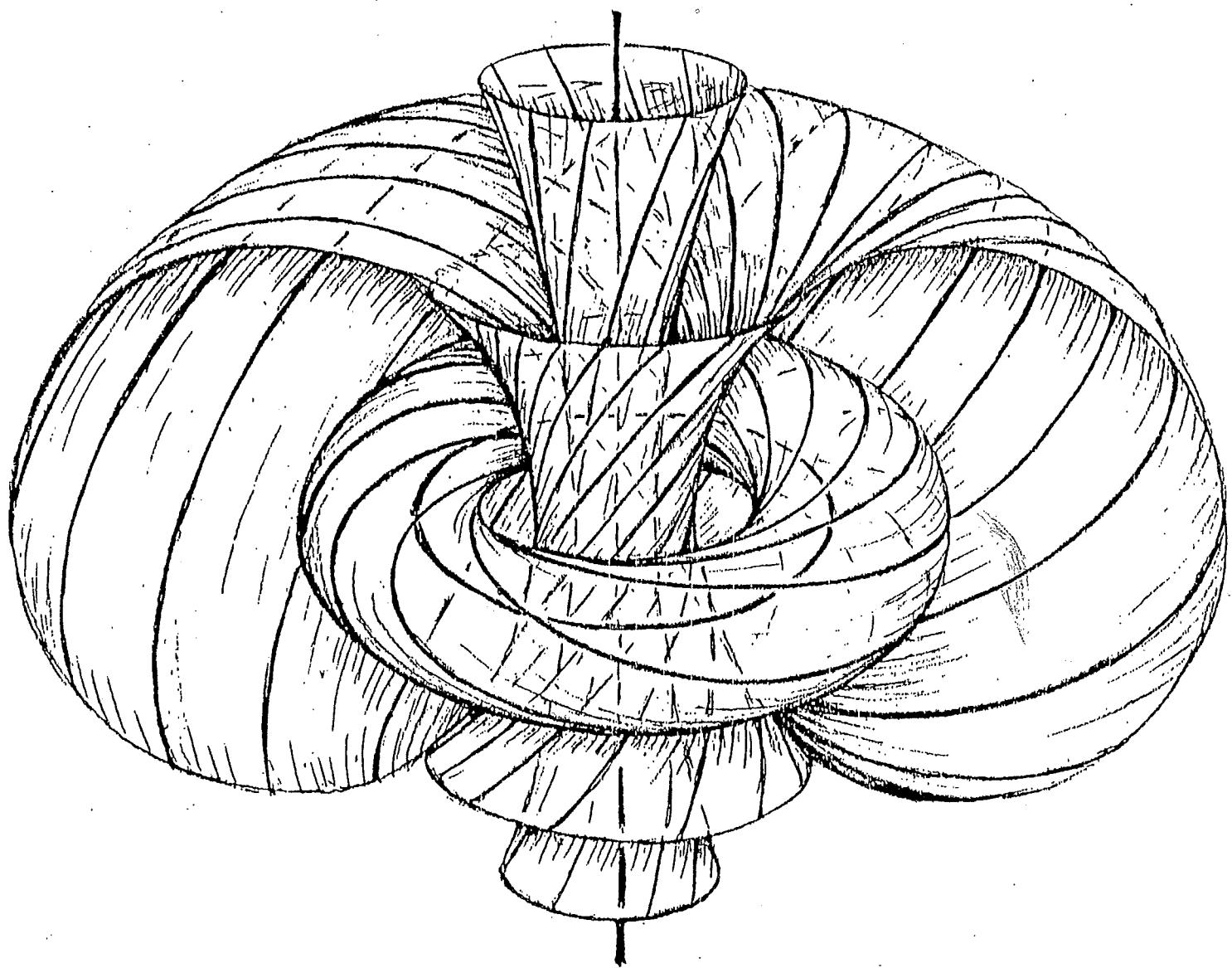


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## More on Quasi-Local Mass.

In [1], I calculated some examples of Penrose's quasi-local mass for a variety of specific two-surfaces in particular space-times. What made the calculations manageable was that each of the two-surfaces considered could be embedded in conformally flat space with the same first and second fundamental forms. Call such a two-surface conformally embeddable (or c.e.). Then evidently on a c.e. two surface the "usual twistor norm":

$$\Sigma = \omega^A \bar{\pi}_A + \bar{\omega}^A \pi_A \quad (1)$$

is actually constant when  $(\omega^A, \pi_A)$  satisfy the two-surface twistor equations, and so  $\Sigma$  defines a norm on  $\mathbb{T}(S)$ , the two-surface twistor space of the two-surface  $S$ . The converse of this statement is also true so that the norm defined by (1) is constant on  $S$  iff  $S$  is c.e.. This has been shown by Ben Jeffreys and myself.

Thus c.e. surfaces can be regarded as understood at least as far as the quasi-local mass is concerned. (There is still e.g. the question of angular momentum)

For non-c.e. surfaces, which Roger Penrose suggests we call "contorted", things are more difficult.

Another question raised in [1] was the calculation of the quasi-local mass for a "small sphere". Recall that to define a small sphere you pick a time-like unit vector  $t^a$  at a point  $p$  in space-time. Now use  $t^a$  to normalise the null vectors at  $p$ . If  $r$  is the affine parameter on the null cone at  $p$  then a small sphere is a surface of constant (small)  $r$ . Write  $\mathbb{T}(S(r))$  for the twistor space of the small sphere  $S(r)$ .

Then one can seek to calculate  $A_{\alpha\beta}(r)$  as a power series in  $r$ . This is a very messy calculation! In [1]

I remarked that the first non-zero term was  $O(r^5)$  and quadratic in the Weyl tensor, though I didn't know at that time the precise form of it. However, I eventually worked it out and the calculation has also been done by Ron Kelly (and we agree on the answer!).

There is an obvious coordinatisation of  $\mathbb{I}(S(r))$  and in this coordinatisation  $A_{\alpha\beta}$  has the form

$$A_{\alpha\beta} = r^5 \begin{pmatrix} \gamma_{AB} & P_B^{A'} \\ P_A^{B'} & 0 \end{pmatrix} + O(r^6)$$

$$\text{with } P_a = \frac{c^2}{9DG} \gamma_{ABCD} X_{A'B'C'D'} t^{B'B'} t^{C'C'} t^{D'D'} \quad (2)$$

$$\text{where } X_{A'B'C'D'} = T_{A'B'C'D'} - 4 \gamma_{ABCD} t^A t^{B'} t^C t^{D'} = -4i H_{AA'BB'} t^A t^{B'} \quad (3)$$

It is tempting to call  $P_a$  the momentum, but this would not have invariant significance. To calculate the norm of  $A_{\alpha\beta}$  one needs the norm on  $\mathbb{I}(S(r))$ . Since (2) is the first non-zero term in  $A_{\alpha\beta}$ , we can use the first term in  $\Sigma$  given by (1) which is constant on  $S$ . We find that  $M_p^2$ , the Penrose mass, is just  $P_a P^a$  so that  $\gamma_{AB}$  is not needed for this. (Note that  $\gamma_{AB}$  is not to be thought of as angular momentum since it is in the "wrong" corner!)

Now, this is rather bad news. To see why, consider the case of small spheres in the Schwarzschild solution.

Any three-surface of spherical symmetry (i.e. of the form  $f(r, t) = 0$ ) can be shown to admit three-surface twistors. Consequently any two-surface in a spherically symmetric three-surface gives one of two answers for  $M_p^2$ : either  $M_s^2$ , the Schwarzschild mass, if it goes round the hole or zero if it doesn't. (In fact it can also be shown that these are precisely the non-contorted two-surfaces). If the vector  $t^a$  used to define the small spheres lies in the  $(t, r)$ -plane then

the small spheres will be non-contorted and  $P_a$  will have to be zero. This is a strong constraint on  $P_a$  but from ② and ③ it can be seen to be satisfied (In ③,  $H_{ab} = H_{AA'B'B'}$  is the magnetic part of the Weyl tensor at the point  $p$ .) However if  $t^a$  does not lie in the  $(t, p)$  plane then ② turns out to define a space-like vector so that  $M_p^2 < 0$ .

What should we make of this? Given the earlier successes of the construction with non-contorted surfaces the obvious idea is to find some kind of modification for contorted surfaces. The small spheres are contorted in that the norm defined by ① is constant at  $O(1)$  but varies at  $O(r)$ . One anticipates that a complex conformal transformation might help matters! Recall that under the conformal transformations  $E_{AB} \rightarrow S E_{AB}$ ;  $E_{A'B'} \rightarrow \tilde{S} E_{A'B'}$  where both  $S$  and  $\tilde{S}$  are complex we find

$$\Sigma \rightarrow \Sigma + i(\Upsilon_{AA'} - \bar{\Upsilon}_{AA'})\omega^A \bar{\omega}^{A'} \text{ where } \Upsilon_a = \nabla_a S$$

and  $E_{ABRS} \rightarrow S \tilde{S}^{-1} E_{ABRS}$  for the twistor four-form.

For the small spheres, Ron Kelly and I find that

$$\Sigma + i(\Upsilon_a - \bar{\Upsilon}_a)\omega^A \bar{\omega}^{A'} = \text{constant} + O(r^2)$$

$$y \quad S = 1 + i \frac{r^2}{2} H_{ab} l^a l^b \quad (4)$$

where  $H_{ab}$  is as before and  $l^a$  is the normalised null vector at  $p$ . Thus the small sphere  $S(r)$  is contorted by an amount related to the magnetic part of the Weyl tensor.

Next we consider the determinant of four solutions of the two-surface twistor equation; i.e. the determinant of the  $4 \times 4$  matrix whose rows are the four sets of  $(\omega^0, \omega^1, T_0^1, T_1^1)$ . If  $S$  is non-contorted, this is a constant on  $S$  and defines  $E_{ABRS}$ . We find

$$\det = \text{const.} \times \left(1 - i \frac{r^2}{3} H_{ab} l^a l^b\right) \quad (5)$$

to this order.

$$\text{Thus } S\tilde{\Omega}^{-1} = 1 + \frac{i\pi^2}{3} H a b l^a l^b + \text{higher order}$$

and again the conformal factors are related to the magnetic part of the Weyl tensor.

Now returning to the quasi-local mass, the suggested modification is to include in the integral a factor  $\tilde{\Omega} \Omega^{-1}$  to undo the effect of the contortedness on the four-form! i.e. to define

$$A_{\alpha\beta} Z^\alpha Z^\beta = -\frac{i\pi^2}{4\pi G} \int \Psi_{ABCD} \omega^{AB} (\tilde{\Omega} \Omega^{-1}) d\sigma^{CD}.$$

Applied to the small sphere by Ron Kelly this precisely eliminates the  $O(r^5)$  term in  $P_a$ ! Thus the mass becomes zero at this order.

Of course, there suggestions (which are under active investigation!) raise almost as many questions as they answer but they do indicate a way to deal with contorted surfaces.

[1]: Some examples of Penrose's quasi-local mass construction.

