epsilon m^2$ , instead of the usual  $J$  versus  $m^2$ . For baryons, a definite assignment of  $\epsilon$  seems to be indicated, whereas for ordinary mesons no definite assignment can be made to cover all cases. In [2] it was suggested that we bring in the exotic mesons. Whereas the ordinary mesons belong to either the octet or the singlet representation of  $SU_3$  (in quark language,  $q\bar{q}$  states), the exotic mesons belong to the 27-dimensional representation (which can be interpreted as  $gg\bar{q}\bar{q}$  states). These latter occur mostly as baryon-antibaryon bound states, and are sometimes called baryoniums. In [2] it was shown that with the assignment

$$\epsilon = \begin{cases} +1 & \text{for } 9 \\ -1 & \text{for } 27 \end{cases}$$

we obtained straight extended Regge trajectories for isospin  $I=0, \frac{1}{2}, 1$ .

Recently, several new resonances were observed in the  $\Lambda\bar{p}$  system [3] in the reaction



at energies of 8.25 GeV/c, 18.5 GeV/c and 50 GeV/c. As can be seen

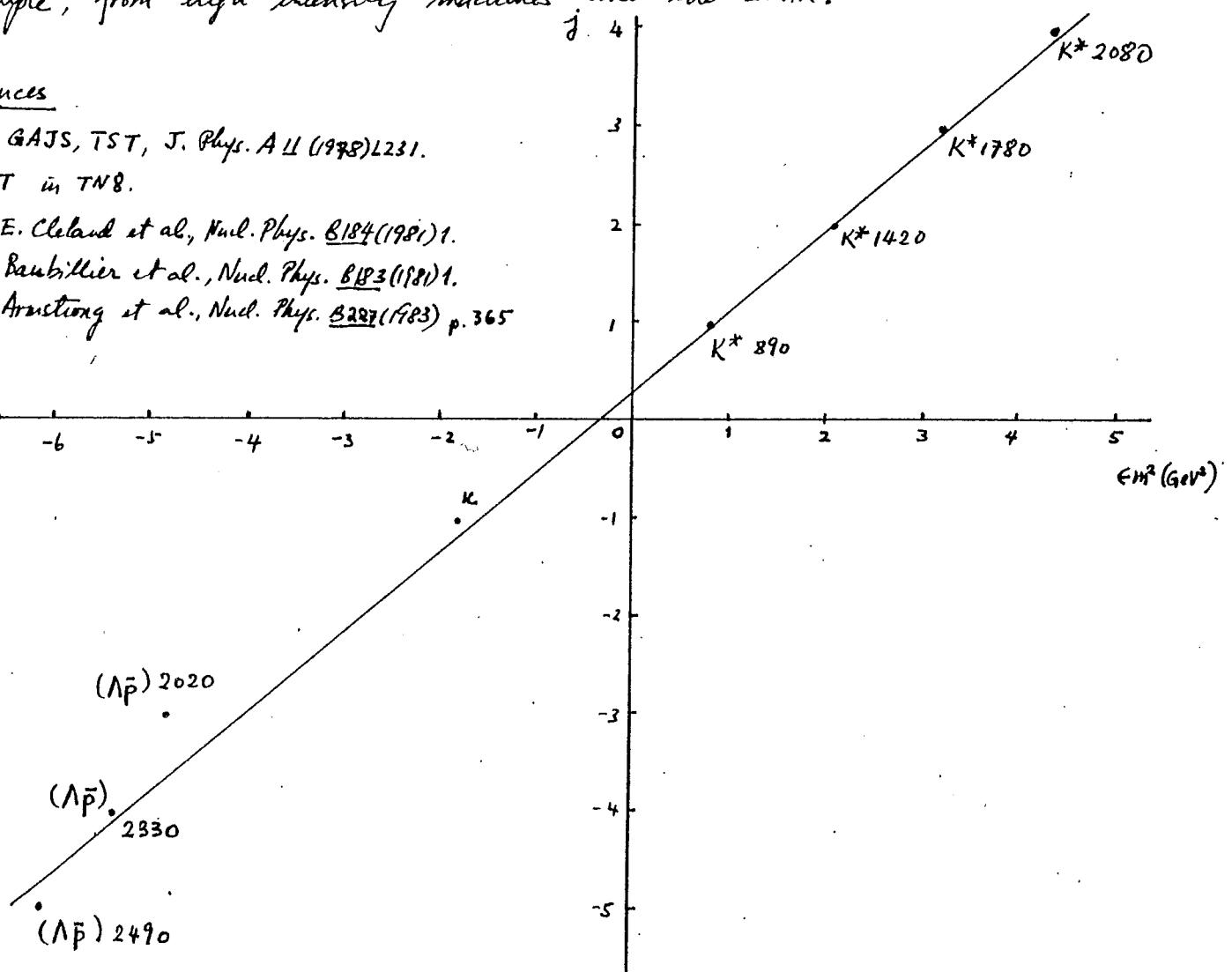
\*short for: theory with infinitely straight trajectories of Regge.

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in the figure, they sit comfortably on our extended  $I = \frac{1}{2}$  trajectory. A conventional trajectory for these resonances would give a slope of  $1.7 \text{ GeV}^2$ , very different from the classical Regge trajectories all of slope  $\sim 1 \text{ GeV}^2$ . We also note that for the non-strange baryon trajectories slope values of  $0.63 \text{ GeV}^2$  and  $0.85 \text{ GeV}^2$  have been suggested. Thus our scheme has the advantage that there is just one universal slope for all the trajectories. This will certainly merit further study once the data on baryonium become more conclusive, for example, from high intensity machines like the LEAR.

#### References

- 1) RP, GAJS, TST, J. Phys. A 11 (1978) L231.
- 2) TST in TN8.
- 3) W. E. Cleland et al., Nucl. Phys. B 184 (1981) 1.  
M. Baubillier et al., Nucl. Phys. B 183 (1981) 1.  
T. Armstrong et al., Nucl. Phys. B 227 (1983), p. 365



Tson Sheung Tsun.

### On Functional Integration - Tom Hurd

I just want to make a few observations about functional integration. Let an action functional  $S[\phi] = \int_M d^4x \mathcal{L}[\phi]$  define the quantum field theory of a real scalar field  $\phi$ . The n-point Wightman functions of the QFT are given by functional derivatives of

$$Z[J] = \int_{\mathcal{S}} d\phi \exp \left\{ i S[\phi] + \int_M J \phi \right\} \quad (*)$$

with respect to  $J(x)$ , evaluated at  $J \equiv 0$ . The functional measure  $d\phi$  lives on a function space  $\mathcal{S}$ , which is usually taken to be something like a space of tempered distributions over  $M$ .

Rather than consider nasty things like that, suppose we think holomorphically and conformally invariantly and take  $\mathcal{S} = \Gamma_R(M^\#; \mathcal{O}(-2))$ , i.e. real analytic functions on compactified Minkowski space  $M^\#$ , with conformal weight -2. Such functions are clearly  $L^2$ -integrable: from  $\phi$  and  $\psi \in \mathcal{S}$  we can construct

$$\langle \phi | \psi \rangle = \frac{1}{2\pi^3} \int_{M^\#} \phi \psi$$

which is actually compact integration of a holomorphic 4-form (as are all other space-time integrals here).

An interesting consequence of the choice  $\mathcal{S} = \Gamma_R(M^\#; \mathcal{O}(-2))$  is that  $Z[J]$  factorizes. In my thesis I noted the following integral formula (known in the literature as the Cauchy-Szegö kernel)

$$\phi^+(x) = \frac{1}{2\pi^3} \int_M \phi^+(y) [(x-y)^2]^{-2} d^4y \quad , \quad x \in \overline{CM^+} \quad (**)$$

which works for any conformal weight -2 function  $\phi^+$  which is holomorphic on  $\overline{CM^+}$ , the closure of the future tube. More generally, given  $\phi \in \Gamma_R(M^\#; \mathcal{O}(-2))$  one can form the integral

$$\phi^+(x) = \frac{1}{2\pi^3} \int_{M^\#} \phi(y) [(x-y)^2]^{-2} d^4y \quad x \in \overline{CM^+}$$

and define  $\phi^+$  to be the 'positive frequency part' of  $\phi$ . In fact  $\phi^+$  is automatically holomorphic on  $\overline{CM^+}$ . Similarly one can define the negative frequency part of  $\phi$

$$\phi^-(x) = \frac{1}{2\pi^3} \int_{M^\#} \phi(y) [(x-y)^2]^{-2} d^4y \quad x \in \overline{CM^-}$$

which automatically extends holomorphically to  $\overline{CM^-}$ . The remainder of

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$\phi$ , i.e.  $\phi^0 = \phi - \phi^+ - \phi^-$ , can be called the space-like part of  $\phi$ . The space-like part of  $\phi$  is real analytic on  $M^\#$  and satisfies

$$\langle 4 | \phi^0 \rangle = 0 \quad (***)$$

for any function  $\psi$  which is holomorphic on either  $\overline{CM^+}$  or  $\overline{CM^-}$ . Thus we find that any  $\phi \in \mathcal{S}$  can be uniquely decomposed as the sum  $\phi = \phi^+ + \phi^- + \phi^0$  and so  $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^- \oplus \mathcal{S}^0$ .

By making use of property (\*\*), one can see that any action  $A[\phi]$  of the ordinary sort can be expressed as a sum of two terms  $A[\phi] = A^\pm[\phi^+, \phi^-] + A^0[\phi^0]$ . This implies that the functional integral (\*) factorizes

$$Z[J] = \int_{\mathcal{S}^+ \oplus \mathcal{S}^-} d\phi^+ d\phi^- \exp \left\{ iA^\pm[\phi^+, \phi^-] + \langle J^+ | \phi^- \rangle + \langle J^- | \phi^+ \rangle \right\}$$

$$x \int_{\mathcal{S}^0} d\phi^0 \exp \left\{ iA^0[\phi^0] + \langle J^0 | \phi^0 \rangle \right\} = Z^\pm[J^+, J^-] Z^0[J^0].$$

The first factor  $Z^\pm$  can be rendered more concrete because the spaces  $\mathcal{S}^+$  and  $\mathcal{S}^-$  both admit a discrete spectral decomposition. The functions

$$e^{pqrs}(x) = \frac{(ptq+r+s)!}{p! q! r! s!} [(x-b)^p][(x-\bar{b})^q][(x-c)^r][(x-\bar{c})^s][(x-\bar{a})^{-(p+q+r+s+2)}]$$

$p, q, r, s \geq 0$ , and their complex conjugates  $\bar{e}_{pqrs}(x)$  turn out complete and orthonormal bases for  $\mathcal{S}^+$  and  $\mathcal{S}^-$  respectively, provided  $a \in CM^+$ ,  $\bar{a} \in CM^-$ , and  $b, \bar{b}, c, \bar{c}$  lie on the intersection of the null cones of  $a$  and  $\bar{a}$ . That is

Proposition:  $\phi \in \mathcal{S}^+ \iff \begin{cases} \phi(x) = \sum_I \phi_I e^I(x), & x \in \overline{CM^+} \\ \text{where } |\phi_I| < K \delta^{p+q+r+s} \quad (I = pqrs) \quad (****) \\ \text{for some } K < \infty \text{ and } \delta < 1. \end{cases}$

The proof of this result is a bit lengthy, and involves using the integral formula (\*\*) in the same sort of way one uses the one-dimensional Cauchy integral formula to prove estimates for Taylor's theorem on a disc.

One can now express  $Z^\pm$  as a function of the infinite collection of complex variables  $J_I^+ := \langle \bar{e}_I | J \rangle$  and  $J_I^- = \overline{J_I^+} = \langle e^I | J \rangle$ :

$$Z^\pm[J_I^+, J_I^-] = \prod_K \left[ \int_C d\phi_K^+ d\phi_K^- \right] \exp iA^\pm[\phi_K^+, \phi_K^-] \exp \sum_I (J_I^+ \phi_I^+ + J_I^- \phi_I^-).$$

I'm not sure how much information of the quantum field theory is contained in the factor  $Z^\pm$ . The factor  $Z^0$  seems much more difficult to handle; certainly no discrete expansion like (\*\*\*\*) exists for general  $\phi^0 \in \mathcal{S}^0$ .

## The index of the 2-twistor equations

One wants to know something about the  $\mathbb{C}$ -dimension of superficial twistor spaces (see [1] for defns) on spacelike 2-surfaces not necessarily homeomorphic to a sphere. In this note, I compute the index of the 2-twistor operator on orientable smooth spacelike 2-surfaces  $\mathcal{S}$ , embedded in a general spacetime  $M$ .

Given such  $\mathcal{S}$ , with a sense of "outward" of  $\mathcal{S}$  in  $M$  (a hence a choice of orientation for  $\mathcal{S}$ ), the metric in  $M$  determines a metric on  $\mathcal{S}$  (assumed smooth) and hence a conformal (holomorphic) structure for  $\mathcal{S}$  - locally  $\exists$  co-ords for  $\mathcal{S}$  in which the metric induced on  $\mathcal{S}$  is of form  $P^2(\zeta, \bar{\zeta}) d\zeta d\bar{\zeta}$ .  $\mathcal{S}$  is thus a compact Riemann surface - genus  $g$ . Let  $\mathcal{D}$  be the  $\mathbb{C}^*$ -principal bundle of orthonormal spinor dyads  $(O^A, i^A)$  at points of  $\mathcal{S}$  such that  $O^A O^{A'} = \delta^{AA'}$  and  $i^A i^{A'} = \eta^{AA'}$  lie in the outward-in-going null directions orthogonal to  $\mathcal{S}$ . Let  $\mathcal{D}(s, w)$  be the complex line bundles induced by the representations  $\rho_{s,w} : \mathbb{C}^* \rightarrow \mathbb{C}^*$  given by  $\rho_{s,w}(\lambda) = (\lambda \bar{\lambda})^w (\lambda/\bar{\lambda})^s$ . These are of course the restriction to  $\mathcal{S}$  of the bundles of  $s$ -spin &  $w$ -conformally weighted functions of Geroch et al [2]. Some facts (obvious):

- (i) The connection on the spin bundle of  $M$  restricted to  $\mathcal{D}$  induces covariant derivatives  $D_{AA'}(s, w)$  on the  $\mathcal{D}(s, w)$ . One has  $\tilde{D}_{s,w} = m^a D_a(s, w)$  and  $\tilde{D}'_{s,w} = \bar{m}^a D_a(s, w)$ .
- (ii)  $\mathcal{D}(s, w_1) \otimes \mathcal{D}(s_2, w_2) \cong \mathcal{D}(s+s_2, w_1+w_2)$
- (iii)  $\mathcal{D}(0, w)$  is trivial  $\forall w$ :  $(t^{AA'} O_A O_{A'})^w$  is a section,  $t^{AA'}$  any nonzero vector field on  $\mathcal{S}$  not parallel to  $i^a$
- (iv)  $\Lambda^{0,0} T^*(\mathcal{S}) \cong \mathcal{D}(1,0)$  and  $\Lambda^{0,1} T^*(\mathcal{S}) \cong \mathcal{D}(0,1)$
- (v)  $S^*|_{\mathcal{S}} \cong \mathcal{B}(-\frac{1}{2}, \frac{1}{2}) \oplus \mathcal{D}(\frac{1}{2}, \frac{1}{2})$  by  $w^A \mapsto (w_A i^A, w_A^A)$ .
- (vi)  $s$  is  $\frac{1}{2}$  integral,  $w$  arbitrary real.
- (vii)  $\overline{\mathcal{D}(s, w)} \cong \mathcal{D}(-s, w)$ .