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## "New improved" quasi-local mass, and the Schwarzschild solution

As has been pointed out by K.P.T. in an accompanying article (TN 18, p. 3), my original definition of quasi-local mass  $m(S)$ , for spacelike  $S \cong S^2$  in a general space-time  $M$  (TN 13, p. 2; Proc. R. Soc. Lond. A 381 (1982) 53-63), seems to run into difficulties in cases when the spacelike 2-surface  $S$  is contorted (i.e. not embeddable in a conformally flat space-time without changing its intrinsic or extrinsic curvatures). In view of the excellent results that K.P.T. had earlier obtained with that definition (Proc. R. Soc. Lond. A 388 (1983) 457-477; cf. also W.T.S., Proc. R. Soc. Lond. A 390 (1984) 191-215) it would seem most unreasonable simply to abandon the quasi-local mass, but rather one should be seeking some simple modification of the original definition which gives agreement with these earlier results and which avoids the anomalies that can arise in contorted cases.

One of the more striking results that K.P.T. had earlier obtained was that for any  $S$  drawn in any 3-surface of revolution  $t=f(r)$  in the Schwarzschild solution  $M_m$ , for mass  $m$ , the (original) quasi-local mass is always zero if  $S$  does not surround the source and is always  $m$  if  $S$  does surround it (just once). These are all uncontorted surfaces  $S$ . The natural conjecture had been that the same result ought to hold if  $S$  does not lie in any  $t=f(r)$  (and is therefore contorted). If one accepts the results for uncontorted  $S$ 's in  $M_m$  then any other result in the contorted cases seems physically unreasonable. The result  $(m(S))^2 < 0$  for (certain) small contorted spheres, that K.P.T. actually obtained (and was confirmed by R.M.K.) simply adds to this physical unreasonableness.

Just before the results of K.P.T.'s calculation were at hand, I had embarked on a seemingly promising-looking

line of reasoning aimed at proving the above conjecture. But then K.P.T.'s results had appeared to invalidate this (then partly conjectural) line of argument. On re-examining my argument somewhat later I was surprised to find that it actually leads to a modification of my original definition for  $A_{\alpha\beta}(\mathcal{S})$ . R.M.K. and K.P.T. then found that with such a modified definition,  $m(\mathcal{S})$  indeed vanishes to 5<sup>th</sup> order for small spheres in (e.g.)  $M_m$ , as it should.

My argument depends upon the existence of a Killing spinor\*

$$X_{AB} = \psi^{-\frac{1}{3}} \alpha_{(A} \beta_{B)} , \text{ where } \square_{ABCD} = \psi \alpha_{(A} \alpha_{B} \beta_{C} \beta_{D)}, \alpha_A \beta^A = 1$$

in  $M_m$  (cf. M.Walker & R.P., Commun. Math. Phys. 18 (1970) 265–274), which satisfies

$$\nabla_{A'(A} X_{BC)} = 0.$$

This much holds also in any {22} vacuum (e.g. Kerr, NUT, C-metric). But I also require:

**Property K** The field  $X^{AB}$ , restricted to  $\mathcal{S}$ , belongs to the symmetric tensor product of the 2-surface twistor space  $\Pi(\mathcal{S})$  with itself.

The argument consists of two parts.

(1) Show that if property K holds, then the modified (i.e. "new improved") quasi-local mass expression vanishes if  $\mathcal{S}$  does not surround the source and gives the Schwarzschild mass  $m$  if  $\mathcal{S}$  does surround it (just once). [The latter still presents problems.]

(2) Show that property K holds for the Schwarzschild solution.

Until a few days ago, (2) was merely a tentative conjecture. Now there appears to be a good plausibility argument which, with some care and attention, can perhaps be made into a full proof.

\* The sign " $\equiv$ " means that the L.H.S and the R.H.S. are related by a constant factor which I haven't been bothered to sort out yet.

Part (1) We work in  $T^*(S)$ , which is a flat (dual) twistor space, and consider the twistor contour integral

$$(A)_{\alpha\beta} = \oint \frac{W_\alpha W_\beta d^4W}{(K^{\mu\nu} W_\mu W_\nu)^3}. \quad (A)$$

Here  $K^{\alpha\beta}$  is a twistor in  $T(S) \odot T(S)$  whose primary part agrees with  $\chi^{AB}$  on  $S$ . For this we need property  $K$ . Also we need  $S$  to be embedded in  $M(S)$  — the complex compact Minkowski space associated with  $T^*(S)$  (i.e. with  $T(S)$ ). The embedding is easily achieved, where we identify each point  $P$  of  $S$  with the linear 2-space in  $T^*_P(S)$  whose elements take the local form  $(\lambda_A, 0)$  at  $P$  (and so annihilate the 2-surface twistor fields  $w^A$  which vanish at  $P$ ). We also need a choice of twistor  $\epsilon^{\alpha\beta\gamma\delta}$  to define the  $d^4W$ :

$$d^4W = \frac{1}{24} dW_\alpha dW_\beta dW_\gamma dW_\delta \epsilon^{\alpha\beta\gamma\delta}. \quad (B)$$

To perform the integral (A) we can introduce a basis  $\delta_0^\alpha, \delta_1^\alpha, \delta_2^\alpha, \delta_3^\alpha$  for  $T(S)$  and take components with respect to it:

$$\lambda_0 = W_0 = W_\alpha \delta_0^\alpha, \lambda_1 = W_1 = W_\alpha \delta_1^\alpha, \lambda^0 = W_2 = W_\alpha \delta_2^\alpha, \lambda' = W_3 = W_\alpha \delta_3^\alpha. \quad (C)$$

Note that each of  $\delta_0^\alpha, \dots, \delta_3^\alpha$  is a 2-surface twistor and so has a description in terms of " $w^0, w^1, \pi_0, \pi_1$ " which varies over  $S$  according to the standard 2-surface twistor equations on  $S$ . If at one point  $Q$  on  $S$  we choose

$$\epsilon^{\alpha\beta\gamma\delta} = 24 \delta_0^\alpha \delta_1^\beta \delta_2^\gamma \delta_3^\delta, \text{ so } \epsilon^{0123} = 1 \quad (D)$$

then this equation will hold at all points of  $S$ . But in terms of the standard local " $w^0, \dots, \pi_1$ " on  $S$  we shall find that the components of the spinor parts ( $\dots, \epsilon^{AB} \epsilon_{RS}, \dots$ ) may vary from point to point on  $S$ . If  $S$  is uncurved then these components are in fact constant over  $S$ . But in the general case they vary. There is just one overall factor to be concerned with — a scalar field  $v$  on  $S$ . Then the components of these spinor parts are just  $\pm v$  and 0. (This  $v$  is just the  $\tilde{\Omega} \Omega^{-1}$  of K.P.T.'s article.) K.P.T. conjectures that  $v$  has constant modulus over  $S$ . This would be important. Then fixing  $v=1$  at  $Q$ , say, we should have  $v$  of unit modulus over  $S$ .

Reverting, for the moment, to twistor components with respect to the "constant" twistor frame of  $\mathcal{C}$ , we can write

$$u^A = -ix^{AA'}\lambda_A \quad (E)$$

where  $x^{AA'}$  are Minkowski coordinates for  $M(\mathcal{S})$ . Then we can write

$$d^4W = d^2\lambda_A dx^{AA'}_A = -\lambda_A \lambda_B dx^{AA'}_A dx^{BB'}_B d\lambda_B d\lambda, \quad (F)$$

and perform the  $\lambda$ -integral first, in (A), to obtain

$$\begin{aligned} (A)_{\alpha\beta} Z^\alpha Z^\beta &= \oint \frac{w^A w^B \lambda_A \lambda_B \lambda_C \lambda_D}{(x^{aa} \lambda_p \lambda_q)^3} dx^{c0}_C dx^{d1}_D d\lambda_C d\lambda_D \\ &= \oint \Psi_{ABCD} w^A w^B dx^{c0}_C dx^{d1}_D \end{aligned} \quad (G)$$

since

$$\Psi_{ABCD} = \oint \frac{\lambda_A \lambda_B \lambda_C \lambda_D}{(x^{aa} \lambda_p \lambda_q)^3} d\lambda_C d\lambda_D \quad \left[ \begin{array}{l} \text{pt. of } \mathcal{S} \\ \text{fixed} \end{array} \right] \quad (H)$$

(by direct calculation, if desired). Now the expression in the final R.H.S. of (G) is just the standard (original) quasi-local mass expression (in the case of vacuum) except for the fact that it is the differentials  $dx^{c0}_C dx^{d1}_D$ , for the embedding of  $\mathcal{S}$  in  $M(\mathcal{S})$ , which appear, rather than the local surface element for  $\mathcal{S}$ . In order to rewrite this integral in terms of this local surface element, we now return to descriptions in terms of the local  $(\mathbb{C}^4, \mathbb{I}^4)$  spinor frame. This entails that the factor  $\nu$  now appears in the integral (G).

Guessing that this applies generally for an arbitrary space-time which need not be vacuum, we are led to the "new improved" quasi-local mass-angular momentum object

$$(A)_{\alpha\beta} Z^\alpha Z^\beta = \frac{-i}{4\pi G} \oint \nu \{ (\bar{\Psi}_1 \cdot \bar{\Psi}_0) w^0 w^0 + 2(\bar{\Psi}_2 \cdot \bar{\Psi}_1 - 1) w^0 w^1 + (\bar{\Psi}_3 \cdot \bar{\Psi}_2) w^1 w^1 \} \underset{\text{surf. area element.}}{\underset{\mathcal{S}}{\int}}. \quad (I)$$

If  $\nu$  is indeed of unit modulus, in accordance with H.P.T.'s conjecture, then (I) seems well-enough defined for its necessary purposes. Otherwise some procedure would be needed to fix  $|\nu|$ , e.g. normalizing so that the average  $|\nu|$  over  $\mathcal{S}$  is unity.

Now, by the methods of standard flat-space twistor theory (cf. R.P. & M.A.H. MacCallum, Phys. Repts. 6C no. 4 (1973) 241) we can evaluate (A), and find

$$A_{AB} = \frac{K^{(-1)} \times \beta}{\sqrt{\det K^{AB}}} \quad \text{or} \quad 0 \quad (J)$$

according as:  $\mathcal{S}$  surrounds the source (once); or else can be shrunk to a point without crossing the source region. To compute the mass from (J) when the source is linked we still have some problems: Perhaps K.P.T.'s method of forming a determinant and then taking  $| \dots |^{1/4}$  may be the best. But seems to need K.P.T. conjecture.