A holomorphic structure on  $\mathcal{D}(s, w)$  is induced by identifying  $\Lambda^{0,1} T^*(\mathcal{S})$  with  $\mathcal{D}(0,1)$  and requiring the exactness of

$$0 \rightarrow \mathcal{O}(s, w) \xrightarrow{i} \mathcal{D}(s, w) \xrightarrow{\tilde{D}'} \mathcal{D}(s-1, w) \rightarrow 0 \quad (1)$$

where  $\mathcal{O}(s, w)$  is the sheaf of germs of holomorphic sections of  $\mathcal{D}(s, w)$ .

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(cf [3]). (i) is a fine resolution of  $\mathcal{D}(s, w)$  and can be used to compute its cohomology. In particular, index  $\delta'_{s,w}$  is  $\dim_{\mathbb{C}} H^0(S, \mathcal{D}(s, w)) - \dim_{\mathbb{C}} H^1(S, \mathcal{D}(s, w)) = c_1(\mathcal{D}(s, w)) + 1 - g$ , where  $c_1$  denotes first (integral) Chern Class, by the Riemann-Roch theorem. Since  $\mathcal{D}(1, 0) \cong \Lambda^{1,0} T^* S$  is the canonical bundle of  $S$ , by Serre duality one can show  $c_1(\mathcal{D}(1, 0)) = 2(g-1)$  by which, using (ii),  $c_1(\mathcal{D}(s, w)) = 2s(g-1)$ , so

$$\text{index } \delta'_{s,w} = (2s-1)(g-1) \quad -②$$

Using a conjugate sequence to (i), one computes

$$\begin{aligned} \text{index } \delta_{s,w} &= c_1(\mathcal{D}(-s, w)) + 1 - g \\ &= (-2s-1)(g-1) \end{aligned} \quad -③$$

Now the 2-twistor operator  $\mathcal{T}$  is given by  $\begin{pmatrix} -\sigma & \delta \\ \delta' & -\sigma' \end{pmatrix}$  mapping

$\mathcal{D}(-\frac{1}{2}, -\frac{1}{2}) \oplus \mathcal{D}(\frac{1}{2}, \frac{1}{2}) \rightarrow \mathcal{D}(\frac{3}{2}, \frac{1}{2}) \oplus \mathcal{D}(\frac{3}{2}, -\frac{1}{2})$ ;  $\mathcal{T}$  has the same top order symbol & hence index as  $\begin{pmatrix} 0 & \delta \\ \delta' & 0 \end{pmatrix}$  so

$$\begin{aligned} \text{index } \mathcal{T} &= \text{index } \delta_{\frac{1}{2}, \frac{1}{2}} + \text{index } \delta'_{-\frac{1}{2}, -\frac{1}{2}} \\ &= 4(g-1) \quad (\leq \dim_{\mathbb{C}} \Pi(S)). \end{aligned} \quad -④$$

Since  $\dim_{\mathbb{C}} \Pi(S) \geq 4$  when  $M$  is Minkowski space, one can expect that the adjoint 2-twistor equations in  $M$  will have a multiplicity of solutions when  $g \geq 1$ , and in general there need not be any non-trivial solutions of the 2-twistor equations for  $g \geq 1$ . KPT has examples of this for the torus.

Thanks to RP, KPT, MGE.

Rob Barton

[1] Penrose (1982) Proc Roy Soc A381, 53

[2] Geroch et al (1973) J Math Phys 14, 874

[3] Eastwood & Tod (1982) Math Proc Camb Phil Soc 92, 317

SYMPLECTIC GEOMETRY OF  $\mathcal{J}^+$  AND  
2-SURFACE TWISTORS

33.

In trying to understand what form of angular momentum flux law may be associated with Penrose's 2-surface twistors (ref 1) defined on cross-sections of  $\mathcal{J}^+$ , one is led to make a comparison with the theory of fluxes given by Ashtekar and Streubel (2), (denoted A-S from here on) who were able to associate fluxes with generators of the B.M.S. group. The purpose of this note is to explain the relationship of the two approaches, and to show how the twistor approach supplies an element missing from the A-S analysis - the construction of charge integrals for the fluxes.

A-S first construct the phase space of radiative modes of the gravitational field on  $\mathcal{J}^+$ . They show that the induced action of the B.M.S. group on this space preserves the symplectic structure thereon, and compute the Hamiltonians generating these canonical transformations. In  $M$ , an analogous analysis of electromagnetic theory suggests that the corresponding Hamiltonians are integrals of a Hamiltonian density which is just the flux associated with the generators  $V$  of the transformations. Transferring the interpretation to full general relativity, one arrives at a Hamiltonian density  ${}^H F_V$  regarded as a local flux. In this symplectic approach the  ${}^H F_V$  arise as primary quantities and one has to integrate them to obtain the 2-sphere charge integrals  ${}^H Q_V [S]$ , with

$${}^H Q_V [S_2] - {}^H Q_V [S_1] = \int_{S_1}^{S_2} {}^H F_V d^3 S$$

Such an integration was carried out by A-S for the case when  $V$  is a B.M.S. super-translation. The general case, for  $V$  a rotation or boost also, was unresolved. It turns out that one can use the theory of 2-surface twistors to solve this problem.

To see what is involved, I need to describe the quantity  ${}^H F_V$  in more detail.  ${}^H F_V$  is constructed from quantities defined intrinsically on  $\mathcal{J}^+$ . Let  $g_{ab}$  denote the pull-back to  $\mathcal{J}^+$  of the (rescaled) space-time metric  $g_{ab}$ , and let  $q^{ab}$  be any symmetric tensor field within  $\mathcal{J}^+$  satisfying

$$g_{am} q^{mn} g_{bn} = q_{ab}$$

Let  $N_{ab}$  denote the News tensor field on  $\mathcal{J}^+$ , that is, the pull-back to  $\mathcal{J}^+$  of

$$-R_{ab} + \frac{R}{6} g_{ab}$$

in conformal frame in which the metric of  $\mathcal{J}^+$  is a unit sphere. Let  $D$  denote the torsion-free connection on  $\mathcal{J}^+$  induced by  $\nabla$ . For any B.M.S. generator  $V$ , then

$${}^H F_V = (16\pi G)^4 N_{ab} [(L_V D_c - D_c L_V) l_d + l_c D_d V] q^{ac} q^{bd},$$

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where, if  $\eta^\alpha$  is the null generator of  $\mathcal{J}^+$ ,  $\ell_\alpha$  is any covector satisfying  $\ell_\alpha \eta^\alpha = 1$ , and  $V$  is defined by

$$\partial_V q_{ab} = 2Vq_{ab}$$

Firstly this must be translated into spin-coefficients. Complete a tetrad by introducing  $m^\alpha, \bar{m}^\alpha$  with

$$m^\alpha \ell_\alpha = 0 = m^\alpha n_\alpha = m^\alpha m_\alpha \quad ; \quad m^\alpha \bar{m}_\alpha = -1$$

$$\text{and let (as usual)} \quad \sigma = m^b m^\alpha D_\alpha b_b \quad ; \quad \tau = m^b n^\alpha D_\alpha b_b$$

The News tensor is then  $N$ , where

$$\bar{N} = \mathfrak{D}\tau - \tau^2 - \bar{\mathbb{B}}' \sigma$$

employing G.H.P. notation for simplicity, and

$$q^{ac} q^{bd} N_{ab} = 2Nm^c m^d + 2\bar{N}\bar{m}^c \bar{m}^d.$$

Now let  $k^\alpha$  be any self-dual B.M.S. generator, that is,

$$k^\alpha = k_m m^\alpha + k_n n^\alpha,$$

$$\mathfrak{D}k_m = 0 = \bar{\mathbb{B}}' k_m$$

$$\bar{\mathbb{B}}' k_n = \frac{1}{2}(\mathfrak{D} + 2\tau) k_m$$

A tedious calculation gives

$${}^H F_k = -\frac{1}{8\pi G} \left[ N (\mathfrak{D}^2 k_n + k_n \bar{N} - \tau/2 \mathfrak{D} k_m - \mathfrak{D}(\sigma k_m)) \right. \\ \left. + \bar{N} (\mathfrak{D}^2 k_n + k_n N - 3/2 k_m \mathfrak{D} \bar{\tau} + 1/2 \mathfrak{D}(k_m \tau)) \right]$$

Recall that the kinematic twistor is defined as follows. Let  $S$  be a 2-surface cross-section of  $\mathcal{J}^+$  with  $\ell^\alpha$  chosen to be orthogonal to  $S$ . Let  $(\sigma, \ell)$  be the associated spin frame. Let  $\omega^A = \omega^A \sigma^A + \omega^B \ell^B$  define a 2-surface twistor, so that

$$\mathfrak{D}\omega = 0 \quad ; \quad \mathfrak{D}\omega^B = \tau \dot{\omega}^B$$

on  $S$ . The kinematic twistor  $A_{\alpha\beta}$  of  $S$  is then given, for a pair of twistors  $Z_1^\alpha$  and  $Z_2^\beta$  labelled by solutions  $\omega_1^A$  and  $\omega_2^B$  of the above by

$$A_{\alpha\beta}^S Z_1^\alpha Z_2^\beta = -\frac{i}{4\pi G} \oint_S [ \dot{\omega}_1^\alpha \dot{\omega}_2^\beta \Psi_1^\circ + (\dot{\omega}_1^\alpha \dot{\omega}_2^\beta + \dot{\omega}_2^\alpha \dot{\omega}_1^\beta)(\Psi_2^\circ - \tau N) ] dS$$

where the  $\bar{\Psi}_i^o$  are the usual components of the (appropriately rescaled) Weyl tensor. Now  $\hat{\omega}_1^A$  and  $\hat{\omega}_2^B$  define a symmetric 2-index twistor with principal part  $\omega^{AB} = \hat{\omega}_1^A \hat{\omega}_2^B$ . In flat space-time  $\omega^{AB}$  would immediately generate a self-dual Killing vector of  $M$ , given, in the unrescaled space-time (denoting this by carets) by the equation

$$\hat{\partial}_{\alpha} \cdot \hat{\omega}_{AB} = -i \hat{E}_{C(A} k_{B)\alpha}$$

The B.M.S. vector associated with  $\omega^{AB}$  is the vector  $k^\alpha$  given by the smooth extension to  $\mathcal{G}^+$  of  $\hat{k}^\alpha$ , viz,  $k^\alpha = \hat{k}^\alpha$ . Applying the required conformal rescaling yields

$$\begin{aligned} k^\alpha &= 2i \hat{\omega}_1^A \hat{\omega}_2^B m^\alpha + i (\hat{\omega}_1^A \hat{\omega}_2^B + \hat{\omega}_2^A \hat{\omega}_1^B) n^\alpha \\ &= K_m m^\alpha + K_n n^\alpha \quad \text{say.} \end{aligned}$$

We may use this to associate, at  $S$ , self-dual B.M.S. vectors with 2-surface twistors. The kinematic twistor, as a function  $L^s[k]$  of these  $k$ , is just

$$L^s[k] = -\frac{1}{8\pi G} \oint_S [K_m \bar{\Psi}_i^o + 2K_n (\bar{\Psi}_i^o - \sigma N)] dS$$

However, these vectors  $k$  are not general, but are constrained by the twistor equations, which imply

$$\bar{\partial} k_m = 0 \quad ; \quad \bar{\partial}^2 k_n = \sigma/2 \bar{\partial} k_m + \bar{\partial}(\sigma k_m) \quad \text{at } S$$

The latter equation constrains  $k$  to lie in a particular Poincare subgroup of the B.M.S. group. For this particular group choice there are some interesting consequences. Dray and Streubel (3) have shown that the above equations on  $k$  imply that, for all values of the constants  $a$  and  $b$ ,

$$L^s[k] = -\frac{1}{8\pi G} \oint_S \left[ K_m (\bar{\Psi}_i^o + a \bar{\partial}(\sigma \bar{\sigma}) + 2a \sigma \bar{\partial} \bar{\sigma}) + 2K_n (\bar{\Psi}_i^o - \sigma N + a \bar{\partial}^2 \bar{\sigma} + b \bar{\partial}^2 \sigma) \right] dS$$

In particular,  $L^s$  is equivalent to  $L_s^s$ , where  $L_s^s$  is given by

$$L_s^s[k] = -\frac{1}{8\pi G} \oint_S \left[ K_m (\bar{\Psi}_i^o + \frac{1}{2} \bar{\partial}(\sigma \bar{\sigma}) + \sigma \bar{\partial} \bar{\sigma}) + 2K_n (\bar{\Psi}_i^o - \sigma N + \frac{1}{2} \bar{\partial}^2 \bar{\sigma} - \frac{1}{2} \bar{\partial}^2 \sigma) \right] dS$$

We can extend this to a  $\mathbb{Q}$ -linear function of any self-dual B.M.S. vector  $k$ .

36.

Consider now  $L^s[k]$  as a function of general B.M.S. vectors  $k$ . That is,  $k$  satisfies the B.M.S. evolution equations

$$E' k_m = 0 \quad ; \quad E' k_n = \frac{1}{2} (\partial + 2\tau) k_m$$

and is not constrained by the twistor equations at any particular 2-surface. Then, if  $S_1$  and  $S_2$  are two 2-sphere cross-sections of  $\mathcal{P}^+$ , a long and tedious calculation shows that

$$L_o^{S_2}[k] - L_o^{S_1}[k] = \int_{S_1}^{S_2} {}^H F_k d^3 S$$

It is straightforward to establish this for the easy case when  $S_2$  is obtained from  $S_1$  by a time translation. For the general case one should note that the formulae for the maps are given in spin frames adapted to the 2-surface, so that in evolving care is required in keeping track of which frames are employed. This resolves the problem of obtaining charge integrals for the A-S flux. The corresponding formula for real B.M.S. vectors can be obtained by taking real parts.

The remaining issue is now clear. If one solves the twistor equations at  $S_2$ , the self-dual vectors so obtained are not the same as those obtained by B.M.S. propagation from vectors obtained from 2-surface twistors at  $S_1$ , unless  $N = 0$  in the region between  $S_1$  and  $S_2$ . This is the old problem - the natural Poincare group for defining angular momentum shifts in the presence of gravitational radiation, relative to the B.M.S. group. The twistor equations extend the notion of "natural" from stationary space-times to general ones. One may say, interpreting the twistor approach in conventional terms, that the flux associated with the Penrose formula is a synthesis of the Hamiltonian flux with the effect of a continual supertranslation of the origins of rotations. However, the bug in this idea is the same as that noted in T.N. 14. The two sets of B.M.S. vectors at  $S_1$  and  $S_2$  can be related, but not, apparently, in a unique conformally invariant way. This corresponds to a translational freedom in identifying the origins in the natural  $\mathcal{P}^M$  associated with  $\mathcal{T}(S_1)$  with the origins in the corresponding  $\mathcal{P}^M$  associated with  $\mathcal{T}(S_2)$ . This freedom can be removed by picking conformal frames associated with the Bondi energy-momentum in which to fix various functions, but it would be satisfying to have an alternative (and conformally invariant) procedure.