Part (2) (Outline of proposed method only.) We need a usable method of distinguishing Schwarzschild case from C-metric (since we know  $A_{AB} \neq 0$  in case:  ) and also from Kerr case (since W.T.S. have shown prop. K false for Kerr). I use the fact that  $M_m$  has 3 rotational Killing vectors  $x^a, y^a, z^a$ , with

$$X_{AB} \tilde{X}_{A'B'} = X_{AA'} X_{BB'} + Y_{AA'} Y_{BB'} + Z_{AA'} Z_{BB'} + \lambda E_{AB} E_{A'B'} \quad (K)$$

(Here  $\tilde{X}_{A'B'}$  is the complex conjugate of  $X_{AB}$ , but will need to be "fixed" from it shortly.) The tangential parts of  $\nabla_{A'}(A'X_{BC}) = 0, \nabla_{A'}(\tilde{X}_{B'C'}) = 0, \nabla_{(A}(X_{B})_{|B}) = 0, \dots, y, \dots, z, \dots$  give  $\partial X_{00} + 2\sigma X_{10} = 0, \partial X_{11} + 2\sigma X_{01} = 0, \dots, \tilde{x}, \dots, \partial X_{01} + \partial X_{11} + \bar{\sigma}' X_{00} = 0, \dots, y, \dots, z, \dots$ , which provide 10 differential equations relating derivatives of the 10 "nice" components  $X_{00}, X_{11}, \tilde{X}_{00'}, X_{11'}, X_{01}, X_{10}, Y_{01}, Y_{10}, Z_{01}, Z_{10}$  to the 9 "nasty" components  $X_{01}, \tilde{X}_{01'}, X_{00'}, Y_{11'}, Y_{00'}, Y_{11}, Z_{00'}, Z_{11}, \lambda$ . From (K) we have 2 equations  $X_{00}\tilde{X}_{01'} = X_{01}^2 + Y_{01}^2 + Z_{01}^2, X_{11}\tilde{X}_{00'} = X_{01'}^2 + \dots + \dots$  relating only nice components and 6 eqns.  $X_{00}\tilde{X}_{01'} = X_{01}X_{00'} + Y_{01}Y_{00'} + Z_{01}Z_{00'}$ , which relate nasty ones to nice ones. 2 more eqns. on nasty components are provided by

$$X_{01}^2 - X_{00}X_{11} = \lambda = \tilde{X}_{01} - \tilde{X}_{00'}\tilde{X}_{11'} \quad (L)$$

and we also have, from the skew parts of (K),

$$X_{01}\tilde{X}_{01'} = X_{01'}X_{10} + Y_{01}Y_{10} + Z_{01}Z_{10} - \lambda = X_{00}X_{11'} + Y_{00}Y_{11} + Z_{00}Z_{11'} + \lambda \quad (M)$$

though only one of these is independent of (L), in effect. In principle, we could use these  $6+2+1$  equations to express all the nasty cpts. in terms of nice ones and substitute back into the 10 differential equations. The question is: how big is the solution space? The idea is to show that it is no bigger than in the flat case, where we know that all such  $X$ 's,  $\tilde{X}$ 's,  $x$ 's,  $y$ 's,  $z$ 's do belong to their proper  $T^1 \otimes T^1, T^1 \otimes T^1, T^1 \otimes T^1, T^1 \otimes T^1$ . Then since, in  $M_m$ , all of  $X_{AB}, \tilde{X}_{A'B'}, X_{AA'}, Y_{AA'}, Z_{AA'}$  actually do satisfy all the equations, it must follow that they belong to the required spaces  $T^1(S) \otimes T^1(S)$ , etc. To see how big the solution space actually is, first solve in the "canonical" case when  $\mathcal{S}$  is spherically symmetrically situated, so  $\sigma = 0, \sigma' = 0$ . The diff. eqns. decouple and can be solved explicitly for the nice cpts. From (L) and (M) we can find  $X_{01}$  and  $\tilde{X}_{01'}$ , in terms of nice cpts. and then solve for the remaining nasty ones. One must then invoke a general argument to show that no new solutions appear on perturbation. Work is in progress. Thanks to Paul Tod (=K.P.T.), Ron Kelly (=R.M.K) & William Shaw (=W.T.S.) & Roger Penrose.

## On Bryant's condition for holomorphic curves in CR-spaces

I am concerned here with 5-dimensional CR-manifolds and the conditions on such a manifold  $\mathcal{C}$  that it be the space of projective null twistors for a hypersurface twistor space:  $\mathcal{C} = \text{PN}(\mathcal{H})$ .

$\mathcal{H}$  is a spacelike hypersurface in a (real) space-time manifold  $M$  (where neither need be assumed analytic). Each point of  $\mathcal{H}$  gives rise to a holomorphic curve ( $\cong S^2$ ) in  $\mathcal{C}$ , so  $\mathcal{C}$  is fibred by this 3-real-parameter family of holomorphic curves. When  $M$  is conformally flat, this extends to a 4-real-parameter family of holomorphic curves (corresponding to the points of  $M$ ). LeBrun has investigated cases of non-conformally flat  $M$ 's where  $\text{PN}(\mathcal{H})$  contains holomorphic curves other than those corresponding to points of  $\mathcal{H}$ ; this does not happen for the generic  $\text{PN}(\mathcal{H})$ .

Robert Bryant has shown that if  $\mathcal{C}$  is merely given as a  $(+ -)$ -Levi-signature CR-5-manifold, then a necessary and sufficient condition that it be (locally) foliated by holomorphic curves is that a certain quartic polynomial  $P$ , constructed from the Chern-Moser "S-tensor" should have a repeated root. Thus it follows that every  $\text{PN}(\mathcal{H})$  must satisfy Bryant's repeated root condition. A rough dimension count indicated that this condition would not be sufficient to characterize  $\text{PN}(\mathcal{H})$ 's, and a complete characterization of such spaces was left open.

I shall give an argument to show that in any  $\text{PN}(\mathcal{H})$ ,  $P$  must actually have (at least) a triple root.

Consider a point  $X$  of  $\mathcal{C} = \text{PN}(\mathcal{H})$ . The roots of  $P$  at  $X$  correspond to the possible choices of 2-surface element  $E$  at  $X$  which, "to 3<sup>rd</sup> order", could be tangent to a holomorphic curve, i.e.,  $E$  is tangent to a smooth 2-surface  $E$ , in  $\mathcal{C}$ , where  $E$  is holomorphic to 3<sup>rd</sup> order

at  $X$ . Corresponding to  $E$ , there is a smooth 2-surface  $\mathcal{S}$  in  $\mathcal{M}$  and a field of null directions (in  $\mathcal{M}$ ) over  $\mathcal{S}$  — or else  $\mathcal{S}$  degenerates at  $X$  (as would be the case when  $E$  actually corresponds to a point of  $\mathcal{M}$ ). When  $\mathcal{S}$  is indeed a smooth 2-surface (at  $X$ ), the condition for holomorphicity at  $X$  is that: (a) the specified null directions at  $\mathcal{S}$  are tangents to a null hypersurface  $\mathcal{L}$  in  $\mathcal{M}$  which intersects  $\mathcal{S}$  locally in  $\mathcal{S}$  (condition of "abreastness" of the null rays, arising since the 1-tangents to  $E$  lie in the holomorphic tangent space to  $\mathcal{L}$  at  $X$ ); and (b) that the complex shear  $\sigma$  of  $\mathcal{L}$  at  $X$ , together with its 1<sup>st</sup> and 2<sup>nd</sup> derivatives in directions tangent to  $\mathcal{S}$ , should vanish.

I use the compacted spin-coefficient (G.H.P.) formalism. The flagpole of  $O^A$  at each point of  $\mathcal{S}$  is chosen in the specified null direction; the flagpole of  $I^A$  is taken in the other null direction orthogonal to  $\mathcal{S}$ . We have

$$\partial\rho - \bar{\partial}'\sigma = (\sigma\bar{\sigma})\tau + (\bar{\sigma}'-\sigma')\pi - \Phi_1 + \bar{\Phi}_{01}.$$

Apply  $\bar{\partial}'$ , then take the difference between the resulting expression and its complex conj., using commutator relations for  $\bar{\partial}', \partial'$ , to obtain:

$$\bar{\partial}'^2\sigma - \bar{\partial}^2\bar{\sigma} = \bar{\partial}'\Phi_1 - \bar{\partial}'\bar{\Phi}_{01} - \bar{\partial}\Phi_1 + \bar{\partial}\bar{\Phi}_{10} + \rho(\sigma\bar{\sigma}' - \bar{\sigma}\sigma' - \Phi_2 + \bar{\Phi}_2)$$

For a "holomorphic curve to 3<sup>rd</sup> order" we require  $\sigma=0, \dots, \bar{\sigma}^2\sigma=0$ , whence

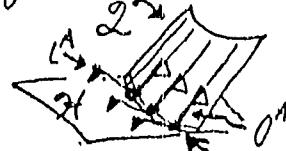
$$\rho = \frac{\text{Im}(\bar{\partial}'\Phi_1 - \bar{\partial}'\bar{\Phi}_{01})}{\text{Im}\Phi_2}$$

Note that in the general case, when  $\text{Im}\Phi_2 \neq 0$  there is only one such  $\rho$ . Now  $\rho (= \bar{\rho})$  and  $\sigma (= 0)$  fix the direction of  $E$  at  $X$ , so in the general case we get just one root of  $P$  corresponding to smooth  $\mathcal{S}$ .

It is easy enough to see that (because  $\sigma=0$  at  $X$ ) the "degenerate" cases of non-smooth  $\mathcal{S}$  at  $X$  arise only when the holomorphic curve is one arising from a point of  $\mathcal{M}$ . These correspond to Bryant's repeated root of  $P$ . Since  $P$  is quartic, the only possible coincidence schemes are  $\{211\}, \{22\}, \{31\}, \{4\}, \{-\}$ . Of these, only  $\{22\}$  and  $\{31\}$  allow just a single root for smooth  $\mathcal{S}$ . But, as LeBrun points out,  $\{2, 2\}$  is ruled out in the general case since we would then have a 2<sup>nd</sup> family of holomorphic curves. Hence,  $\{31\}$  is, indeed, the general case (with  $\{4\}$  or  $\{-\}$  in special situations).

Thanks to Peter Thompson and Claude LeBrun. ~ Roger Penrose

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A NEW VIEWPOINT ON THE ANGULAR MOMENTUM  
TWISTOR IN G.R.

The angular momentum twistor of a 2-surface  $S$  in a spacetime  $M$  is usually taken to describe the momentum and angular momentum of the source and gravitational field on a 3-surface spanning  $S$ .

In symplectic mechanics we have a direct correspondance between observables and canonical transformations of the phase space in question. This is given by lie dragging along a vector field  $X_H$  given by:

$$X_H \lrcorner \omega + dH = 0$$

Where  $\omega$  is the symplectic form on the phase space and  $H$  is the observable in question.

Observables such as angular momentum and momentum generate canonical transformations that correspond to lie dragging the field along a killing vector. (The phase space we are interested in is that of the gravitational field and source fields, this is usually given by initial data on some 3-surface).

The purpose of this note is to show how an analogous viewpoint can be taken for the angular momentum twistor in G.R..

In curved space we, of course, don't in general have any killing vectors. However a 2-surface twistor provides us with a notion of a killing vector at the 2-surface:  $Z^A = (\omega^A, \tau_{AA'}) \rightarrow K^{AA'} = \omega^A \tau_{AA'}$

It turns out that this is in fact enough to define the corresponding component of momentum/angular momentum of the gravitational field and source linking the 2-surface. This is because the total hamiltonian density is, in fact, an exact 3-form and therefore the

value of the hamiltonian only depends on the values of the "Quasi killing vector" on the bounding 2-surface.

In the usual hamiltonian treatment of G.K. one casts off boundary terms at will. Clearly we must keep track of them for the above approach to make sense. Regge & Teitelboim established a sound criteria for finding the correct surface integral:

If  $H$  has the wrong surface integral attached to it

$$\delta H = \int_S g_{ij} S T^{ij} - \pi^{ij} S g_{ij} \sqrt{g} d^3x + \text{a surface integral}$$

Clearly Hamilton's equations will only hold if the surface integral vanishes. As explained in TN 17 the Sparling 3-form provides the hamiltonian density satisfying this criteria. (The Sparling 3-form,  $\Gamma$ , is defined on the spin bundle, to evaluate the hamiltonian we restrict  $\Gamma$  to a section of the spin bundle,  $\bar{\Pi}_A = \alpha_A(x)$  where  $\alpha_A \bar{\alpha}_A$  defines the vector field corresponding to the desired Lapse and shift, and then we integrate  $\Gamma$  over this section, restricted to a hypersurface  $\Sigma$  of the spacetime).