Very many thanks to T. Dray, with whom these calculations were done, and also to R. Penrose, K.P. Tod, and G.T. Horowitz.

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William Shaw

### Dual Two-Surface Twistor Space

This paper is a short treatment of the space dual to two-surface twistor space, and how this relates to 'norms' and conformal embeddings of the two-surface in flat space-time. We assume throughout only 4 solutions exist to the two-surface twistor equations.

#### Introduction

The two-surface twistor  $\tilde{z}^* = (\omega^A, \Pi_A)$  is defined as solutions to the following equations

$$\partial' \omega^0 = \sigma' \omega^1 \quad \partial' \omega^1 = \sigma \omega^0 \quad 1.$$

$$-i\Pi_1 = \partial \omega^0 - p' \omega^1 \quad -i\Pi_0 = \partial' \omega^1 - p \omega^0 \quad 2.$$

Two  $\bar{\Pi}_A$  derivatives may be obtained by applying the commutator  $[\partial, \partial']$  to

$$-\iota(\partial \Pi_0 + p \Pi_1) = X_2 \omega^1 + X_1 \omega^0$$

$$-\iota(\partial' \Pi_1 + p' \Pi_0) = X_2 \omega^0 + X_3 \omega^1 \quad 3.$$

$$\text{where } X_1 = \psi_1 - \phi_{01}, \quad X_2 = \psi_2 - \phi_{11} - \Lambda, \quad X_3 = \psi_3 - \phi_{21}$$

Since there are 4 solutions the other  $\bar{\Pi}_A$  derivatives serve to define 8 functions  $s, B, C, D$  and their primed versions.

$$-i\partial \bar{\Pi}_1 = i(\bar{s}' \bar{\Pi}_0 - B' \bar{\Pi}_1) + C' \omega^1 + D' \omega^0 \quad 4.$$

$$-i\partial' \bar{\Pi}_0 = i(\bar{s} \bar{\Pi}_1 - B \bar{\Pi}_0) + C \omega^0 + D \omega^1$$

applying  $[\partial, \partial']$  to  $\bar{\Pi}_1, \bar{\Pi}_0$  gives equations satisfied by  $s, B, C, D$

$$\partial B + (\bar{s}\bar{s}' - \bar{\sigma}\bar{\sigma}') = X_2 - \bar{X}_2 \quad 5.$$

$$C + \partial' p - \partial \bar{s} = B p + B' \bar{s} \quad 6.$$

$$\partial C + \sigma D - \bar{s} D' - \partial' X_1 = -B X_1 \quad 7.$$

$$\partial D - \bar{s} C' + p' C - \sigma' X_1 + p X_3 - \partial' X_2 = -B X_2 \quad 8.$$

and their primed versions.

38.

### The Dual

There are two approaches to the dual; the algebraic and the differential. Algebraically we may pick a basis for the twistor space and hence have at a matrix of functions over the two-surface.

$$i\tilde{\Pi}^\alpha = \begin{pmatrix} \omega^0 & \Pi_1 & \omega^1 & \Pi_0 \\ {}_2\omega^0 & {}_2\Pi_1 & {}_2\omega^1 & {}_2\Pi_0 \\ {}_3\omega^0 & {}_3\Pi_1 & {}_3\omega^1 & {}_3\Pi_0 \\ {}_4\omega^0 & {}_4\Pi_1 & {}_4\omega^1 & {}_4\Pi_0 \end{pmatrix} \quad 9.$$

To obtain the dual space we simply invert the matrix, obtaining (to give agreement with the flat case)

$$i\tilde{\Pi}_\alpha = \begin{pmatrix} {}_2\tilde{\Pi}_c & {}_2\tilde{\omega}^1 & {}_2\tilde{\Pi}_1 & {}_2\tilde{\omega}^0 \\ {}_2\tilde{\Pi}_0 & {}_1\tilde{\omega}^1 & {}_1\tilde{\Pi}_1 & {}_1\tilde{\omega}^0 \\ {}_4\tilde{\Pi}_0 & {}_4\tilde{\omega}^1 & {}_4\tilde{\Pi}_1 & {}_4\tilde{\omega}^0 \\ {}_3\tilde{\Pi}_0 & {}_3\tilde{\omega}^1 & {}_3\tilde{\Pi}_1 & {}_3\tilde{\omega}^0 \end{pmatrix} \quad 10.$$

With a basis dual to our original choice.

Differentially we note that

$$i\tilde{\Pi}_c = \frac{\epsilon^{ijk}}{\epsilon^{pqr}} \omega^p \Pi_q \Pi_r \quad 11.$$

etc

and may differentiate, obtaining

$$\partial \tilde{\omega}^0 = \bar{s}' \tilde{\omega}^1 \quad \partial' \tilde{\omega}^1 = \bar{s} \tilde{\omega}^0 \quad 12.$$

$$i\tilde{\Pi}_0 = \partial \tilde{\omega}^0 - p \tilde{\omega}^0 - B \tilde{\omega}^1 \quad i\tilde{\Pi}_1 = \partial' \tilde{\omega}^0 - p' \tilde{\omega}^1 - B \tilde{\omega}^0 \quad 13.$$

We note also that

$$B' = \partial \ln (\epsilon^{pqr} \omega^p \omega^q \Pi_r \Pi_s) \quad 14.$$

and similarly for B.

We now note that  $\sigma=0$  implies that for 2 solutions  $\omega^0=0$ . If we take  $p=p'=0$  (we may so do by making a conformal transformation such that  $\frac{\partial}{\partial z} = 1$ ,  $\frac{\partial}{\partial \bar{z}} = p$ ,  $\frac{\partial}{\partial \bar{z}} = p'$ ) This does not affect  $\sigma, \sigma', s, s'$  and we see that the upper L.H. corner of  $i\tilde{\Pi}^\alpha$  vanishes, and thus the bottom R.H. corner

of  $\bar{\Pi}_\alpha$  vanishes, implying that  $s' = 0$ . The converse is obviously also true, as is the equivalence of  $\sigma$  vanishing and  $s$  vanishing.

### Norms and embedding

#### Lemma 1

The following are equivalent

1.  $\sum_0 = \omega^0 \bar{\Pi}_0 + \omega^1 \bar{\Pi}_1 + c.c. = \text{constant}$
2.  $\psi_2 = \bar{\Psi}_2, s = \sigma, s' = \sigma'$
3. The two-surface is embeddable in real conformally flat space

#### Proof

1. implies 2. by differentiation. 2. implies 3. by observing that  $B = B' = 0$  since  $\nabla B = \nabla(\ln D\bar{\Pi}) = 0$  and  $\nabla$  is a conformal Laplacian on the Riemann sphere, and that equations 6, 7, 8 are then precisely the Bianchi identities in a conformally flat space-time. This tells that a conformal factor exists such that  $X_1, X_2, X_3$  may be made to vanish, and that this two-surface may be placed in flat space-time. 3. implies 1. because 1. is true in flat space-time and is conformally invariant.  $\square$

#### Lemma 2

$$1. \sum_1 = \omega^0 \bar{\Pi}_0 + \omega^1 \bar{\Pi}_1 + iB' \omega^0 \bar{\omega}^1 + c.c. = \text{constant}$$

$$2. s = \sigma, s' = \sigma'$$

3. The two-surface is embeddable in real conformally flat space-time, with

torsion introduced by a complex conformal factor. ( $g_{ab} \rightarrow g_{ab}, E_{ab} \rightarrow e^{i\theta} E_{ab}, E_{ab} \rightarrow e^{-i\theta} E_{ab}$ )

#### Proof

As for Lemma 1.  $\square$

#### Lemma 3

All spacelike <sup>two</sup>-surfaces of spherical topology with  $\sigma = \sigma' = 0$  are conformal to a metric two-sphere in flat space-time.

#### Proof

We have  $s = \sigma, s' = \sigma'$ , thus 2. in Lemma 2 is satisfied and our two-surface is conformal to one with  $\sigma = \sigma' = 0$  in flat space-time. All such two-surfaces

have scalar curvature =  $\rho\rho'$  constant, and our two-surface is thus a sphere.  $\square$

The generic embedding of the two-surface in complex conformally flat space-time has been introduced before ( Tod 1983 ). This can be made explicit by noting that in general equations 6,7,8 are the Bianchi identities in a conformally flat complex space-time with conformal torsion generated by

$$\mathcal{D} = (\text{Det } \Pi^{\alpha\beta})^{1/2}, g_{ab} \rightarrow g_{ab}, E_{ab} \rightarrow D E_{ab}, E_{ab'} \rightarrow \frac{1}{2} E_{ab'}$$

The curvature on unprimed spinors is as for the two-surface in its curved space time, but the shears of the primed basis spinors are given by  $\bar{s}, \bar{s}'$  not  $\bar{\sigma}, \bar{\sigma}'$ .

#### Dual Angular Momentum

Starting from the dual twistor space one may seek to define a dual angular-momentum twistor  $\tilde{A}_{\alpha\beta}$ . Writing  $\tilde{Z}_\alpha = (\tilde{\omega}^\alpha, \tilde{\pi}_\alpha)$  one might write a similar expression to the Iod form of  $A_{\alpha\beta} Z^\alpha Z^\beta$ . Just as

$$A_{\alpha\beta} Z^\alpha Z^\beta \propto \int (\tilde{\Pi}_0 \cdot \tilde{\Pi}_1 + \tilde{\Pi}_0 \cdot \tilde{\Pi}_1) dS \quad 15.$$

$$\text{we define } \tilde{A}^{\alpha\beta} \tilde{Z}_\alpha \tilde{Z}_\beta \propto \int (\tilde{\Pi}_0 \cdot \tilde{\Pi}_1 + \tilde{\Pi}_0 \cdot \tilde{\Pi}_1) dS \quad 16.$$

This may be rewritten

$$\tilde{A}^{\alpha\beta} \tilde{Z}_\alpha \tilde{Z}_\beta \propto \int (C \tilde{\omega}_1^\alpha \tilde{\omega}_2^\beta + X_2 (\tilde{\omega}_1^\alpha \tilde{\omega}_2^\beta + \tilde{\omega}_2^\alpha \tilde{\omega}_1^\beta) + C' \tilde{\omega}_2^\alpha \tilde{\omega}_2^\beta) dS \quad 17.$$

If one considers instead a surface for which equations 12,13 are the complex conjugate angular-momentum twistor, with curvature terms defined as if  $\bar{s}, \bar{s}'$  were the complex conjugate shears, instead of  $\bar{\sigma}, \bar{\sigma}'$ . Thus one might define.

$$\tilde{A}^{\alpha\beta} \tilde{Z}_\alpha \tilde{Z}_\beta = \int ((\bar{s}\bar{s} - \bar{s}'\bar{s}') \tilde{\omega}_1^\alpha \tilde{\omega}_2^\beta + (\bar{s}_2 - \bar{s}_2' + \bar{\sigma}\bar{\sigma}') (\tilde{\omega}_1^\alpha \tilde{\omega}_2^\beta + \tilde{\omega}_2^\alpha \tilde{\omega}_1^\beta) + (\bar{s}'\bar{s} - \bar{s}'\bar{s}') \tilde{\omega}_2^\alpha \tilde{\omega}_2^\beta) dS \quad 18.$$

Integration by parts shows that expressions 17 and 18 are equal. One may now define a quantity of dimension mass by

$$\frac{1}{2} \mu^2 = - \tilde{A}^{\alpha\beta} A_{\alpha\beta} \quad 19.$$

Though how useful this is remains to be seen. Thanks to KPT, WIS, RP.

Tod 1983 'Some Examples..' Proc Roy Soc A388

Ben Jeffys

## An Occurrence of Pell's Equation in Twistor Theory.

The index of the two-surface twistor equations on a two-surface has been shown by a number of people to be  $4(1-g)$  where  $g$  is the genus of the two-surface.

On a sphere  $g=0$  and there is always at least a four-parameter family of solutions.

However, on a torus  $g=1$  and there may not be any solutions or the number of solutions may behave in a strange way. For a torus in flat space there will always be at least four solutions defined by restriction. I want to describe some simple tori not in flat space, where the number of solutions does indeed behave in a strange way.

I shall work in toroidal coordinates  $(\xi, \eta, \phi)$  defined from cylindrical polars  $(r, z, \phi)$  by: set  $w = r + iz$ ;  $\zeta = \xi + i\eta$  and  $\frac{w+a}{w-a} = e^{-i\zeta}$  for some real  $a$ . The metric becomes

$$dr^2 + dz^2 + r^2 d\phi^2 = \frac{a^2}{X^2} (d\eta^2 + d\xi^2 + \sinh^2 \eta d\phi^2)$$

where  $X = \cosh \eta - \cos \xi$  (see e.g. Margenau and Murphy)

The surfaces of constant  $\eta$  are tori obtained from circles centre  $r = a \coth \eta$ ,  $z = 0$  radius  $a \operatorname{cosech} \eta$  and the surfaces of constant  $\xi$  are spheres centre  $r = 0$ ,  $z = a \cot \xi$  radius  $a \operatorname{cosec} \xi$ . The coordinate ranges are  $0 \leq \eta < \infty$ ;  $0 \leq \xi < 2\pi$ ;  $0 \leq \phi < 2\pi$ .

Focus attention on  $T_0$ , the torus  $\eta = \eta_0$ . This has unit normal  $N = \frac{X}{a} \partial_\eta$  and null tangent  $S = \frac{X}{a\sqrt{2}} (\partial_\xi + i \operatorname{cosech} \eta \partial_\phi)$ .

Form a null tetrad with  $S, \bar{S}, D = \frac{1}{\sqrt{2}} (\partial_\xi + N)$  and  $\Delta = \frac{1}{\sqrt{2}} (\partial_\xi - N)$ .

It is a simple matter to write down the two-surface twistor equations on  $T_0$  and separate the  $\phi$ -dependence:

let  $w^0 = X^{-1/2} \exp(i\eta_0) F(\xi)$  and  $w^1 = X^{1/2} \exp(i\eta_0) G(\xi)$

$$\text{then } \left( \frac{d}{d\xi} + \frac{m}{\sinh \eta_0} \right) F = -\frac{1}{2} \coth \eta_0 G; \left( \frac{d}{d\xi} - \frac{m}{\sinh \eta_0} \right) G = \frac{1}{2} \coth \eta_0 F \quad (1)$$

42.

$$\text{whence } \frac{d^2 F}{d \xi^2} = -\omega_m^2 F ; \omega_m^2 = \frac{\cosh^2 \gamma_0 - 4m^2}{4 \sinh^2 \gamma_0} \quad — (2)$$

To get well-defined spinor fields on  $T_0$  we require  $\omega^0$  and  $\omega'$  to change sign when  $\phi \rightarrow \phi + 2\pi$  or  $\xi \rightarrow \xi + 2\pi$ . Thus  $m = \frac{1}{2}k$  and  $\omega_m = \frac{1}{2}n$  for odd integers  $k, n$  and (2) becomes

$$\frac{(n^2-1)}{2} \sinh^2 \gamma = 1 - k^2 \quad — (3)$$

The only solution of (3) is  $k^2=1, n^2=1$  and all  $\gamma$ . This gives four solutions to (1) on each torus of constant  $\gamma$  (and no more!).