The important equation satisfied by the Sparling 3-form is:

$$\Gamma = i d\bar{\Pi}_A \wedge d\bar{\Pi}_A \wedge dx^{AA'} = d(i\bar{\Pi}_A d\bar{\Pi}_A \wedge dx^{AA'}) + \bar{\Pi}_A \bar{\Pi}_{A'} G^{AA'}_L d^3x^b$$

Adding on the matter hamiltonian density  $\bar{\Pi}_A \bar{\Pi}_{A'} T^{AA'}_b d^3x^b$  yields the total hamiltonian density,  $\delta H_{tot}$ . Applying the Einstein field equations yields:

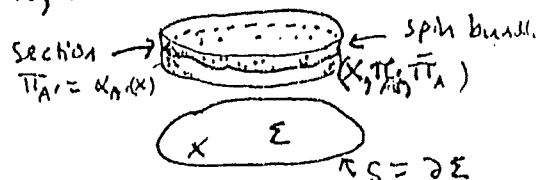
$$\delta H_{tot} = d(i\bar{\Pi}_A d\bar{\Pi}_A \wedge dx^{AA'})$$

So the hamiltonian,  $H_{tot} = \int_S \delta H_{tot} = \oint_S i\bar{\Pi}_A d\bar{\Pi}_A \wedge dx^{AA'}$  as promised. The "Quasi-killing vector" at  $S$  is complex, so we must complexify the spin bundle, let  $\bar{\Pi}_A \rightarrow \tilde{\Pi}_A$ , and we must set  $\bar{\Pi}_A = \Pi_A$  (abus of notation above) and  $\tilde{\Pi}_A = \omega_A$ .

$$\text{Now } d\tilde{\Pi}^A |_{S, \tilde{\Pi}^A = \omega_A} = \nabla_{AB} \omega^A dx^{BB'} \approx \epsilon^A_S \tilde{\Pi}_{S'} dx^{BB'}$$

$$\text{So: } H_{tot} = \oint_S i\bar{\Pi}_A \bar{\Pi}_{A'} d^2x^{AA'}$$

Which is K.P.T's formula for  $A_{AB} Z^A Z^B$ , as promised. L.J.Mason



## The Hill-Penrose-Sparks C.R.-folds

In "Physical space-time and non-realizable C.R.-structures", Bull. A.M.S. 8 (1983), 427-448, R.P. gives a construction (due to C.D.H., R.P., & G.A.J.S.) of non-realizable C.R.-manifolds as follows. Suppose  $M$  is a 3-dimensional manifold with C.R.-structure defined by a vector field  $X$  and suppose that  $g$  is a smooth function on  $M$  such that  $Xf = g$  has no solution for smooth  $f$  even locally on  $M$  (there are lots of these (H. Lewy)). Then  $g$  represents a non-zero local  $\bar{\partial}_b H'$  which can be "exponentiated" to a C.R.-line-bundle the total space of which provides a non-realizable example. A simple proof is as follows. The total space is  $M \times \mathbb{C}$  with C.R.-structure defined by the vector fields  $\{V + g\bar{z} \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\}$  for  $\bar{z}$  the coordinate on  $\mathbb{C}$ . In other words C.R.-functions must be annihilated by these. But  $\frac{\partial f}{\partial \bar{z}} = 0 \Rightarrow f$  has an expansion  $f = \sum_{j=0}^{\infty} f_j \bar{z}^j$  where  $f_j$  are functions on  $M$ . Thus  $Vf_j + jgf_j = 0$  and so  $f_j = 0$  for  $j > 0$  for otherwise take logs to solve  $Xf = g$ . It follows that  $f$  is independent of  $\bar{z}$  and, in particular, C.R.-functions don't locally separate points.

## Superambitwistor

The first formal neighbourhood  $A_{(1)}$  of ambitwistor space may be regarded, roughly speaking, as the space of variables  $(Z^\alpha, W_\alpha, X)$  under the homogeneous equivalence relation  $(Z^\alpha, W_\alpha, X) \sim (\lambda Z^\alpha, \mu W_\alpha, \lambda\mu X)$  and constraints  $Z.W = X \propto X^2 = 0$ . Superambitwistor space (for  $N=1$ ) (cf. Witten: "An interpretation of classical Yang-Mills theory", Phys. Lett. 77B (1978), 394-398) is similarly  $(Z^\alpha, J, W_\alpha, \psi) \sim (\lambda Z^\alpha, \lambda J, \mu W_\alpha, \mu\psi)$  s.t.  $Z.W = 2J\psi$  but  $J \propto \psi$  are "Fermionic" i.e. anticommuting. Denote this space by  $A_{[1]}$ . It can be more rigorously defined and, in particular, it is possible to do calculus on such supermanifolds in formally much the same way as on ordinary "Bosonic" manifolds. I shall assume that this has been done and indicate an application. Since  $J \propto \psi$  are anticommuting  $(2J\psi)^2 = 0$  so there is a function  $A_{[1]} \rightarrow A_{(1)}$  by setting  $X = 2J\psi$ . Indeed, there are, more generally,  $A_{[N]} \rightarrow A_{(N)}$  and it may well be illuminating to investigate the curved analogue thereof. This note sticks to the case  $N=1$ . SuperMinkowski space  $\mathbb{M}_{[1]}$  has variables  $(x^\alpha, \theta^\alpha, \tilde{\theta}^\alpha)$  where  $\theta$  and  $\tilde{\theta}$  are Fermionic spinor variables.  $\mathbb{M}$  sits inside  $\mathbb{M}_{[1]}$  as  $\{\theta = 0 = \tilde{\theta}\}$ .

Just as there is a Penrose transform from ambitwistor to Minkowski space so there is a superPenrose transform from superambitwistor to superMinkowski space. It is based on the incidence relation

$$\left. \begin{array}{l} w^A = (\chi^{AA'} - \theta^A \bar{\theta}^{A'}) \pi_{A'} \\ \zeta = \bar{\theta}^{A'} \pi_{A'} \\ \xi^{A'} = (-\chi^{AA'} - \theta^A \bar{\theta}^{A'}) \eta_A \\ \psi = \theta^A \eta_A \end{array} \right\} \quad \begin{array}{l} \zeta^\alpha = (w^A, \pi_{A'}) \\ W_\alpha = (\xi^{A'}, \eta_A) \end{array}$$

Let  $M$  be a region in Minkowski space. This can be supercharged to  $M_{[1]}$  (no restriction on  $\theta \propto \bar{\theta}$ ) and gives rise to corresponding regions  $A_{(1)} \subset A_{(1)} \times A_{[1]} \subset A_{[1]}$ . In the ambitwistor description of Yang-Mills fields (Isenberg-Yasskin-Green, Witten,...) it is a crucial fact that  $H^1(A_{(1)}, \Omega(-1, -1)) = 0$  and this can be tricky to prove. Supercalculus gives a very simple proof as follows. One can Penrose transform to  $M$  but this factors through  $H^1(A_{[1]}, \Omega(-1, -1))$  i.e. first pull-back to  $A_{[1]}$ , then superPenrose transform to  $M_{[1]}$  and finally set  $\theta = 0 = \bar{\theta}$ . It therefore suffices to show that  $H^1(A_{[1]}, \Omega(-1, -1)) = 0$ . This follows by applying the superPenrose transform which can be effected by the "splitting" method. In other words, suppose  $F_{ij}(\zeta^\alpha, \zeta, W_\alpha, \psi)$  is a representative cocycle and set  $f_{ij} = F_{ij}((\chi^{AA'} - \theta^A \bar{\theta}^{A'})\pi_{A'}, \pi_{A'}, \bar{\theta}^{A'} \pi_{A'}, (-\chi^{AA'} - \theta^A \bar{\theta}^{A'})\eta_A, \eta_A, \theta^A \eta_A)$ . Then note that  $\eta^A \pi^{A'} \nabla_{AA'} f_{ij} = 0$  and hence, if we split  $f_{ij} = f_j - f_i$ , then  $\eta^A \pi^{A'} \nabla_{AA'} f_j$  defines a global homogeneous zero function and so is independent of  $\eta \propto \pi$  i.e. defines  $\phi = \phi(x, \theta, \bar{\theta})$ . I claim, however, that  $\phi = 0$ . To see this note that  $f_{ij}$  also satisfies

$$\left. \begin{array}{l} \eta^A \partial_A f_{ij} = 0 \\ \pi^{A'} \tilde{\delta}_{A'} f_{ij} = 0 \end{array} \right\} \text{where } \partial_A = \frac{\partial}{\partial \theta^A} + \bar{\theta}^{A'} \nabla_{AA'} \text{ & } \tilde{\delta}_{A'} = \frac{\partial}{\partial \bar{\theta}^{A'}} + \theta^A \nabla_{AA'}.$$

Thus  $\eta^A \partial_A f_j$  is globally defined but, being homogeneous of degree  $-1$  in  $\pi$ , must vanish. Similarly  $\pi^{A'} \tilde{\delta}_{A'} f_j = 0$ . Hence:

$$\begin{aligned} 0 &= \eta^A \partial_A (\pi^{A'} \tilde{\delta}_{A'} f_j) + \pi^{A'} \tilde{\delta}_{A'} (\eta^A \partial_A f_j) \\ &= \eta^A \pi^{A'} (\partial_A \tilde{\delta}_{A'} + \tilde{\delta}_{A'} \partial_A) f_j = 2 \eta^A \pi^{A'} \nabla_{AA'} f_j = 2\phi, \text{ q.e.d.} \end{aligned}$$

Other applications (using  $N \geq 1$  and other homogeneities) together with more rigor will appear in a preprint soon.

Ambitwistors and Yang-Mills Fields in  
Self-Dual Space-Times

Claude LeBrun

Abstract. A generalization of the Witten/Iseñberg-Yasskin-Green correspondence for non-self-dual Yang-Mills fields is proposed for space-times with self-dual Weyl curvature.

### §1. Ambitwistors and Twistors

Let  $\mathbb{M}$  be a complex 4-manifold with holomorphic conformal structure; and for the sake of simplicity, assume that  $\mathbb{M}$  is geodesically convex. The ambitwistor space  $N$  of  $\mathbb{M}$  is defined to be the space of null geodesics in  $\mathbb{M}$ . If the Weyl curvature happens to be self-dual, we may also introduce the twistor space  $P$  of  $\mathbb{M}$ , defined to be the space of anti-self-dual 2-surfaces ("twistor surfaces") in  $\mathbb{M}$ . Both  $N$  and  $P$  encode the conformal geometry of  $\mathbb{M}$ ; what is the precise relationship between them? The answer is, roughly, that the complex 5-manifold  $N$  is the projectivized cotangent bundle of the 3-manifold  $P$ .

With the end of making this more precise, recall that to every  $x \in \mathbb{M}$  corresponds a Riemann sphere  $L_x \subset P$ , namely the set of twistor surfaces through  $x$ . If  $y \in L_x$ , and if  $\varphi \in T_y^*P$  is a non-zero holomorphic covector at  $y$ , then we shall say that the projective class  $[\varphi]$  of  $\varphi$  is explored by  $x$ .

iff  $\varphi$  is orthogonal to  $L_x$  -- i.e. iff  $\langle \varphi, v \rangle = 0 \quad \forall v \in T_y^! L_x$ .