Now we wish to perturb the space-time containing  $T_0$  and see what happens to the solutions. The simplest way to do this is to decree that  $\phi$  no longer have period  $2\pi$  but, say,  $2\pi(1+\delta)$ . In other words, we introduce a conical singularity in the  $tz$ -plane. This changes the condition on  $m$  and (3) becomes

$$(n^2-1) \sinh^2 \gamma = 1 - k^2 (1+\delta)^{-2} \quad — (4)$$

At once we see that, if  $\delta < 0$ , there are no solutions to (4) and thus no 2-surface twistors. This small addition of curvature eliminates all solutions!

If instead  $\delta > 0$ , then for each odd  $n \geq 1$  with  $k^2=1$  there is one choice of  $\gamma$  satisfying (4) and so one particular torus with four solutions. Most tori have none but there is a sequence with 4, with increasing eigen-value  $n$ .

The next case of interest is when  $\delta$  reaches 2. (Note that this is a very large periodicity in  $\phi$ !) Solutions with  $n=1, k=\pm 3$  exist on all tori but special ones exist with  $k^2=1$  and  $n$  and  $\gamma$  related by (4). These special tori have 8 solutions.

When  $\delta$  reaches 4, solutions with  $n=1, k=\pm 5$  exist on all tori and extra ones exist with  $k=\pm 1, k=\pm 3$  respectively at solutions of

$$(n_1^2-1) \sinh^2 \gamma = \frac{24}{25} ; (n_2^2-1) \sinh^2 \gamma = \frac{16}{25} \quad — (5)$$

This gives two different families of tori each with 8 solutions. We ask: can these two families intersect to give a torus with 12 solutions? For this we need a simultaneous solution of (5) i.e.

$$3n_1^2 - 2n_2^2 = 1 \quad — (6)$$

(b) has the obvious solution  $(n_2, n_1) = (1, 1)$ . This is not acceptable as a solution of (5), but its existence is important! The way to proceed is to regard (b) as the condition for  $(n_2, n_1)$  to be a unit time-like vector in an integral Lorentzian metric and to seek an integral Lorentz transformation L. Since  $(1, 1)$  is known to be a unit vector, the result of applying L to it any number of times will be also and this will give infinitely many solutions to (b).

In general, with the metric  $A\ell^2 - Bz^2$ , the required Lorentz transformation is  $\begin{pmatrix} a & bB \\ bA & a \end{pmatrix}$  where

$$a^2 - ABb^2 = 1 \quad — (7)$$

which is Pell's equation!

Solutions to (7) are guaranteed to exist provided  $AB$  is not a perfect square and are in fact related to the continued fraction of  $\sqrt{AB}$ .

In this example  $AB = 6$  so  $(a, b) = (5, 2)$  will do. Now  $L = \begin{pmatrix} 5 & 4 \\ 6 & 5 \end{pmatrix}$  and we generate infinitely many solutions to (b) and so infinitely many tori with 12 solutions, by repeatedly applying this to  $(1, 1)^T$ .

At the next stage ( $S=6$ ) there are three equations like (5) and so three infinite families of tori with 8 solutions. Each pair of families gives an equation like (b) so that there are three infinite families with 12 solutions. One might hope that these families would intersect giving infinitely many tori with 16 but in fact this doesn't happen. To see why, consider some of the equations which arise with  $S=8$ :

$$10n_2^2 - 7n_1^2 = 3 ; \quad 5n_3^2 - 2n_1^2 = 3 \quad — (8)$$

We know that each of these has infinitely many solutions in integers, each of which gives a torus with 12 solutions to (1). If we write them as

$$\frac{p^2}{q^2} - \frac{7}{10} = \frac{3}{10q^2} ; \quad \frac{r^2}{q^2} - \frac{2}{5} = \frac{3}{5q^2}$$

then we see that the solutions provide rational approximations to the irrationals  $(\frac{7}{10})^{1/2}$  and  $(\frac{2}{5})^{1/2}$  respectively with accuracy

$O(q^{-2})$ . Simultaneous solutions to (8) would provide

simultaneous rational approximations to the same order, but this is forbidden by a theorem of W.M. Schmidt\*. The simultaneous rational approximation of  $n$  irrationals is allowed precisely to order  $q^{-1-\frac{1}{n}}$ ! Thus (8) and similar sets of simultaneous equations can have at most finitely many simultaneous solutions.

In conclusion, from this particular way of choosing tori in curved space we can't get infinite families with 16, 20, ... solutions. However, there may well be odd ones with more. The largest number I have found corresponds to the solution  $(n_1, n_2, n_3) = (49, 41, 31)$  of (8) which gives a torus with 16 independent 2-surface twistors.

I am grateful to Tony Scholl for telling me about the work of Schmidt and for discussions on number theory.

\* see e.g. the article by him in vol 1, 1970 Int. Cong. of Math. Nice

Paul Tool

### Complexification, Twistor Theory, and Harmonic Maps from Riemann Surfaces

Michael Eastwood

Abstract: Explains complexification so as to motivate, for example, the Ward correspondence for Yang-Mills as a four-dimensional analogue of Cauchy-Riemann. A similar analogy in reverse leads into "twistor theory and harmonic maps from Riemann surfaces" (TM15) with more detail.

### A Duality for Homogeneous Bundles on Twistor Space

Michael Eastwood

Abstract: Explains in more detail "the Penrose transform for homogeneous bundles" (TM16). In particular, the conjecture made in TM16 is nearly true i.e. fails only for certain special (and interesting) cases.

## MORE ON GOOGLIES

This article is (hopefully) an elucidation of some of the points in "A Prosaic Approach to Googlies" (ADH, TN16).

The basic idea is to define a kind of twistor transform which translates the holomorphic leg-break structure on  $\mathbb{P}^*$  to holomorphic googly structure on  $\mathbb{M}$ . Since the leg-break space, even if curved, is locally flat (biholomorphic to nice regions in  $\mathbb{C}^3$  or flat  $\mathbb{P}^*$ ), we concentrate on the local structure of  $\mathbb{P}^*$  to start with. Later, we shall investigate the patching-together of these flat leg-break regions, and what googly operations correspond to this.

### The Twistor Transform

We first review the usual twistor transform. This is defined on  $H^1$ 's, which are essentially non-local objects; they are defined on regions large enough that they are affected by the curvature of the twistor space. (Typically, they are defined on the complement of a neighborhood of a line.) Thus, although this twistor transform is defined on flat twistor spaces, we do not expect it will be easy to generalize it to the case when curvature is present. Nevertheless, it is useful to review this case, because the twistor transform we will eventually define on local functions ( $H^0$ 's, which do not see the curvature of the twistor space) will be an analog of it.

So, suppose  $f(w_\alpha)$  is a dual twistor function with two separated singularities (so it represents an  $H^1$ ), homogeneous of degree -2. Its twistor transform is

$$\oint f \underset{\substack{L \cdot w = 0 \\ M \cdot w = 0}}{=} f(w_\alpha) \frac{Dw}{(w \cdot z)^2} = \text{Diagram} \quad (*)$$

where the contour has an  $S^1$  separating the singularities of  $f(w_\alpha)$ , an  $S^1$  around  $z^\alpha$ , and has boundary on  $\{L \cdot w = 0\} \cup \{M \cdot w = 0\}$ . The twistors  $L, M^\alpha$  are arbitrary (so long as they are generic); any particular values for them yield a particular representative for  $f$ . For example,

$$\oint \frac{1}{c \cdot w D \cdot w} = \frac{(CDLM)}{(CDLz)(CDMz)}$$

The space-time field corresponding to this is

$$\frac{1}{(x-p)^2} \quad \text{where} \quad p^\alpha \leftrightarrow [C^{\mu\nu} D^\beta]$$

independent of  $L^\alpha$  and  $M^\alpha$ .

Although the singularities of  $\mathcal{J} f$  will vary with  $L^\alpha$  and  $M^\alpha$ , the cohomology class determined by  $\mathcal{J} f$  remains the same. Call a point in  $\mathbb{P}$  where  $\mathcal{J} f$  is singular independent of  $L^\alpha$  and  $M^\alpha$  an invariant singularity of  $\mathcal{J} f$ . Then (modulo some technical questions about the complex-analytic structures of the various regions involved),  $\mathcal{J} f$  is an element of  $H^1(\mathbb{P}\text{-invariant singularities}, \mathcal{O}(-2))$ . In the example above, the invariant singularities are those twistors on the line  $CD$ .

If the twistor space is curved, (\*) is not well-defined on  $H^1$ 's, because the integral does not annihilate coboundaries. This is our previous observation that a curved-space analog of (\*) is not obvious.

(\*) will not serve to define a twistor transform on  $H^0$ 's, since arbitrary functions do not have separated singularities and hence no contour exists for (\*) in general. There is a simple generalization of (\*) that will serve, though; for arbitrary  $f(w_\alpha)$ , define

$$\hat{\mathcal{J}} f \equiv \oint_{\substack{L \\ M \\ N}} f(w_\alpha) \frac{dw}{(z-w)^2} = \langle f \rangle \text{---} \begin{array}{c} L \\ M \\ N \\ z \end{array} \quad (\star\star)$$

where the contour surrounds  $z^\alpha$  with an  $S^1$ , and has boundary on  $\{L.W=0\} \cup \{M.W=0\} \cup \{N.W=0\}$ . Again,  $L^\alpha$ ,  $M^\alpha$ ,  $N^\alpha$  are arbitrary (generic) twistors, and any particular values for them yield a particular representative for  $\hat{\mathcal{J}} f$ . Define the invariant singularities of  $\hat{\mathcal{J}} f$  as before. Then,  $\hat{\mathcal{J}} f$  can be interpreted as a kind of  $H^2$  on  $\mathbb{P}$ -invariant singularities. For example,

$$\hat{\mathcal{J}} \frac{1}{(A \cdot w)^2} = \frac{(ALMN)(ZLMN)}{(AZLM)(AZMN)(AZNL)}$$

$$\hat{\mathcal{J}} \frac{1}{A \cdot w B \cdot w} = \frac{(ABLM)}{(ABLZ)(ABMz)} \log \left( \frac{ABLM}{BZLM} \right) + \text{cyc. perm. } \begin{array}{c} L \\ M \\ N \\ Z \end{array}$$

In the first example, the invariant singularity is the point  $A^\alpha$  in  $\mathbb{P}$ , and  $\hat{\mathcal{J}} f$  is an ordinary  $H^2$ . In the second example,

the invariant singularity consists of two points:  $A^\alpha, B^\alpha$  in  $\mathbb{P}$ . In this case the  $H^2$  is slightly more exotic. It is defined on  $\mathbb{P}$ -a cut from  $A^\alpha$  to  $B^\alpha$ . The cut must lie on the line  $AB$  but is otherwise arbitrary, and the  $H^2$  is required to be defined no matter where the cut is taken. (It may be interpreted as a kind of equivariant cohomology element.) These cuts are a general feature of  $\hat{\mathcal{A}}f$  when the function has separated singularities. One can imagine starting from  $(*)$  and pushing the twistor  $z^\alpha$  between the singularities and back to its original value; the integral will then acquire a period, which will be the residue surrounded by the path  $z^\alpha$  traced out.

Finally, note that  $\hat{\mathcal{A}}$  has an inverse:

$$\hat{\mathcal{A}}^{-1} g = \oint_{s' \times s' \times s'} g(z^\alpha) \frac{\partial z}{(z \cdot w)^2} = \text{Diagram showing a loop with a circle containing } g \text{ and arrows indicating integration paths.}$$

### The Googly Space

The results above enable us to identify the structure on  $\mathbb{P}$  corresponding to the local holomorphic structure of  $\mathbb{P}^*$  (knowledge of holomorphic functions on Stein sets in  $\mathbb{P}^*$ ). For each function  $f(w_\alpha)$  on a region in  $\mathbb{P}^*$ , we have an  $H^2$ ,  $\hat{\mathcal{A}}f$ , on a region in  $\mathbb{P}$ . If the complement of the region in  $\mathbb{P}$  is not connected, the  $H^2$  will really be defined on a space with cuts running between the various omitted regions of  $\mathbb{P}$ . Although these  $H^2$ 's might appear a bit frightening at first, they are represented by not-too-unusual twistor functions with certain singularity structures (they must have three separated singularities).

In the leg-break space, we may patch two functions together if they agree over the intersection of their domains. This induces a "patching" on the  $H^2$ 's of the googly space. (It is possible to give an intrinsically googly description of this, even when there is curvature present.) This will be reported more fully in the (near, I hope) future.

Note: The formula announced at the end of the article in TN16 has been shown to be correct to all orders; it gives an intrinsically googly description of  $z^m$  charged fields minimally coupled to a  $s_d$  background.

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A CP<sup>5</sup> CALCULUS FOR SPACE-TIME FIELDS

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ABSTRACT

Compactified Minkowski space can be embedded in projective five-space CP<sup>5</sup> (homogeneous coordinates  $x^i$ ,  $i = 0, \dots, 5$ ) as a four-dimensional quadric hypersurface given by  $\Omega_{ij}x^i x^j = 0$ . Projective twistor space (homogeneous coordinates  $z^\alpha$ ,  $\alpha = 0, \dots, 3$ ) arises via the Klein representation as the space of two-planes lying on this quadric. These two facts of projective geometry form the basis for the construction of a global space-time calculus which makes use of the coordinates  $x^i \longleftrightarrow x^{ab}$  ( $= -x^{ba}$ ) to represent spinor and tensor fields in a manifestly conformally covariant form. This calculus can be regarded as a synthesis of work on conformal geometry by Veblen, Dirac, and others, with the theory of twistors developed by Penrose.

NON-LINEAR CONNECTIONS FOR

CURVED TWISTOR SPACES

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We provide here a systematic review of the basic framework: the underlying projective geometry; the calculus of tensor fields; the characterization of spinors as twistor-valued fields  $\psi^\alpha(X)$  which satisfy a geometrical condition ( $\psi^\alpha X_{\alpha\beta} = 0$  on  $\Omega$ ); and the introduction of the conformally invariant Laplacian operator  $\nabla^2 = \Omega^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j}$ . In addition a number of subsidiary topics are discussed which illustrate the general scheme, including: a derivation of the zero rest mass equations for all helicities; and a new and manifestly conformally covariant form of the twistor contour integral formulae for massless fields.

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