We shall also say that  $z \in \mathbb{P}(T^* \mathcal{P})$  is explored iff it is explored by some  $x \in \mathcal{M}$ .

Theorem. The ambitwistor space  $N$  is canonically biholomorphically equivalent to the space  $\mathcal{U}$  of explored points in  $\mathbb{P}(T^* \mathcal{P})$ .

Proof. Let  $y \in \mathcal{P}$ , and let  $S_y \subset \mathcal{M}$  be the corresponding twistor surfaces; thus  $x \in S_y \Leftrightarrow y \in L_x$ . Kodaira's theorem identifies  $T_x^! \mathcal{M}$  with the space of holomorphic sections of the normal bundle of  $L_x$ ; and as this normal bundle is isomorphic to  $\mathcal{O}(1) \oplus \mathcal{O}(1)$ , it follows that the map

$$\Phi_y : S_y \rightarrow \mathbb{P}(T_y^! \mathcal{P})$$

$$: x \rightarrow T_y^! L_x$$

is of maximal rank. Let  $\pi : \mathcal{U} \rightarrow \mathcal{P}$  be the canonical projection, and suppose that  $z \in \pi^{-1}(y)$ . The set  $\gamma_z \subset \mathcal{M}$  of points exploring  $z$  is a non-singular curve; indeed, if  $\ell_z \subset \mathbb{P}(T_y^! \mathcal{P})$  is the dual line of  $z = [\varphi]$ , so that

$$\ell_z := \{[v] \in \mathbb{P}(T_y^! \mathcal{P}) \mid \langle \varphi, v \rangle = 0\},$$

then  $\gamma_z = \Phi_y^{-1}(\ell_z)$ . Moreover, the curves  $\gamma_z$  are null curves, since each is contained in a (totally null) twistor surface.

In any twistor surface  $S_y$ , we may find a unique  $\gamma_z$  tangent to any chosen direction, since  $\Phi_y$  maps these curves to lines in  $\mathbb{P}_2$ . Since any non-zero null vector  $v \in T_y^! \mathcal{M}$  is

tangent to precisely one twistor surface, it follows that there is exactly one  $\gamma_z$  tangent to  $v$ . Since this  $\gamma_z$  depends holomorphically on  $[v]$ , it follows [3] that the  $\gamma_z$  are the null geodesics of a torsion metric connection. We must now show that the torsion may be taken to vanish.

The space of these torsion null geodesics is  $\mathcal{U} \subset \mathbb{P}(T^* \mathfrak{p})$ . The conformal torsion therefore vanishes [3] iff  $\mathcal{U}$  admits a holomorphic contact form (which would necessarily be unique). But any projective cotangent bundle has a contact form--namely the projectivization of the canonical form  $\theta = \sum p_j dq^j$ . Hence our "torsion" null geodesics  $\gamma_z$  are honest-to-goodness null geodesics, and  $\mathcal{U}$  is thus identified with  $\mathbb{N}$ .

Q.E.D.

Remark. It is also useful to have an explicit description of the quadric  $Q_x \cong \mathbb{P}_1 \times \mathbb{P}_1$  of null geodesics through  $x \in \mathbb{M}$ . In fact  $Q_x \subset \mathcal{U}$  is precisely the projectivized conformal bundle of  $L_x \subset \mathfrak{p}$ .

## §2. Formal Neighborhoods and Yang-Mills Fields

When  $\mathbb{M}$  is conformally flat, the ambitwistor space  $\mathbb{N}$  is an open subset of

$$\mathbb{A} = \{([Z], [W]) \in \mathbb{P}_3 \times \mathbb{P}_3^* \mid \langle Z, W \rangle = 0\};$$

with the present orientation conventions,  $\mathfrak{p} \subset \mathbb{P}_3^* = \mathbb{P}(T)$ , and the reader may note that  $\mathbb{A} = \mathbb{P}(T^* \mathbb{P}_3) = \mathbb{P}(T^* \mathbb{P}_3^*)$  in accordance

# 21

with the results of §1. Indeed,  $T^* \mathbb{P}_3 \subset T^* Q$  as the orthogonal space of the universal  $(\mathcal{O}(-1))$  line bundle  $L \subset T \times \mathbb{P}_3$ .

There is an immediate generalization of this to the general self-dual case. Let  $L \rightarrow P$  be the universal  $(\mathcal{O}(-1))$  line bundle defined by

$$\mathcal{O}(L^{\otimes 4}) = \Omega_P^3,$$

and let  $\tau := L - O_P$  be its frame bundle; here  $O_P \subset L$  is the zero section.  $\tau$  is a holomorphic principal  $\mathbb{C}_*$ -bundle, which is to say that for every non-zero complex number  $\zeta \in \mathbb{C}_*$  we are given a biholomorphism

$$\begin{aligned} m_\zeta: \tau &\rightarrow \tau \\ &: z \mapsto \zeta z, \end{aligned}$$

thus providing a representation of  $\mathbb{C}_*$  on  $\tau$  with  $P = \tau/\mathbb{C}_*$ .

Taking adjoints of derivatives, we have a family of induced maps

$$m_\zeta^*: T'^* \tau \rightarrow T'^* \tau,$$

and the quotient  $E = T'^* \tau / \mathbb{C}_*$  is naturally a rank 4 holomorphic vector bundle over  $P$ ; thus,  $E \rightarrow P$  is a pushed-down version of  $T'^* \tau \rightarrow \tau$ . The pull-back

$$\mathcal{O}(T'^* P) \hookrightarrow \mathcal{O}(T'^* \tau)$$

is  $\mathbb{C}_*$  invariant, and hence induces an inclusion  $T'^* P \hookrightarrow E$ . We thus have a naturally defined imbedding

$$\mathbb{N} \hookrightarrow \mathbb{P}(E)$$

generalizing the flat-case imbedding  $\mathbb{M} \hookrightarrow \mathbb{P}_3 \times \mathbb{P}_3^*$ .

Definition. For  $\mathbb{M}$  any self-dual space-time, the  $n$ th-order neighborhood of the ambitwistor space  $\mathbb{N}$  is the ringed space

$$\mathbb{N}^{(n)} = (\mathbb{N}, (\mathcal{O}_{\mathbb{P}(E)}/I^{n+1})|_{\mathbb{N}}),$$

where  $I \subset \mathcal{O}_{\mathbb{P}(E)}$  is the ideal of holomorphic functions vanishing on  $\mathbb{P}(T^*p)$ .

In the case of  $n = 1$ , this agrees with the alternate definition given in [4]--though the verification involves some quite tedious calculations; the basic ingredient is that  $E \otimes L \rightarrow p$  correspond to the dual of the bundle of local twistors on  $\mathbb{M}$  via the usual Ward correspondence. It appears, moreover, that this fact alone determines  $\mathbb{N}^{(n)}$  uniquely, provided that  $n \geq 3$ .

The machinery developed in [1] would seem to generalize to the present circumstances without any essential modification. This would then prove the following:

Conjecture. There is a natural 1-1 correspondence between

- (a) Yang-Mills fields on  $\mathbb{M}$ ; and (b) holomorphic vector bundles on  $\mathbb{N}^{(3)}$  which are trivial when restricted to quadrics  $Q_x \subset \mathbb{N}$ .

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## TWISTORS AND FIELDS WITH SOURCES ON WORLDLINES

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ABSTRACT

It is shown that zero-rest-mass fields with sources on an analytic wordline are naturally defined on a double cover of some region of Minkowski space.

Twistor spaces are constructed which correspond to such regions and these turn out to be non-Hausdorff spaces obtained by identifying two copies of regions in ordinary twistor space, except on a ruled surface which corresponds to the worldline.

It is shown that cohomology classes on the twistor space correspond to sourced fields on Minkowski space thus extending the twistor description of massless fields.

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# IS THE PLEBANSKI VIEWPOINT RELEVANT TO THE GOOGLY PROBLEM?

There are two ways to set about hunting the googly. The first is to take a standard 'legbreak' graviton and hold up a mirror to it ("dualize!") so that the googly thinks it is one of its own kind & comes trotting tamely towards you. The second is to dress up a linear graviton (i.e. a field  $A^{\mu\nu}{}_{\alpha\beta}$ ) on flat spacetime obtained from an  $f_{-6}(z)$ ) so that it looks like the remains of a living (non-linear) googly which has been steamrollered, and hope the googly is attracted by this.

Plebanski on the other hand took the halfflat graviton & turned it into a single function of 4 space-time variables obeying a non-linear differential equation [P]. In fact he produced alternative forms of this transformation, of which I will be concerned with the second, in terms of the function  $\Theta(p, q, x, y)$ . In fact we won't pay much attention to  $\Theta$  itself and its equation (since ultimately we seek an equation-less point of view) but rather use it to suggest new aspects, & we shall see that it leads to a way of "dressing up" the linear graviton.

First let us recall how P's "second form" works (see also [P] but the language is different!). Let  $p, q, x, y$  be spacetime coords. Consider a null tetrad of the following form:

$$\begin{aligned} o^{\alpha} o^{\alpha'} \nabla_{\alpha\alpha'} &= D = \partial_{/\partial x}, \quad o^{\alpha} o^{\alpha'} \nabla_{\alpha\alpha'} = \bar{s} = \frac{\partial}{\partial p} + \Theta_{xy} \frac{\partial}{\partial x} - \Theta_{xx} \frac{\partial}{\partial y}, \\ o^{\alpha} l^{\alpha'} \nabla_{\alpha\alpha'} &= s = \partial_{/\partial y}, \quad l^{\alpha} l^{\alpha'} \nabla_{\alpha\alpha'} = \Delta = \frac{\partial}{\partial y} + \Theta_{yy} \frac{\partial}{\partial x} - \Theta_{yx} \frac{\partial}{\partial y}. \end{aligned} \quad (1)$$

where  $\Theta_{xy} = \frac{\partial^2 \Theta}{\partial x \partial y}$  etc. Consider the commutator

$$\begin{aligned} [\bar{s}, \Delta] &= \left\{ \begin{array}{l} \Theta_{yy} p + \Theta_{xy} \Theta_{yyx} - \Theta_{xx} \Theta_{yy} \\ - \Theta_{xyy} - \Theta_{yy} \Theta_{xyx} + \Theta_{yx} \Theta_{xyy} \end{array} \right\} \partial_{/\partial x} + \left\{ \dots \right\} \partial_{/\partial y} \\ &= \left\{ \Theta_{py} - \Theta_{qy} + \Theta_{xy} \Theta_{xy} - \Theta_{xx} \Theta_{yy} \right\}_y \partial_{/\partial x} + \left\{ \text{ditto} \right\}_x \partial_{/\partial y} \end{aligned}$$

which will vanish if  $\text{this} \cdot \left\{ \downarrow \right\} = 0 \dots \dots (2)$

The other commutators are

$$\begin{aligned} [D, \bar{s}] &= \Theta_{xy} D - \Theta_{xx} s \\ [s, \Delta] &= \Theta_{yy} D - \Theta_{xy} \Delta \end{aligned}$$

$$[D, \bar{s}] = 0$$

$$[D, \Delta] = \Theta_{xyy} D - \Theta_{xyy} \Delta$$

$$[\bar{s}, \bar{s}] = \Theta_{xyy} D - \Theta_{xyy} \bar{s}$$

From the three on the right it follows that

$[\bar{s} + kD, \Delta + k\bar{s}] = 0$  for any constant  $k$ , so the unprimed spinors integrate to form  $\beta$ -planes. The non-zero components of the spinor connection are:

$$\bar{s} o^{\alpha'} = \Theta_{xx} l^{\alpha'} - \Theta_{xy} o^{\alpha'}$$

$$\Delta o^{\alpha'} = \Theta_{xy} l^{\alpha'} - \Theta_{yy} o^{\alpha'}$$

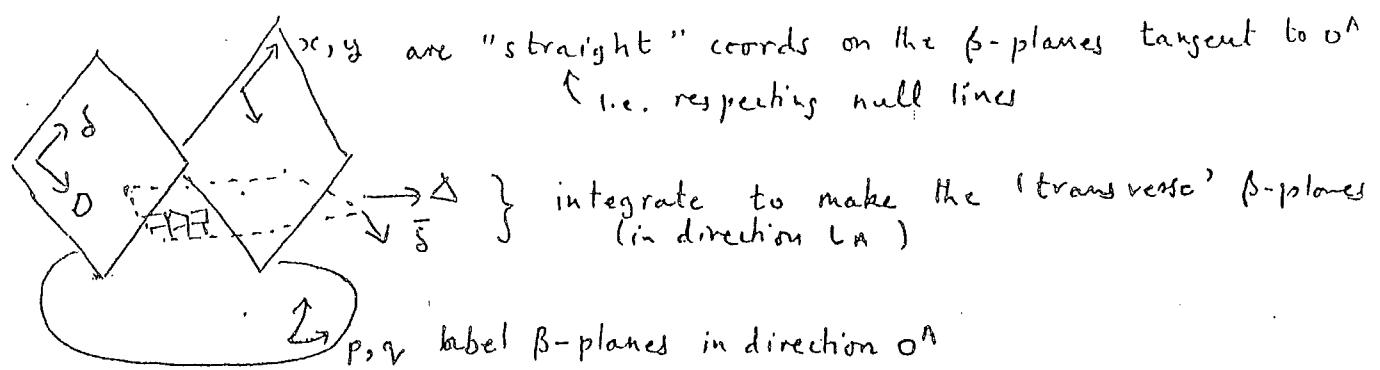
$$\bar{s} l^{\alpha'} = \Theta_{xy} l^{\alpha'} - \Theta_{yy} o^{\alpha'}$$

$$\Delta l^{\alpha'} = \Theta_{yy} l^{\alpha'} - \Theta_{yy} o^{\alpha'}$$

and typical curvature components

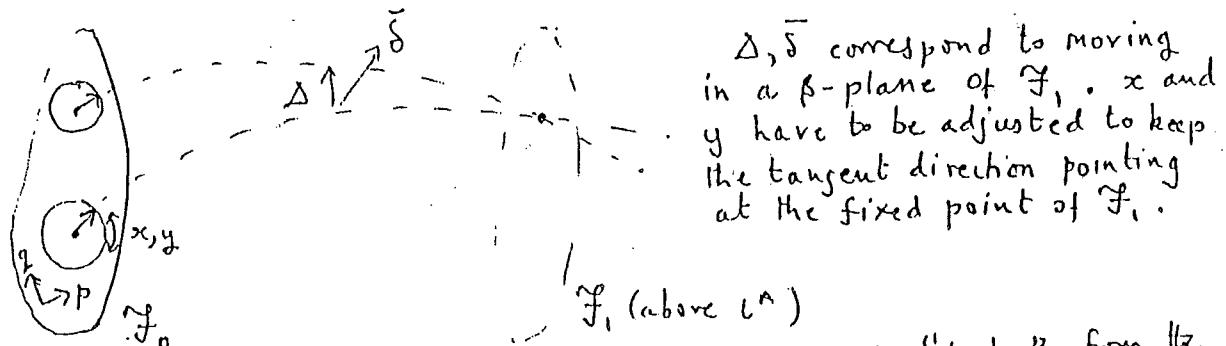
$$D \bar{s} o^{\alpha'} - \bar{s} D o^{\alpha'} - [D, \bar{s}] o^{\alpha'} = \Theta_{xx} o^{\alpha'} - \Theta_{xyy} o^{\alpha'} \dots \dots (3)$$

Thus we have arrived at a  $\frac{1}{2}$  flat graviton. In a picture:



Note that equation (2) on  $\Theta$  ensures that the other  $\beta$ -planes which go across at intermediate angles are also integrable.

How does one go in the opposite direction? It is more illuminating to start from the legbreak (lower index) twistor picture rather than space-time itself (see also [KLNT]). Let  $p, q$  be coords in the fibre above  $O_A, \mathcal{F}_0$  (in this context we will always assume that  $p, q$  respect the canonical 2-form on the fibres). Let  $x, y$  be coords on the projective tangent bundle  $PT(\mathbb{P}\mathbb{T}^n)$ , respecting the restriction to  $\mathcal{F}_0$ , which respect the  $P^2$  structure with fixed line at  $\infty$  above each point of  $\mathcal{F}_0$ . Then  $p, q, x, y$  constitute coords on the space of global lines (consider their intersections with  $\mathcal{F}_0$ )



We claim that  $\Delta$  and  $\bar{\delta}$  which describe how  $\mathcal{F}_1$  "looks" from its point of view of the tangents at  $\mathcal{F}_0$  (light travels along global lines!) can be expressed by a function  $\Theta$  as in (1). In fact let us broaden our view and consider  $T(\mathbb{P}\mathbb{T}^n)$  over an open region (not restricted to  $\mathcal{F}_0$ ) with coords  $p, q, \bar{y}$  (with  $\bar{y}$  standard coord on base). For coords on the fibres of  $T(\mathbb{P}\mathbb{T}^n)$  let  $(x, y, t)$  denote the derivative  $\frac{\partial}{\partial p} + \frac{\partial}{\partial q} + t \frac{\partial}{\partial \bar{y}}$ . Then once  $O_A$  is chosen we have a non-linear connection (NLC) given by: "keep pointing at the same point of  $\mathcal{F}_1$ , and keep  $t$  constant". Then we have:

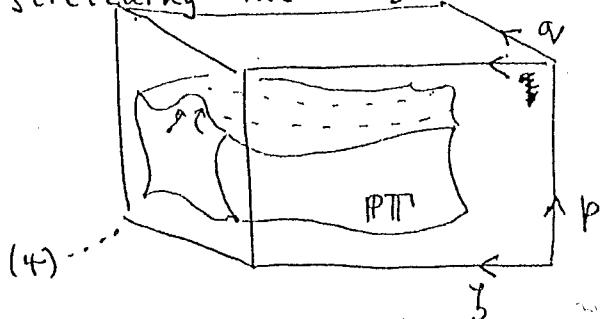
Proposition. For any choice\* of  $p, q$  there is a  $\Theta(pq\bar{y}; xyt)$  homog. of degree 3 in  $x, y, t$  such that the NLC is given by the horizontal vector fields  $\bar{\delta}, \Delta$  (as in (1)) and

$$\delta = \frac{\partial}{\partial \bar{y}} + \Theta_{xy} \frac{\partial}{\partial x} - \Theta_{yt} \frac{\partial}{\partial y}$$

and  $\Theta$  satisfies equation (2) and two analogous equations which ensure the integrability of  $\bar{\delta}$  and  $\delta$ ,  $\Delta$  and  $\delta$ . (\*: as mentioned above we take for granted that  $d\rho dq$  is correct)

To what extent does  $\Theta$ , considered as a fn on the tangent bundle depend on the choice of  $p, q$ ? Note first from (3) that the 4th derivatives  $\Theta_{xxxx}$  etc. equal the components of  $\Theta^{ab} \delta^{cd}$  which cannot depend on the coords, so  $\Theta$  can change at most by a 3rd degree polynomial in  $x$  and  $y$ , i.e. a cubic in  $xy$  & - which can also be characterised as a global change. (10 degrees of freedom). On the other hand in so far as  $\Theta$  expresses an invariant connection on  $T(\text{PTT})$  it will depend on the second derivatives of the coord. change. On the face of it this is 12 degrees of freedom (the 2nd derivs. of 2 fns of 3 variables) but the 3 derivatives of the area preserving condition  $\frac{\partial(p, q)}{\partial(p, q)} = 1$  reduce this to 9. Still doesn't agree! This is because the tenth degree of freedom  $\Theta \rightarrow \Theta + at^3$  is completely immaterial as regards the connection. (In fact the cubic change in  $\Theta$  corresponds precisely to the 3rd derivatives  $F_{ppp}$  etc. of the function which describes the coord change locally as  $w \rightarrow w + \partial w \mid F_{xx}$ ; again  $F_{yyy}$  has no material significance).

One can imagine  $\Theta$  as a kind of non-isotropic stress in this slab of PTT, which can be altered by bending and stretching the slab relative to a fixed framework of coords.



If the slab happens to be "pre-stressed" (with  $\Theta_{xxxx}$  etc.  $\neq 0$ ) then not even the most judicious deformation can eliminate the stress altogether. However, what one can do is to straighten out all the

lines which point towards a chosen point of  $F$  (which we take to be outside the slab covered by  $p, q, r$ ), then  $\Theta$  reduces to zero on the corresponding section of  $T(\text{PTT})$ . and its derivatives up to 3rd order:  $\Theta_{xxx}$  etc.

Clearly this "pre-stressing" contains the information of the graviton, and I hope to show that it seems to be strangely amenable to being represented in googly or "dual" (with respect to the legbreak) fashion, while at the linear level it is directly related to the information of an  $f_6$ .

What do I mean by "dual"? In this case the following seems appropriate: Think of picture (4) as a map from the twistor slab to the coordinate grid  $P^3$ . Then follow it by the canonical map from points of  $P^3$  to planes of the (ordinary) dual  $P^3$  (which I will call  $P^{3*}$ ). Now if you deform the slab, the planes will flap around in  $P^{3*}$ . Each one will be anchored at its intersection with the line  $I$  (the area-preserving character of the deformation

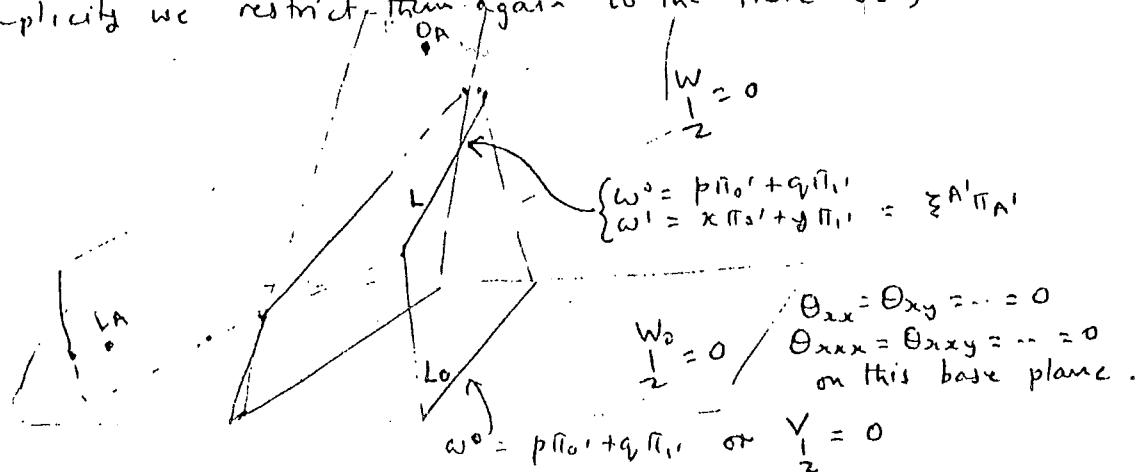
will also have some image). Note that the planes will not actually slide over one another - they are "sticky" in this sense. The nearest neighbours of a plane each intersect it in a line & this relationship between neighbours & lines cannot be altered by a deformation. It is only a bit further away ("at 2nd order") that things can change, and we will use  $\Theta$  to investigate this.

First, however let us simply translate the NLC into information in this picture. We must know how to transport lines from one plane to another (the dual of "tangent vectors from point to point"). If  $\Theta = 0$  this is done by casting shadows from the point  $L_A$  (on the line  $I$ ).

If  $\Theta \neq 0$ , we can only say:

The shadows go all wrong! But let us investigate how  $\Theta$  distorts the shadow at any one plane. Suppose we have straightened out the lines in the slab which point at one particular point of  $\mathcal{F}_1$ . In  $P^3*$  this gives one plane through  $L_A$ ,

$w^1 = 0$  say (or  $\frac{w^0}{2} = 0$ ) which is a true cross-section of the NLC, unlike the other such planes  $w^1 = x\pi_{01} + y\pi_{11} = \xi \cdot \pi$  or  $\frac{w^0}{2} = 0$ ; they have to be jiggled up and down (i.e.  $x$  and  $y$  have to change) as they cut across the planes  $w^0 = p\pi_{01} + q\pi_{11}$  (for simplicity we restrict them again to the fibre  $\mathcal{F}_0$ )



Now consider  $\Theta$  (a function of lines) on the plane  $V_\alpha$ . Since the 2nd and 3rd derivatives of  $\Theta$  are zero at the base line  $L_0$ ,  $\Theta_{xx}, \Theta_{xy}, \Theta_{yy}$  at  $L$  can be obtained by twice integrating the 4th deriv., in other words the components of  $\Theta_{AB'C'B'}$ , from  $L_0$  to  $L$ . Now any such  $\Theta_{AB'C'B'}$ , so long as we consider one  $\beta$ -plane at a time can be expressed as the restriction of an  $f_{\beta\gamma}$  on  $P^3*$  to the plane  $V$ . Thus

$$\Theta_{xxxx} = \oint (\pi_{01})^4 f_{\beta\gamma} (p\pi_{01} + q\pi_{11}, x\pi_{01} + y\pi_{11}, \pi_{01}, \pi_{11}) d^2\pi$$

and so on. Hence  $\Theta_{xx} = \oint (\pi_0)^2 d^2\pi \int (\xi \cdot \pi - \omega^1) d\omega^1 f_{-c}$ ,  
 $\Theta_{xy} = \oint \pi_0' \pi_1 d^2\pi \dots$ ,  $\Theta_{yy} = \dots$ . Now from (1) the  
change in  $y$  as one moves over  $p, q$  is given by:

$$dy = -\Theta_{xx} dp - \Theta_{yy} dq$$

$$= \oint -\pi_0' (\pi_0 dp + \pi_1 dq) d^2\pi \int_{L_0}^L \dots = \oint \pi_0' d\omega^0 d^2\pi \int \dots$$

Similarly for  $dx$ . Combining, we have:

$$d\xi_A = \int \pi_A (\xi \cdot \pi - \omega^1) f_{-c} d\omega^0 d\omega^1 d^2\pi$$

or,  $dW_1 = \int \frac{1}{2} z \bar{z} f_{-c} d^4z$ . The contour is a cylinder, bounded by  $L_0$  and  $L$ . See what has happened: the property of the NLC, that it is given by a  $\Theta$  as in (1) has turned out to have a surprisingly natural-looking expression in the googly picture. The question arises: what is the meaning of

$$\phi = \int_{L_0}^L \frac{1}{2} z \bar{z} f_{-c} \frac{1}{2} \bar{z} d^3z \dots \quad (5)$$

note  
 $\boxed{\phi} = 0$   
 $\boxed{\partial} = 0$

? This expresses the difference of the appearance of the NLC when you change from coordinates based on the cross-section  $W_0$  to co-ords based on  $W$ .  $\phi$  can be interpreted as the twisting (around their points at  $\infty$ ) of neighbouring planes which results from this deformation. The condition for this twisting at each plane to integrate to a deformation must be something analogous to " $d\phi = 0$ ".

In the linear case this property follows from the fact that the  $f_{-c}$ 's on the various planes are all restrictions of a single  $f_{-c}$  on  $P^3$ ; in fact  $\int \phi$  from one plane to another is the volume integral of  $\frac{1}{2} z \bar{z} f_{-c} \frac{1}{2} \bar{z} d^3z$  between the planes (contour  $\boxed{z} \times S^1$ ).

For the full googly something more geometrical is needed to express the integrability (in the appropriate sense) of  $\phi$ . A useful first step would be to answer the question: what kind of a deformation of the environment of a plane is determined by an  $H^1(\mathcal{O}(-6))$  on that plane (more accurately: the restriction of a 3-dimensional  $H^1$ ), that forces the plane to twist as it moves, according to (5)?

G. Burnett - Stuart.

References [P] = "Some solutions of complex Einstein solutions"  
JMP 16 2395 (1975)

[KLNT] = "Theory of H-space" by Ko, Ludvigsen, Newman, Tod  
Phys. Rep. 71 53 (1981)

[H] = "Non-linear connections for curved twistor spaces" by A. Helfer  
(preprint)

COHOMOLOGICAL INTERPRETATION OF  $t(z^\alpha) = \frac{R}{z} \log \frac{R}{z}$

In TMN 14 (p.20-21) it was shown how  $f(z^\alpha) = \log \left\{ \frac{R}{z} \frac{R}{z} / (\alpha^A \pi_{A'})^2 \right\}$ , which is known to be a twistor function for the Coulomb field of a charge moving on the straight world-line corresponding to the quadric  $Q$  ( $\{ \frac{R}{z} \frac{R}{z} = 0 \}$ ), could be described cohomologically. Briefly, one exponentiates and uses  $\frac{R}{z} \frac{R}{z} / (\alpha^A \pi_{A'})^2$  etc. as a representative cocycle for an element of  $H_Q^1(\mathbb{P}^1, \mathcal{O}^*)$ .

The analogous spin-2 field is the linearized Schwarzschild solution, the left handed part of which is generated by the twistor function  $\frac{R}{z} \log \left\{ \frac{R}{z} \frac{R}{z} / (\alpha^A \pi_{A'})^2 \right\}$ , where  $\alpha^A$  is any constant spinor. We now give a cohomological interpretation of this.

Define the quotient sheaf  $G(2)$  by:-

$$0 \longrightarrow \frac{R}{z} \frac{R}{z} \longrightarrow \mathcal{O}(2) \longrightarrow G(2) \longrightarrow 0.$$

We note first that first cohomology with coefficients in  $G(2)$  generates massless spin-2 fields just as  $\mathcal{O}(2)$  does, because the  $\frac{\partial}{\partial w^A}$ 's in the contour integral are (more than) sufficient to kill off functions of the form  $\frac{R}{z} \frac{R}{z}$ . If we are dealing with ordinary cohomology on a region for which the Penrose transform is an isomorphism, then we don't describe any new fields by using  $G(2)$  in place of  $\mathcal{O}(2)$ .

In the case of  $H_Q^1(\mathbb{P}^1, G(2))$  (to which  $\frac{R}{z} \log \left\{ \frac{R}{z} \frac{R}{z} / (\alpha^A \pi_{A'})^2 \right\}$  will shortly be shown to belong) the long exact sequence

corresponding to the defining sequence of  $G(2)$  contains the segment:-

$$0 \rightarrow H^1_Q(\mathbb{P}^I, \theta(2)) \longrightarrow H^1_Q(\mathbb{P}^I, G(2)) \longrightarrow H^2_Q(\mathbb{P}^I, \mathbb{T}_{(\alpha\beta)}^I) \rightarrow 0.$$

It is fairly easy to see that  $H^2_Q(\mathbb{P}^I, \mathbb{T}_{(\alpha\beta)}^I) \cong \mathbb{T}_{(\alpha\beta)}^I$  and it would seem that the map to this group is a "charge" map (c.f. TN14 p.20) giving the ten complex conserved quantities of the field; only if all these vanish can a representative in  $H^1_Q(\mathbb{P}^I, \theta(2))$  be found.

To realise the twistor function as a Čech cocycle for  $H^1_Q(\mathbb{P}^I, G(2))$  we first define a relative cover (TN14. p.12) of  $\{\mathbb{P}^I, \mathbb{P}^I \setminus Q\}$ , depending on two constant spinors  $\alpha^A, \beta^A$  as:

$$U_\gamma = \mathbb{P}^I \setminus Q \quad \text{to cover } \mathbb{P}^I \setminus Q$$

augmented :  $U_\alpha = \mathbb{P}^I \setminus \{\alpha^A \pi_{A1} = 0\}$

by :  $U_\beta = \mathbb{P}^I \setminus \{\beta^A \pi_{A1} = 0\}$ .

The cocycle is now given by:

$$f_{\alpha\bar{\alpha}} = \frac{Q}{z\bar{z}} \log \left\{ \frac{Q}{z\bar{z}} / (\alpha^A \pi_{A1})^2 \right\}; \quad f_{\beta\bar{\beta}} = \frac{Q}{z\bar{z}} \log \left\{ \frac{Q}{z\bar{z}} / (\beta^A \pi_{A1})^2 \right\};$$

$f_{\alpha\beta} = \frac{Q}{z\bar{z}} \log \left( \frac{\beta^A \pi_{A1}}{\alpha^B \pi_{B1}} \right)^2$ . The point of the whole construction is that the sheeting of the logarithms is absorbed in the definition of  $G(2)$ .

One can play this game with other non-negative homogeneities; in particular  $G(0)$  defined in the obvious way allows (unfortunately (?)) a cohomological description of covariant fields of non-integral charge.

It may also be possible to make sense of other twistorial logarithms, for example in the scalar product, by analogous constructions. The problem is deciding how to define ones up "int. products" since  $G(2)$  (for example) is not a sheaf of  $\theta$ -modules.

## Colouring Donaldson's Moduli Space

Penrose has suggested<sup>1)</sup> that Donaldson's moduli space<sup>2)</sup> may have some spacetime significance. The points of this moduli space are 1-instanton solutions to the  $SU(2)$  Yang-Mills theory on a compact differentiable 4-manifold  $M$ . Roughly speaking, it is a smooth 5-manifold-with-boundary together with certain singularities. The boundary is diffeomorphic to the original spacetime  $M$ . The singularities can be considered as the vertex of cones whose cross-section is  $\mathbb{CP}^2$ . The  $\mathbb{CP}^2$ 's occur as a result of general cobordism theory. I wish to make a trivial speculative remark on the possible significance of these  $\mathbb{CP}^2$ 's.

The group  $U(3)$  acts transitively on  $\mathbb{CP}^2$  preserving both its metric and its complex structure. Furthermore,  $\mathbb{CP}^2$  is the symmetric space  $U(3)/U(1) \times U(1)$  (also  $SU(3)/S(U(1) \times U(2))$ ). Put in a slightly different way,  $\mathbb{CP}^2$  is the space with the lowest dimension having  $SU(3)$  as its symmetry group.<sup>3)</sup> (We ignore the difference between  $U(3)$  and  $SU(3)$  for the moment. We shall see that this does not matter for our purpose.) Given that spacetime is 4-dimensional, if Penrose's suggestion gives us any glimpse of what a true picture of the world looks like, then these  $\mathbb{CP}^2$ 's will arise naturally. It then follows (?) that  $SU(3)$  is a natural symmetry of the 4-dimensional world.

Most physicists now take it as proved that strong interaction is described by the symmetry group  $SU(3)$ . If one incorporates electromagnetism in the accepted way, the symmetry group is actually  $U(3)$ . Hence we can consider either of them. Except that it seems to "work", we have no clue as to why it should be this particular group.

The above observations, though both vague and wild to the extreme, might be a clue. The one good (?) point about it is that the symmetry actually depends just on the dimensionality of spacetime and not its detailed structure.

If I were allowed to be even more speculative, I would point out that in representing  $\mathbb{C}P^2$  as a symmetric space, one calls into play all the known symmetries of particle physics. But this is perhaps really going too far.

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Ton Sherry Tsou.

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#### A D V E R T I S E M E N T

Forthcoming book

#### SPINORS AND SPACE-TIME

by Roger Penrose and Wolfgang Rindler

Cambridge University Press 1984

SOON TO BE PUBLISHED

# Entropy, Uncertainty, and Nonlinearity

by D.P. Hughston

Stimulated by talks and discussions at the Oxford Quantum Gravity Workshop at Lincoln College earlier this year, I had the following thoughts.

Perhaps we should take a Bayesian attitude towards the wave function in quantum mechanics. In probabilistic terms this means we are faced with the task of formulating "best bets with the information given". When we have exact knowledge of the Hamiltonian and exact knowledge of the initial conditions or boundary conditions, then nothing is gained by the use of Bayesian methods. But suppose there is uncertainty — suppose we only have partial knowledge of the conditions under which a solution is sought. Then, providing we have a suitable means at our disposal for assigning prior probabilities to the unknown data, we can proceed by Bayesian methods to construct an exact wave function for the system under consideration — only now we must interpret the wave function in a more general way as providing us with information about the "best bet" as to the particle's behavior (rather than determining an exact statistical distribution for an ensemble). The problem of prior probabilities can be approached through the 'principle of maximum entropy' (cf. Jaynes 1957, Grandy 1980, Rosenkrantz 1983) which can be regarded as a refinement on the Laplacian 'principle of indifference' (cf. Keynes 1929 chap. II). Needless to say in most realistic circumstances we do indeed lack complete information in advance about details of the Hamiltonian, initial conditions, etc., therefore we must enquire how to formulate quantum mechanics under such conditions.

Let us consider first the Schrödinger equation for a single particle:

$$i\frac{\partial}{\partial t}\psi(x, t) = [-\frac{1}{2m}\Delta + U(x, t)]\psi(x, t). \quad (1)$$

Imagine a circumstance where the potential  $U(x, t)$  is not known exactly; all we are told is, say, its expectation value  $w(t)$ , defined by

$$w(t) = \langle U(x, t) \rangle = \int U(x, t) \psi \bar{\psi} d^3x. \quad (2)$$

We are also given that  $U(x, t)$  falls off sufficiently rapidly towards infinity, as does  $\psi(x, t)$ .

With this information alone at our disposal we wish to determine  $\psi(x, t)$ . The problem looks hopeless! Given initial data  $\psi(x, t_0)$  at  $t = t_0$  we evidently cannot use the Schrödinger equation to evolve it, since we do not know  $U(x, t)$ .

I would argue, however, that we can indeed determine  $\psi(x, t)$ , by the following method:

The configurational entropy  $S_\xi(t)$  of the wave function is defined by

$$S_\xi(t) = - \int \psi \bar{\psi} \ln(\psi \bar{\psi} / \xi) d^3x, \quad (3)$$

where  $\xi$  is an arbitrary parameter with dimensions of length<sup>-3</sup>

Note that under a change of  $\xi$  we have

$S_\xi - S_{\xi'} = \ln(\xi/\xi')$ . Thus the actual value of an entropy  $S_\xi$  is not necessarily meaningful — but the difference between two entropies [e.g.  $S_\xi(t_1) - S_\xi(t_2)$ ] has an 'absolute' meaning, and can be thought of as a

measure of the 'relative information content' of the two states under consideration: a gain in entropy corresponds to a loss in information. Roughly speaking the more spread out a wave function is, the higher its entropy.

The principle of maximum entropy amounts to the reasonable maxim that, as best as can be managed, we should not assume any information other than what we are given. Thus  $U(x, t)$  is to be chosen in such a way that  $S(t)$  is maximized, for each value of  $t$ , subject to the constraints of the given information.

Let us write  $p = \psi^* \bar{\psi}$ . Then we wish to maximize

$$S_p(t) = - \int p \ln(p/\xi) d^3x, \quad (4)$$

subject to

$$\int U(x, t) p d^3x = w(t) \quad (5)$$

and

$$\int p d^3x = 1, \quad (6)$$

the latter of these being the normalization condition on  $\psi$ .

Now by Galilean invariance  $U(x, t)$  can only depend upon  $x$  and  $t$  through  $p(x, t)$ , since the only information given about  $U(x, t)$  is in its relation to  $p(x, t)$ . Thus we have a variational problem, to maximize the integral expression

$$\int [-p \ln(p/\xi) + \lambda U(p) p + \mu p] d^3x,$$

together with (5) and (6), where  $\lambda$  and  $\mu$  are Lagrange multipliers. Varying this integral with respect to  $p$  we obtain

$$-\ln(p/\xi) - 1 + \lambda \frac{\partial}{\partial p} (U p) + \mu = 0$$

whence

$$-p \ln(p/\xi) + \lambda U p + \mu p = c$$

for some constant  $c$ . But  $\lim_{x \rightarrow \infty} p = 0$ ,

so we get

$$\lambda U = \ln(\rho/\xi) - u$$

Taking the expectation of each side of this relation we obtain

$$\lambda w(t) = -(S + u)$$

whence

$$U(x, t) = \frac{-w(t)}{S_\xi(t) + u} (\ln(\rho/\xi) - u) \quad (7)$$

The Lagrange multiplier  $u$  is to be determined by the normalization condition (6). Suppose, for a given value of  $\xi$ , we fix  $u$  by (6). Then  $u$  may be absorbed into the definition of  $\xi$  in (7) to yield a new value for  $\xi$ . With this new value of  $\xi$  (determined by the condition  $\psi$  be normalized) we have:

$$U(x, t) = -\frac{w}{S} \ln(\rho/\xi) \quad (8)$$

Note that by virtue of (4) the 'best bet' potential (8) satisfies the constraint (5), as desired.

Inserting expression (8) for  $U(x, t)$  into the Schrödinger equation (1) we obtain

$$i \frac{\partial \psi}{\partial t} = -\frac{1}{2m} \Delta \psi - \beta \psi \ln(\psi \bar{\psi}/\xi) \quad (9)$$

where  $\beta(t) = w(t)/S(t)$ . Thus we see that the effect of statistical uncertainty in the form of the potential is to introduce a non-linear term into Schrödinger's equation.

It should be stressed that the non-linearities here do not arise directly from any physical process, but

rather represent an accurate impression of the original uncertainty in the specification of the physical problem.

As an example, let us consider the case when  $\psi$  is in an energy eigenstate, with  $i\frac{\partial}{\partial t}\psi = E\psi$ . For simplicity we work in just one space dimension, and assume  $w(t)$  to be constant. After an intricate but straightforward calculation we obtain:

$$\psi(x,t) = \left(\frac{4m(E-W)}{\pi}\right)^{1/4} e^{-iEt} e^{-2m(E-W)(x-a)^2}$$

where  $a$  is an arbitrary constant. The other parameters of the theory turn out, in this example, to be:

$$\beta = 2(E-W), \quad S = \frac{1}{2} \frac{W}{E-W}, \quad \xi = \sqrt{\frac{4m(E-W)}{\pi}} e^{\frac{2E}{E-W}-1};$$

and the probability density is:

$$p(x) = 2 \sqrt{\frac{m(E-W)}{\pi}} e^{-2m(E-W)(x-a)^2}.$$

The mean-square deviation of the distribution is given by  $\sigma = 1/\sqrt{8m(E-W)}$ .

In this example we see that, apart from the constant  $a$ , the problem is completely determined. It is not surprising that  $a$  cannot be fixed — since this would amount to a violation of Galilean invariance — but the original information specified had such invariance. If we supply one extra bit of information, e.g. the value of  $\langle x \rangle$ , then  $a$  can be determined as well.

It should be pointed out that the modified Schrödinger equation (9) is identical in form to

the non-linear wave equation proposed by Iwo Bialynicki-Birula and Jerzy Mycielski, who argue persuasively in favour of its attractive properties. (In their theory  $\beta$  is a constant, and is not necessarily related to entropy as it is here.)

I hope to discuss this work in greater detail elsewhere, and to give some more examples of how we can solve Schrödinger's equation\* under conditions of 'minimal information'.

I wish to express my gratitude to J. Bekenstein, who in conversations at Aspen last summer drew my attention to Jaynes' papers.

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(\* and also, possibly, other quantum mechanical wave equations)

## APPLICATIONS OF SO(8) SPINORS

by

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ABSTRACT

The purpose of this article is to present a brief but relatively complete and self-contained account of the geometry of SO(8) spinors. These spinors are appropriate for the description of compactified flat six-space. As an application we construct a conformally invariant contour integral formula for solutions of Laplace's equation in six dimensions.

## Contents:

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  2. The Fundamental Quadric  $\Omega^6$
  3. Laplace's Equation on  $\Omega^6$
  4. SO(8) Spinors
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- Appendix A: On Laplace's Equation in Even Dimensions  
Appendix B: Polarization of  $\Omega_+$

To appear in the I. Robinson Festschrift (W. Rindler and A. Trautman, eds.: Bibliopolis)

The Projective Geometry of Simple Cosmological Models

TWISTOR THEORY AND THE EINSTEIN EQUATIONS

by

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University College, Oxford OX1 4BH and  
The Mathematical Institute, Oxford OX1 3LB, U.K.

December 1983

ABSTRACT

The conformal properties of flat space-time can be described in terms of the projective geometry of a 4-dimensional quadratic hypersurface  $\Omega$  embedded in projective 5-space  $P^5$ .

R. Penrose (1979, 1980) has argued that the goal of twistor theory with regard to the vacuum Einstein equations ought to consist of some kind of unification of twistor-theoretic descriptions of anti-self-dual (ASD) and self-dual (SD) space-times. SD space-times currently possess a description only in terms of dual twistor space, however, rather than twistor space. In this paper, suggestions due to Penrose for providing a purely twistor space description of SD space-times are investigated. It is shown how the points of certain SD space-times define mappings on twistor space and the geometry of these mappings is studied. The families of mappings for two particular SD space-times are presented explicitly.

Abstract

In this paper it is shown how an arbitrary conformally flat metric can be defined by the selection of a single scalar field on  $\Omega$ . The curvature tensor associated with this metric can then be calculated. The procedure is illustrated with this metric self-contained treatment of the geometry of Friedmann-Robertson-Walker models within the  $P^5$  framework. The essential properties of these space-times, such as the location of conformal infinity, curvature singularities, matter flow lines, etc., are all incorporated in geometric diagrams which arise naturally from the projective geometric construction.

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